Askey-Wilson algebra and a generalization of the Heun operators

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Bispectral pairs

Bispectral pair is a generalization of Leonard pair.

Formal definition:
The pair of operators X, Y which satisfy $AW(3)$ algebra relations

$$[X, Y] = Z,$$

$$[Y, Z] = -2\nu YXY + A_2 Y^2 + A_1\{X, Y\} + BY + C_1 X + E_1,$$
$$[Z, X] = -2\nu XYX + A_1 X^2 + A_2\{X, Y\} + BX + C_2 Y + E_2,$$

$A_{1,2}, C_{1,2}, E_{1,2}, B, \nu$ are **structure constants** of the AW-algebra (Granovskii, Lutzenko, AZ (1992)).
The Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [Z, X], Y] = 0$$

holds for AW-algebra.

In finite-dimensional case **bispectral pair** $\Longleftrightarrow$ **Leonard pair**

Casimir operator

$Q = Z^2 - \nu\left(XY^2X + YX^2Y\right) + (2+\nu)\left(A_1XYX + A_2YXY\right) +$ quadratic terms

commutes with generators

$$[Q, X] = [Q, Y] = [Q, Z] = 0$$

Affine property:

change

$$X \to \alpha_1 X + \alpha_2,\ Y \to \beta_1 Y + \beta_2$$

leads to the same algebra with "shifted" parameters

q-algebra and q-polynomials

When $q \neq 0, \pm 1$ the parameter $\nu \neq 0$. New parameter q instead of ν

$$2\nu = q + q^{-1} - 2$$

Then AW-algebra can be presented in terms of two relations (Terwilliger, 2000)

$$X^2Y + YX^2 - (q + q^{-1})XYX = A_1Y + BX + C_1,$$

$$Y^2X + XY^2 - (q + q^{-1})YXY = A_2X + BY + C_2$$

Another form in terms of "linear q-algebra" (Granovskii, AZ 1993):

$$[X, Y]_q = Z, \; [Y, Z]_q = A_1Y + BX + C_1, \; [Z, X]_q = A_2Y + BX + C_2$$

where $[X, Y]_q = XY - qYX$ is "q-mutator"

Z_3 **form** (Wiegmann, Zabrodin, 1995, Terwilliger, 2004)

$$[X,Y]_q = a_3 Z + \omega_3, \; [Y,Z]_q = a_1 X + \omega_1, \; [Z,X]_q = a_2 Y + \omega_2$$

All q-polynomials from Askey scheme are described by representations of above algebra.

In case if $a_1 a_2 a_3 \neq 0$ it is possible to put

$$a_1 = a_2 = a_3 = 1$$

Then there are 4 independent parameters: $\omega_1, \omega_2, \omega_3$ and the value Q of the Casimir operator. They correspond to 4 parameters of Askey-Wilson polynomials.

Degenerations

When some parameters a_i are zero then the Aksey-Wilson polynomials degenerate to other polynomials from the Askey scheme.

For example when $a_1 = 0$ one has **big q-Jacobi** polynomials (or **q-Hahn** polynomials in finite-dimensional case).

The case $a_1 = \omega_1 = 0$ corresponds to **little q-Jacobi** polynomials etc.

The maximal degeneration $a_1 = a_2 = a_3 = 0$ leads to the algebra

$$[X, Y]_q = 1, \ [Y, Z]_q = 1, \ [Z, X]_q = 1$$

Each relation is the **q-oscillator**: $XY - qYX = 1$. This algebra is called equitable $sl_q(2)$ algebra (Terwilliger, 2005)

Bannai-Ito algebra

Put $q = -1$:

$$\{X, Y\} = a_3 Z + \omega_3,\ \{Y, Z\} = a_1 X + \omega_1,\ \{Z, X\} = a_2 Y + \omega_2$$

Describes **Bannai-Ito polynomials** (=limit $q \to -1$ of Askey-Wilson polynomials)

Degenerations: $a_1 = 0$ corresponds to **big -1 Jacobi** polynomials (L.Vinet, AZ, 2010);

$a_1 = \omega_1 = 0$ corresponds to **little -1 Jacobi** polynomials.

