KP maps and context-free languages

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OUTLINE

Outline

Consider the linear problem

$$\boldsymbol{\Psi}_{k+1} - \boldsymbol{\Psi}_{k(i)} = \boldsymbol{\Psi}_k u_{i,k}, \qquad i = 1, \ldots, N, \qquad k \in \mathbb{Z}$$

[Kajiwara, Noumi, Yamada 2002]

NOTATION: $\boldsymbol{\Psi}_{k(i)}(n_1, \ldots, n_i, \ldots, n_N) = \boldsymbol{\Psi}_k(n_1, \ldots, n_i + 1, \ldots, n_N)$

The compatibility conditions give equations

$$u_{j,k} u_{i,k(j)} = u_{i,k} u_{j,k(i)}, \qquad u_{i,k(j)} + u_{j,k+1} = u_{j,k(i)} + u_{i,k+1}$$

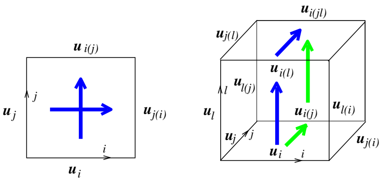
The system is equivalent to the non-Abelian Hirota–Miwa system by *[Nimmo 2006]*

THEOREM

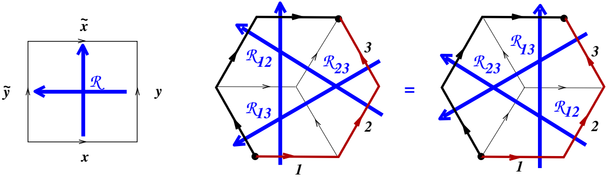
The non-commutative KP map

$$u_{i,k(j)} = (u_{i,k} - u_{j,k})^{-1} u_{i,k}(u_{i,k+1} - u_{j,k+1}), \qquad 1 \leq i \neq j \leq N,$$

is multidimensionally consistent



$$\boldsymbol{u}_i = (u_{i,k}), \qquad k \in \mathbb{Z} \quad \text{or} \quad k \in \mathbb{Z}_P, \quad u_{i,k+P} = u_{i,k}$$



OBSERVATION

[Adler, Bobenko, Suris 2004]

The COMPANION MAP $\mathcal{R}$ of a reversible three-dimensionally consistent map satisfies set-theoretical Yang–Baxter equation

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}$$

The companion map

In terms of the companion variables

$$x_k = u_{i,k}, \quad y_k = u_{j,k(i)}, \quad \tilde{x}_k = u_{j,k}, \quad \tilde{y}_k = u_{i,k(j)}$$

the NC discrete KP system (in periodic reduction) reads

$$x_k y_k = \tilde{y}_k \tilde{x}_k, \qquad y_k + x_{k+1} = \tilde{x}_k + \tilde{y}_{k+1}, \qquad k = 1, 2, \ldots, P$$

Lemma

The auxiliary functions h_k defined by

$$\tilde{x}_k = y_k - h_k^{-1} \qquad \text{or equivalently by} \quad \tilde{y}_{k+1} = x_{k+1} + h_k^{-1}$$

satify the system

$$y_k h_k = 1 + h_{k-1} x_k, \qquad k \mod P$$

whose solution (by successive approximation technique starting from $h_k^{(0)} = 0$) is

$$h_k = y_k^{-1}\left(1 + y_{k-1}^{-1} x_k + y_{k-1}^{-1} y_{k-2}^{-1} x_{k-1} x_k + y_{k-1}^{-1} y_{k-2}^{-1} y_{k-3}^{-1} x_{k-2} x_{k-1} x_k + \ldots\right)$$

By construction $\mathcal{R}$ is "involutive"/invertible, i.e. $\mathcal{R}_{21} \circ \mathcal{R}_{12} = \mathrm{id}$

Given alphabet $\mathcal{A} = \{a_1, b_1, \ldots, a_P, b_P\}$, consider the system

$$Z_k = 1 + a_k Z_{k+1} b_k, \qquad k \mod P$$

Given integer $n \geq 0$, denote

$$[ab]_k^0 = 1, \qquad [ab]_k^n = a_k [ab]_{k+1}^{n-1} b_k, \qquad \text{with subscripts modulo } P$$

EXAMPLE

For $P = 3$, $k = 1$ and $n = 5$ we have $[ab]_1^5 = a_1 a_2 a_3 a_1 a_2 b_2 b_1 b_3 b_2 b_1$

$$Z_k = \sum_{n \geq 0} [ab]_k^n = 1 + a_k b_k + a_k a_{k+1} b_{k+1} b_k + \ldots$$

