Continued fractions & nonlinear recurrences

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Dedicated to the memory of Jon Nimmo

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Outline

1. Continued fractions in $\mathbb{R}$
   - Regular continued fractions
   - Some famous irrationals
   - Linear recurrences for convergents
   - Nonlinear recurrences of 2nd order with the Laurent property
   - Transcendental numbers with explicit continued fraction expansions
2. Continued fractions in $\mathbb{C}((X^{-1}))$
   - Motivation: Hankel determinants for Somos sequences
   - Continued fractions in hyperelliptic function fields (van der Poorten)
   - Generating functions and orthogonal polynomials
   - Hankel formula for Somos-4 (Chang, Hu & Xin)
   - Integrable 4D map from continued fraction of sextic

Regular continued fractions

Every non-zero number $\alpha \in \mathbb{R}$ has a regular continued fraction expansion,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [a_0; a_1, a_2, \ldots],$$

obtained recursively by setting $r_0 = \alpha^{-1}$, and recursively calculating

$$a_k = \left\lfloor r_k^{-1} \right\rfloor, \quad r_{k+1} = \frac{1}{r_k} - a_k, \quad k \geq 0 \qquad (\text{STOP if } r_{k+1} = 0).$$

- Infinite and unique iff $\alpha \in \mathbb{R} \setminus \mathbb{Q}$
- Eventually periodic iff $\alpha \in \mathbb{Q}(\sqrt{d})$ for non-square $d \in \mathbb{Z}_{>0}$
- Khinchin: $\lim_{n\to\infty} \left(\prod_{j=1}^{n} a_j\right)^{1/n} = K_0 \approx 2.685452001$ a.a. $\alpha \in [0, 1]$

Some famous irrational numbers

Continued fraction expansions of some famous numbers in $\mathbb{R} \setminus \mathbb{Q}$:

- $\varphi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \ldots]$ (quadratic irrational)
- $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ (transcendental)
- $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots]$ (transcendental)

Fractional parts of φ and e belong to a set of measure zero in $[0, 1]$: the geometric mean of the coefficients tends to 1 in the first case, and ∞ in the second. What about π? Unknown! But cf. Wallis:

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cdots}}}}$$

Linear recurrences for convergents

The nth convergent of the continued fraction is

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n],$$

with the same 3-term linear recurrence for numerators/denominators, i.e.

$$p_n = a_n p_{n-1} + p_{n-2},$$

and similarly for q_n. In matrix form this is encoded by

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Nonlinear recurrences with the Laurent property

Theorem [H] Given a polynomial $f \in \mathbb{Z}[x]$ of degree d, the Laurent property holds for the 2nd order recurrence

$$x_{n+1}x_{n-1} = f(x_n),$$

i.e. $x_n \in \mathcal{R} = \mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}]\,\forall n \in \mathbb{Z}$ iff one of the following cases holds:

- (i) $x^d f(\lambda x^{-1}) = \kappa f(x)$, $\lambda = f(0) \neq 0$;
- (ii) $f(x) = xF(x)$, $x^{d-1}F(\lambda^2 x^{-1}) = \kappa F(x)$, $\lambda = F(0) \neq 0$;
- (iii) $f(x) = x^M G(x)$ with $M \geq 2$ and G is arbitrary.

Case (i): cluster algebras/LP algebras (adjacent iterates coprime, cf. Kanki et al.); case (ii) also via lift to 3rd order: $x_n = X_n X_{n+1}$; but not case (iii).
Example: $x_{n+1}x_{n-1} = x_n^2(x_n+1)$ has the Laurent property, with $x_n, y_n = x_{n+1}/x_n$ and $y_{n+1}/y_n \in \mathcal{R}$. Thus $x_0 = x_1 = 1 \implies$ integer sequence A112373 in the OEIS:

$$1, 1, 2, 12, 936, 68408496, 342022190843338960032, \ldots$$

A curious transcendental number

The birational maps defined by case (iii) are far from integrable: they have positive entropy. Consider sequences A112373 and A114552:

$1, 1, 2, 12, 936, 68408496, 342022190843338960032, \ldots,$ and
$1, 2, 6, 78, 73086, 4999703411742, 171000951445091523071194028090 7486,$ etc.

The latter are the ratios $y_n = x_{n+1}/x_n$. From the recurrence

$$x_{n+1}x_{n-1} = x_n^2(x_n + 1)$$

there is some $C > 0$ such that $\log x_n \sim C\left(\frac{3+\sqrt{5}}{2}\right)^n$. Then it turns out that

$$S = \sum_{j=0}^{\infty} \frac{1}{x_j} = 1 + 1 + \frac{1}{2} + \frac{1}{12} + \frac{1}{936} + \cdots \approx 2.5844017240$$

is transcendental. How to see this?

A curious continued fraction

Every positive real $\alpha \in \mathbb{R}_{>0}$ has an Engel expansion, obtained recursively by setting $s_0 = \alpha$, and

$$y_k = \left\lceil s_k^{-1} \right\rceil, \quad s_{k+1} = y_k s_k - 1, \quad k \geq 0 \qquad (\text{STOP if } s_{k+1} = 0).$$

In the previous example take $S_\infty = S - 1$, and since $x_j | x_{j+1}$,

$$S_\infty = \sum_{j=1}^{\infty} \frac{1}{x_j} = \sum_{j=0}^{\infty} \frac{1}{y_0 y_1 \cdots y_j} = \frac{1 + \frac{1 + \frac{1+\cdots}{y_2}}{y_1}}{y_0} \approx 1.5844017240,$$

which is an Engel series. Surprisingly, its continued fraction expansion has the explicit form

$$\begin{aligned} S_\infty &= [1; 1, 1, 2, 2, 6, 12, 78, 936, 73086, 68408496, 4999703411742, \ldots] \\ &= [x_0; y_0, x_1, y_1, x_2, y_2, x_3, \ldots] \end{aligned}$$