Algebraic Heun pencil

Generic bilinear operator constructed from X and Y:

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 \mathcal{I}$$

We call M the **algebraic Heun operator**

Properties

(i) M is generic **bilinear** operator with respect to X and Y.

(ii) M possesses a **bispectrality** property.

Why Heun operator?

Example: **Jacobi algebra**

$$X = x, \quad Y = x(1-x)\partial_x^2 + (\nu_1 x + \nu_2)\partial_x$$

The Jacobi polynomials are eigenfunctions of the operator Y

$$Y P_n^{(\omega_1,\omega_2)}(x) = \lambda_n P_n^{(\omega_1,\omega_2)}(x)$$

The operator X is 3-diagonal:

$$X P_n(x) = P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x)$$

The operator M is 2nd-order differential operator which is tridiagonal with respect to Jacobi polynomials

$$M P_n(x) = \xi_{n+1} P_{n+1}(x) + \eta_n P_n(x) + \zeta_n P_{n-1}(x)$$

Main statements (A.Grünbaum, L.Vinet, AZ, 2016)

Proposition 1. The most general 2nd-order differential operator which is 3-diagonal on Jacobi polynomials coincides with M.

Proposition 2. The operator M coincides with generic Heun operator.

Assume that X and Y are finite-dimensional. Then

$$Xe_n = \lambda_n e_n, \quad Yd_n = \mu_n d_n, \; n = 0, 1, 2, \ldots, N$$

and

$$Xd_n = a_{n+1}d_{n+1} + b_n d_n + a_n d_{n-1}, \quad Ye_n = \xi_{n+1}e_{n+1} + \eta_n e_n + \xi_n e_{n-1}$$

It is clear that M is 3-diagonal with respect to **both** bases e_n and d_n:

$$Me_n = \{e_{n-1}, e_n, e_{n+1}\}, \quad Md_n = \{d_{n-1}, d_n, d_{n+1}\}$$

Inverse statement: the **most general** operator with this (bispectrality) property is bilinear in X and Y and hence = algebraic Heun operator (Nomura and Terwilliger, 2007)

Time and band limiting in Fourier analysis

Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

Inverse transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega$$

In practice we always have restrictions $-T < t < T$ and $-\Omega < \omega < \Omega$. For example $\Omega = 20KHz$ for sound.

Corresponding projection operators $D(T)$ and $B(\Omega)$. They defined as

$$D(T)f(t) = \begin{cases} f(t) & \text{if } |t| \le T \\ 0 & \text{if } |t| > T \end{cases}$$

$$B(\Omega)F(\omega) = \begin{cases} F(\omega) & \text{if } |\omega| \le \Omega \\ 0 & \text{if } |\omega| > \Omega \end{cases}$$

What is action of $B(\Omega)$ on functions $f(t)$? The **integral operator**:

$$B(\Omega)f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \Omega(t-s)}{t-s} f(s)ds$$

$D(T)$ and $B(\Omega)$ are **time- and band-limiting operators**.

Is it possible to achieve simultaneous restriction of Fourier analysis to the intervals $[-T, T]$ and $[-\Omega, \Omega]$? The answer is NOT.
Uncertainty principle

$$\sigma_f \sigma_F \geq 1/2$$

where

$$\sigma_f^2 = \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt$$

$$\sigma_F^2 = \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 dt$$

I.e. if the signal is restricted on small interval $[T, T]$ then the Fourier transform $F(\omega)$ should be unrestricted.

However, one can consider the **concentration** of the signal on the prescribed interval

$$\alpha_f^2 = \frac{\int_{-T}^{T} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}$$

if the frequency function is already restricted $F(\omega) = 0, \ |\omega| > \Omega$.

The problem: find the band-limited functions $f(t)$ which provide maximal value of the concentration α_f^2.