The set $\mathcal{L}_P(k) = \{[ab]_k^n, \quad n \geq 0\}$ can be called a KP language

COMPOSITION $\alpha \models n$ is a finite sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of positive integers with $n = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, the length of the composition is $|\alpha| = m$. Define

$$[ab]_k^{\alpha} = [ab]_k^{\alpha_1} \ldots [ab]_k^{\alpha_m} \, .$$

PROPOSITION

The inverse of the series $Z_k = \sum_{n\geq 0} [ab]_k^n$ is given by

$$Z_k^{-1} = \sum_{n\geq 0} \sum_{\alpha \models n} (-1)^{|\alpha|} [ab]_k^{\alpha} = 1 - a_k b_k + a_k b_k a_k b_k - a_k a_{k+1} b_{k+1} b_K + \ldots \, .$$

Proof: Notice that $Z_k^{-1} = \sum_{n\geq 0} c_n$, where c_n is homogeneous polynomial of degree $2n$

$$Z_k^{-1} Z_k = 1 \quad \Rightarrow \quad c_0 = 1, \qquad c_0 [ab]_k^n + c_1 [ab]_k^{n-1} + \cdots + c_n = 0, \quad n > 0$$

The identification

$$c_n = \sum_{\alpha \models n} (-1)^{|\alpha|} [ab]_k^{\alpha}$$

follows from separation of the last component α_m from the composition α

$$\sum_{\alpha \models n} (-1)^{|\alpha|} [ab]_k^{\alpha} = - \sum_{i=1}^{n} \sum_{\beta \models n-i} (-1)^{|\beta|} [ab]_k^{\beta} [ab]_k^{i} \, .$$

Outline

$\mathcal{A} = \{a, b, c, \dots\}$ — finite set (alphabet)
$\mathcal{A}^*$ — set of finite sequences (words) over $\mathcal{A}$ including the empty sequence λ
$\mathcal{A}^* \supset \mathcal{L}$ — formal language
A FORMAL GRAMMAR — set of production rules together with some auxiliary symbols which allow to construct words of a given language

A grammar (or rewriting system) $G(N, A, R, S)$ is given by

1. an alphabet N whose elements are called variables or nonterminals,
2. an alphabet A (disjoint from N) whose elements are called terminals,
3. a finite set R of rewritting rules or productions, each rule being a pair

$$(N \cup A)^* N (N \cup A)^* \to (A \cup N)^*$$

4. an element $S \in N$ called the initial variable (axiom)

A CONTEXT-FREE GRAMMAR — the left-hand side of each production rule consists of only a single auxiliary symbol

$$N \to (A \cup N)^*$$

EXAMPLE

$\mathcal{A} = \{a, b\}$, $\mathcal{L} = \{\lambda, ab, aabb, aaabbb, \dots\} = \{a^n b^n\}_{n \in \mathbb{N}_0}$
Productions (here S is an auxiliary start symbol)

$$(1)\ S \to \lambda, \quad (2)\ S \to aSb$$

derivation of $aabb$: $S \overset{(2)}{\Rightarrow} aSb \overset{(2)}{\Rightarrow} aaSbb \overset{(1)}{\Rightarrow} aabb$

type	language	abstract machine (acceptor)
0	recursively enumerable	Turing machine
1	context-sensitive	non-deterministic linear-bounded TM
2	context-free	non-deterministic push-down automaton
3	regular	finite state automaton

The context-sensitive grammar G where $N = \{S, B\}$, $A = \{a, b, c\}$, S – the start symbol, and productions:

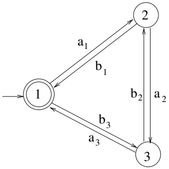
$$1.\quad S \to aBSc, \qquad 2.\quad S \to abc$$
$$3.\quad Ba \to aB, \qquad 4.\quad Bb \to bb$$

defines the language $\mathcal{L}(G) = \{a^n b^n c^n \mid n \geq 1\}$

Example (continued)

The characteristic series $\Sigma = \lambda + ab + aabb + aaabbb + \dots$ of the language $\mathcal{L}$ satisfies equation