Answer: we should solve the eigenvalue problem

$$V_{T,\Omega} f(x) = \lambda f(x)$$

where $V_{T,\Omega}$ is the **integral operator**

$$V_{T,\Omega} f(x) = \frac{1}{\pi} \int_{-T}^{T} \frac{\sin \Omega(x-s)}{x-s} f(s) ds$$

Equivalent presentation

$$V_{T,\Omega} = B_\Omega D_T B_\Omega$$

The operator $V_{T,\Omega}$ is compact and self-adjoint and has only discrete spectrum $0 < \lambda < 1$.

The **maximal** eigenvalue λ_0 yields the solution:

$$\max\{\alpha_f^2\} = \lambda_0$$

Because $V_{T,\Omega}$ is integral operator, the eigenvalue problem is complicated to solve. There are no effective methods to find solutions.

Main idea of BTL

Find 2nd-order differential operator L

$$L = A(x)\partial_x^2 + B(x)\partial_x + C(x)$$

which commutes with $V_{T,\Omega}$

$$LV_{T,\Omega} = V_{T,\Omega}L$$

Then all eigenfunctions of $V_{T,\Omega}$ are also eigenfunctions of L and can be found (at least approximately) by standard methods.
Solution was found by Slepian, Pollak and Landau:

$$L = \partial_x(1 - x^2)\partial_x - \Omega^2 x^2,$$

where $T = 1$ (standard normalization of t).

Equation $Lf(x) = \mu f(x)$ has **prolate spheroidal functions** as solutions. Why?

Explanation: L is a special case of algebraic Heun operator.

Indeed, choose

$$X = x^2, \quad Y = \partial_x^2$$

Then X, Y is a bispectral pair because

$$[X,[X,Y]] = 8X, \quad [Y,[Y,Z] = 8Y$$

Restriction of the operator X corresponds to time-limiting. Restriction of Y corresponds to band-limiting.

Try to construct generic **symmetric bilinear** operator from X and Y

$$L = \alpha\{X, Y\} + \beta X + \gamma Y$$

such that $[L, D_T] = [L, B_\Omega] = 0$. Then L commutes with $V_{T,\Omega} = B_\Omega D_T B_\Omega$.

Easily verified that

$$L = -\frac{1}{2}\{X, Y\} - \Omega^2 X + Y$$

Prolate spheroidal equation = special case of confluent Heun equation.

Finite dimensional BTL

Consider finite-dimensional case: $n = 0, 1, \dots, N$. Two eigenbases e_n and d_n and two projectors π_1 and π_2

$$\pi_1 e_n = \begin{cases} e_n, & \text{if} \quad n \le J_1 \\ 0, & \text{if} \quad n > J_1 \end{cases}$$

$$\pi_2 d_n = \begin{cases} d_n, & \text{if} \quad n \le J_2 \\ 0, & \text{if} \quad n > J_2 \end{cases}$$

They satisfy

$$\pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2.$$

If $J_1 = N$ then $\pi_1 = \mathcal{I}$. Similarly, if $J_2 = N$ then $\pi_2 = \mathcal{I}$.

Combination of projectors yields two **restriction operators**:

$$V_1 = \pi_1\pi_2\pi_1 = K_1K_2, \quad V_2 = \pi_2\pi_1\pi_2 = K_2K_1,$$

where

$$K_1 = \pi_1\pi_2, \quad K_2 = \pi_2\pi_1$$

Operators V_1 and V_2 are symmetric and hence are diagonalizable and have the same eigenvalues (may be degenerate).

When $J_1 = J_2 = N$ both V_1 and V_2 are identity operators $V_1 = V_2 = I$. When only $J_2 = N$ and J_1 arbitrary then $V_1 = V_2 = J_1$ and the operator V_1 has $J_1 + 1$ eigenvalues equal to 1 and $N - J_1$ zero eigenvalues.

But what are eigenvectors and eigenvalues of operators V_1, V_2 for arbitrary J_1, J_2?

The operators V_1, V_2 are **highly nonlocal**. For example:

$$V_{ik}^{(1)} = \sum_{s=0}^{J_2} \tilde{w}_s \chi_i(\mu_s) \chi_k(\mu_s)$$

Hence the problem of funding eigenvectors and eigenvalues is **very complicated**.

Main idea - to find a **3-diagonal matrix** M which commutes with both V_1 and V_2

$$[M, V_1] = [M, V_2] = 0.$$

Because eigenvalue problem for 3-diagonal matrices is much easier to solve.

Perline's bispectral BTL operator

Assume that X and Y is a Leonard pair

Try the **symmetric bilinear** operator (idea of Perline, 1987)

$$M = \tau_1\{X, Y\} + \tau_3 X + \tau_4 Y + \tau_0$$

such that M commutes with both projectors

$$[M, \pi_1] = [M, \pi_2] = 0$$

Then M will also commute with V_1, V_2.

$$
Me_n = \begin{pmatrix}
* & * & & & & & \\
* & * & * & & & & \\
 & * & * & * & & & \\
 & & * & * & \color{red}{0} & & \\
 & & & \color{red}{0} & * & * & \\
 & & & & * & * & * \\
 & & & & & * & *
\end{pmatrix}
\qquad
Md_n = \begin{pmatrix}
* & * & & & & & \\
* & * & * & & & & \\
 & * & * & \color{red}{0} & & & \\
 & & \color{red}{0} & * & * & & \\
 & & & * & * & * & \\
 & & & & * & * & * \\
 & & & & & * & *
\end{pmatrix}
$$

This leads to restrictions on coefficients τ_3, τ_4

$$\tau_1 \left(\lambda_{J_1} + \lambda_{J_1+1}\right) + \tau_4 = 0, \quad \tau_1 \left(\mu_{J_2} + \mu_{J_2+1}\right) + \tau_3 = 0$$

It is **always** possible to find coefficients τ_3, τ_4.

The **only exception** - Bannai-Ito spectrum:

$$\lambda_n = (-1)^n \left(n\alpha + \beta\right)$$

In this case degeneration happens

$$Me_n = \begin{pmatrix} * & * & & & & & \\ * & * & & & & & \\ & & * & * & & & \\ & & * & * & & & \\ & & & & * & * & \\ & & & & * & * & \\ & & & & & & \ddots \end{pmatrix}, \quad Md_n = \begin{pmatrix} * & & & & & & & \\ & * & * & & & & & \\ & * & * & & & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ & & & & & * & * & \\ & & & & & * & * & \\ & & & & & & & \ddots \end{pmatrix}.$$

BTL for anti-Krawtchouk polynomials

For BI polynomials the above method does not work - there are no nontrivial 3-diagonal commuting operator M commuting with V_1, V_2.

Instead, one can try **5-diagonal** operator M.

Consider special example: anti-Krawtchouk polynomials (V.Genest, L.Vinet, AZ, 2014).

Anti-spin algebra

$$\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2,$$

Casimir operator

$$Q = L_1^2 + L_2^2 + L_3^2$$

takes the value

$$Q = (N + 1/2)(N + 3/2), \quad N = 1, 2, \dots$$

Search commuting operator M in terms of **quartic and cubic** expressions

$$M = \{L_1^2, L_2^2\} + \alpha_1\{L_1^2, L_2\} + \alpha_2\{L_2^2, L_1\} +$$

$$\alpha_3 L_1^2 + \alpha_4 L_2^2 + \alpha_5 L_1 + \alpha_6 L_2$$

M is **pentadiagonal**

$$M e_n = G_n e_{n-2} + F_n e_{n-1} + H_n e_n + F_{n+1} e_{n+1} + G_{n+2} e_{n+2}$$

Projector

$$\pi_{N_1} e_n = \begin{cases} e_n, & n \le N_1 \\ 0, & n > N_1 \end{cases}.$$

Commutativity $[M, \pi_{N_1}] = 0$ holds iff

$$G_{N_1+1} = G_{N_1+2} = F_{N_1+1} = 0$$

Similalrly, "dual" projector

$$\pi_{N_2} d_n = \begin{cases} d_n, & n \le N_2 \\ 0, & n > N_2 \end{cases} .$$

Commutativity $[M, \pi_{N_2}] = 0$ conditions

$$\tilde{G}_{N_2+1} = \tilde{G}_{N_2+2} = \tilde{F}_{N_2+1} = 0$$

6 equations for 6 unknowns. Solution:

$$\alpha_1 = (-1)^{N_1},\ \alpha_2 = (-1)^{N_2},\ \alpha_3 = -1 - \kappa_1,$$

$$\alpha_4 = -1 - \kappa_2,\ \alpha_5 = (-1)^{N_2}\kappa_1,\ \alpha_6 = (-1)^{N_1}\kappa_2,$$

where

$$\kappa_1 = 2{N_1}^2 + 4N_1 + 5/2, \quad \kappa_2 = 2{N_2}^2 + 4N_2 + 5/2.$$

Classification of algebraic Heun operators

Askey scheme ⇒ Algebraic Heun operator

- **$su(1,1)$ and Heisenberg-Weyl algebras ⇒ Hermite and Laguerre polynomials ⇒ confluent Heun operators** (F.A.Grünbaum, L.Vinet, AZ, 2016)
- **Jacobi algebra ⇒ Jacobi polynomials ⇒ ordinary Heun operator** (F.A.Grünbaum, L.Vinet, AZ, 2016)
- **Hahn algebra ⇒ Hahn polynomials ⇒ Heun-Hahn and Heun-Krawtchouk operators** (L.Vinet, AZ, 2018)
- **q-Hahn algebra ⇒ big (little) q-Jacobi, q-Hahn polynomials ⇒ big (little) q-Heun operators** (P.Bseilhac, L.Vinet, AZ, 2018)
- **AW algebra ⇒ AW polynomials ⇒ Heun-AW operators** (P.Bseilhac, S.Tsujimoto, L.Vinet, AZ, to appear)

Evolution of algebraic approach to Heun operators

1. A.Erdelyi (40-th) - tridiagonal property of solutions of Heun equation with respect to Jacobi polynomials.

2. R.Perline (1987) - first appearance of special finite-dimensional AHO.

3. A.Turbiner (end of 80-th - beginning of 90-th) - description of the ordinary Heun operator in terms of quadratic combination of sl_2 generators $\Rightarrow$ equivalent to Krawtchouk or Meixner AHO.

4. K.Nomura, P.Terwilliger (2007) - finite-dimensional analogue of AHO and generic bispectrality property.

5. M.Ismail, E.Koelink (2010-2016) - Method of 3-diagonalization and construction of special cases of AHO.

6. A.Grümbaum, L.Vinet, AZ (2016) - ordinary Heun operator as bilinear pencil on Jacobi algebra.

7. A.Grümbaum, L.Vinet, AZ (2017) - generic AHO and BTL problem

Perspectives and open problems

- **Elementary reductions of algebraic Heun operators**. When equation $M\psi = 0$ is reduced to (generalized) hypergeometric equation?
- **Elliptic Heun operator**. Krichever and Zabrodin (1995) constructed **elliptic Lamé** equation on the base of Sklyanin algebra. How to construct corresponding Heun pencil?
- **Multivariate analogs of AHO**. Ruijsenaars and Van Diejen considered multivariate exactly solvable models related with generalized Heun operators. What is their algebraic description?

- **Heun pencil and biorthogonal functions**. The AHO is a **linear pencil**

$$M = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0$$

Consider the equation

$$M\psi = 0$$

If eigenvalue $\lambda = -\tau_0$ then we have the **ordinary** eigenvalue problem. Other choices lead to **generalized eigenvalue problem**

$$M_1\psi = \lambda\, M_2\psi$$

where M_1, M_2 - 3-diagonal operators. This leads to theory of **biorthogonal rational functions** (AZ, 1999). What are the most general BRF arising in this problem?

- **Applications to integrable systems**. It is known that after separation of variables for some integrable systems (i.e. Kepler problem) in elliptic coordinates the Heun operators appear as integrals. How to understand this phenomenon from algebraic point of view?

- **BTL and random matrix models**. Integral equations (and commuting differential operators) for some RMM are **the same** as for special examples of BTL (Mehta, Tracy, Widom). What is more deep relation between two problems?

Thank you for your attention!