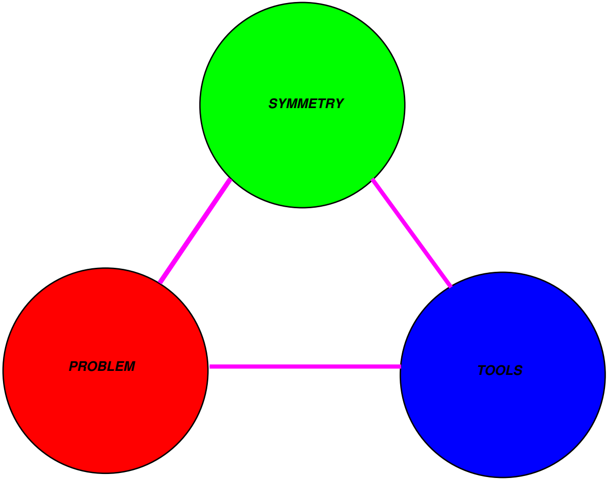
$$\Sigma = \lambda + a\Sigma b$$

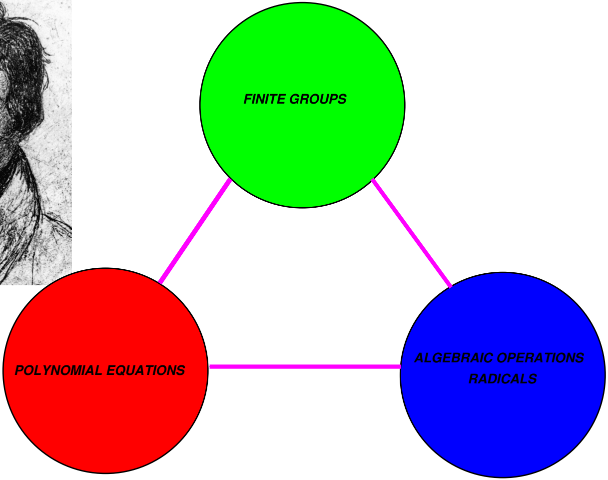


Example (The cyclic regular language C_3)

The characteristic series $\Sigma_1 = \lambda + a_1 b_1 + b_3 a_3 + a_1 a_2 a_3 + b_3 b_2 b_1 + \dots$ of the regular language accepted by the above finite state cyclic automaton satisfies the linear system

$$\begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 & b_3 \\ b_1 & 0 & a_2 \\ a_3 & b_2 & 0 \end{pmatrix} \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix}$$

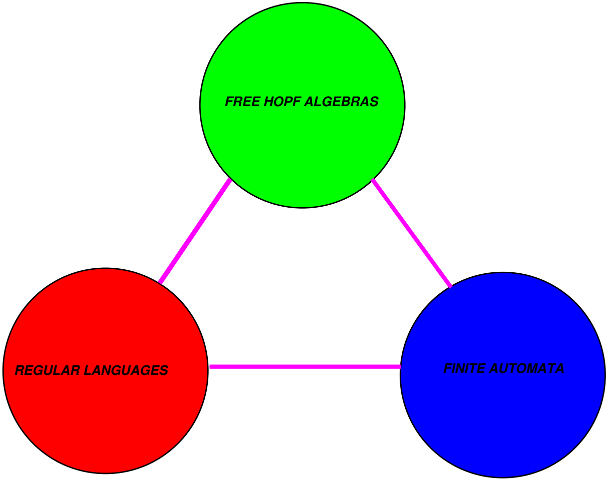

SYMMETRY
PROBLEM
TOOLS



A polynomial equation can be solved using algebraic operations and radicals iff its Galois group is solvable

THE TRIAD FOR REGULAR LANGUAGES

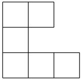
Kleene
Schützenberger
Reutenauer



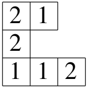
A language $\mathcal{L} \subset A^*$ over finite alphabet A is regular iff its characteristic series belongs to the Sweedler's dual of the free Hopf algebra $\mathbb{Q}\langle A \rangle$

Outline

Composition $\alpha = (\alpha_1, \ldots, \alpha_m) \longleftrightarrow$ Young composition diagram

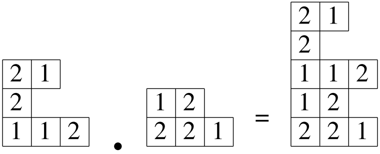
$\alpha = (2,1,3) \longleftrightarrow$ 

Given set $C = \{1, 2, \ldots, P\}$ of colors, consider colored Young diagrams (colored compositions, multisequences of colors)

$\mathcal{J} = ([2,1],[2],[1,1,2]) \longleftrightarrow$ 

Fix a field $\mathbb{k}$, the algebra NSym_P over $\mathbb{k}$ has:

- linear basis – colored compositions
- muliplication – juxtaposition of such compositions: $\mathcal{I}.\mathcal{J} = \mathcal{I} \sqcup \mathcal{J}$
- unity – empty composition



PROPOSITION

The algebra NSym_P can be equipped with the compatible coproduct defined on generators (one-line diagrams) by the left–right splitting, and then extended by homomorphism

$$\Delta\left(\boxed{1}\boxed{1}\boxed{2}\right) = \emptyset \otimes \boxed{1}\boxed{1}\boxed{2} + \boxed{1} \otimes \boxed{1}\boxed{2} + \boxed{1}\boxed{1} \otimes \boxed{2} + \boxed{1}\boxed{1}\boxed{2} \otimes \emptyset$$

and with the counit

$$\epsilon(\mathcal{J}) = \begin{cases} 0 & \mathcal{J} \neq \emptyset \\ 1 & \mathcal{J} = \emptyset \end{cases}$$

The antipode map in this case (graded and connected bialgebra) is given automatically

REMARKS

- In the monochromatic $P = 1$ case, we obtain the Hopf algebra NSym of non-commutative symmetric functions by GELFAND ET AL. [1995]
- The above one-line Young composition diagram generators become then the complete homogeneous non-commutative symmetric functions

The contect-free language of colored compositions and KP languages

Remarks

- Colored compositions $\longleftrightarrow$ (special) words over alphabet $\mathcal{A} = \{a_1, b_1, \dots, a_P, b_P\}$

$$\begin{array}{|c|c|c|} \hline 2 & 1 & \\ \hline 2 & & \\ \hline 1 & 1 & 2 \\ \hline \end{array} \quad \longleftrightarrow \quad \mathrm{a}_2 \mathrm{a}_1 \mathrm{b}_1 \mathrm{b}_2 \; \mathrm{a}_2 \mathrm{b}_2 \; \mathrm{a}_1 \mathrm{a}_1 \mathrm{a}_2 \mathrm{b}_2 \mathrm{b}_1 \mathrm{b}_1$$

- The corrresponding language $\mathcal{L}_P^*$ is context-free and its characteristic series Σ^* satisfies equations

$$\Sigma = a_1 \Sigma b_1 + \cdots + a_P \Sigma b_P$$
$$\Sigma^* = \lambda + \Sigma \Sigma^*$$

  here Σ is the characteristic series of the language $\mathcal{L}_P$ of one-line generators of the algebra

- The support $\{[ab]_1^\alpha, \alpha \models n, \; n \geq 0\}$ of the series Z_1^{-1} is intersection of $\mathcal{L}_P^*$ and the cyclic regular language C_P

THE GRADED AND SWEEDLER'S DUAL TO NSym_P

The Hopf algebra $\mathrm{NSym}_P = \oplus_{n\geq 0}\mathrm{NSym}_P^{(n)}$ is

- graded: the weight is the number n we decompose
- locally finite: $\dim \mathrm{NSym}_P^{(n)} = n^P 2^{n-1} < \infty$
- and connected: $\dim \mathrm{NSym}_P^{(0)} = 1$

therefore by general construction there exists its graded dual $(\mathrm{NSym}_P)^{gr} = \mathrm{QSym}_P$

By $(\mathcal{I}^*)$ denote the basis of QSym_P dual to $(\mathcal{I})$. The dual coproduct δ to the concatenation product "$\cdot$" is

$$\delta(\mathcal{I}^*) = \sum_{\mathcal{I}=\mathcal{J}\sqcup\mathcal{K}} \mathcal{J}^* \otimes \mathcal{K}^*$$

Σ^*, Z_k BELONG TO THE SWEEDLER DUAL $(\mathrm{NSym}_P)^\circ$

$$\delta(\Sigma^*) = \Sigma^* \otimes \Sigma^*, \qquad \delta(Z_k) = 1 \otimes Z_k + Z_k \otimes 1$$

The product in QSym_P is the colored version of the quasi-shuffle product AD [2016]
(natural if you represent elements of QSym_P as power series in infinite number of partially commuting variables)
known from the theory of quasi-symmetric functions ($P = 1$) by GESSEL [1984]

Conclusion and research problems

- The key element in derivation of the Yang-Baxter map from non-commutative discrete KP hierarchy is characteristic series of certain context-free language
- The language can be obtained from generators of the Hopf algebra of generalized non-commutative symmetric functions (in fact the characteristic series belongs to the restricted dual of the algebra)
- The graded dual to the Hopf algebra NSym_P is the Hopf algebra of generalized quasi-symmetric functions, whose monochromatic version has numerous applications in algebraic combinatorics
- It is well known that Sato approach to the KP hierarchy is deeply embedded in the theory of symmetric functions — generalize to NSym_P and QSym_P (or to the Foissy self-dual Hopf algebra of colored trees)
- The corresponding theory of Schur functions (the case $P = 1$ is known)
- ...

THANK YOU FOR YOUR ATTENTION

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