Proof and generalizations: other curious continued fractions

The partial sums of the Engel series are equal to the even convergents:

$$S_{n+1} = \sum_{j=1}^{n+1} \frac{1}{x_j} = \frac{p_{2n}}{q_{2n}} = [x_0; y_0, x_1, y_1, x_2, \ldots y_{n-1}, x_n]$$

The transcendence of S_∞ follows by Roth's theorem: if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is algebraic then $\forall \delta > 0$ there exist only finitely many p/q for which

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{2+\delta}}.$$

For any $r \in \mathbb{Q}_{>0}$ and sequence of $z_n \in \mathbb{Z}_{>0}$ this result generalizes to

$$S_\infty = r + \sum_{j \geq 2} \frac{1}{x_j},$$

with Engel series coming from the non-autonomous recurrence

$$x_{n+1} x_{n-1} = x_n^2 (z_n x_n + 1).$$

J-fractions in quadratic function fields

Motivation: Somos-4 recurrence

$$\tau_{n+4}\tau_n = \alpha\,\tau_{n+3}\tau_{n+1} + \beta\,(\tau_{n+2})^2.$$

(Hard hexagon, dimer models, cluster algebras, discrete KP reductions...)

Analytic formula: General solution of the initial value problem is

$$\tau_n = AB^n \frac{\sigma(v_0 + nv)}{\sigma(v)^{n^2}} \quad \text{with} \quad \sigma(z;\Lambda) \leftrightarrow E:\ y^2 = 4x^3 - g_2 x - g_3 \cong \mathbb{C}/\Lambda.$$

QRT: biquadratic curve

$$H = d_{n+1}d_n + \frac{\alpha}{d_{n+1}} + \frac{\alpha}{d_n} + \frac{\beta}{d_{n+1}d_n} \quad \text{with} \quad d_{n+1} = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

but cf. van der Poorten: continued fraction from quartic model

$$Y^2 = (X^2 + f)^2 + 4v(X - w).$$

Hyperelliptic function fields

Let $\mathcal{F} = \mathbb{C}(X, Y)/(\, Y^2 = A(X)^2 + 4R(X)\,)$, for polynomials

$$A(X) = X^{\mathrm{g}+1} + \cdots\,, \qquad R(X) = u\,X^{\mathrm{g}} + \cdots\,,$$

where g is the genus. Pick $Y_0 = (Y + P_0)/Q_0 \in \mathcal{F}$, with

$$P_0(X) = A(X) + 2d_0\,X^{\mathrm{g}-1} + \cdots\,, \quad Q_0(X) = u_0\,X^{\mathrm{g}} + \cdots\,,$$

and, with linear coefficients $a_n(X) = 2(X + v_n)/u_n$, expand Y_0 as

$$Y_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = \lfloor Y_0 \rfloor + \mathrm{remainder}$$

For the recursion, write

$$Y_n = a_n + \frac{1}{Y_{n+1}}, \quad \text{with} \quad Y_n = \frac{Y + P_n}{Q_n} = \frac{Q_{n-1}}{Y - P_n}.$$

Discrete Lax pair

Projective version of recursion: $Y_n = \psi_n/\phi_n$, $\mathbf{\Psi}_n = (\psi_n, \phi_n)^T$ gives

$$\mathbf{L}_n(X)\,\mathbf{\Psi}_n = Y\mathbf{\Psi}_n, \qquad \mathbf{M}_n\,(X)\mathbf{\Psi}_{n+1} = \mathbf{\Psi}_n,$$

$$\mathbf{L}_n = \begin{pmatrix} P_n & Q_{n-1} \\ Q_n & -P_n \end{pmatrix}, \quad \mathbf{M}_n \sim \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} X + v_n & \frac{1}{2}u_n \\ \frac{1}{2}u_n & 0 \end{pmatrix}$$

Compatibility condition is the discrete Lax equation

$$\mathbf{L}_n\mathbf{M}_n = \mathbf{M}_n\mathbf{L}_{n+1}$$

(cf. Kuznetsov-Vanhaecke: BT for even Mumford system).

Preserves spectral curve $Y^2 = A(X)^2 + 4R(X)$, and leading order gives

$$d_{n+1} = -u_n u_{n+1}/4.$$

Series expansion and orthogonal polynomials

Expand near one of 2 points at infinity, $Y \sim A(X) \sim X^{g+1}$ as $X \to \infty$, to find moment generating function and its J-fraction:

$$Y_0 - a_0 = \sum_{j=1}^{\infty} s_j X^{-j} = \cfrac{-2{u_0}^{-1} d_1}{v_1 + X - \cfrac{d_2}{v_2 + X - \cfrac{d_3}{v_3 + X - \cdots}}}.$$

Convergents $p_n(X)/q_n(X) \in \mathbb{C}(X)$ via 3-term linear recurrence

$$q_n = (X + v_n)\, q_{n-1} - d_n\, q_{n-2}, \quad n \geq 2$$

with $p_0 = 0$, $p_1 = s_1$ and $q_0 = 1$, $q_1 = X + v_1$.
For linear functional $< \cdot >$ with moments $s_j =< X^{j-1} >$, the sequence $(\, q_n(X)\,)$ consists of monic orthogonal polynomials, i.e.

$$< q_m\, q_n >= c_n\, \delta_{mn}.$$

Elliptic case and Somos-4

For $\mathrm{g} = 1$ have $Y^2 = (X^2 + f)^2 + 4u(X - v)$, where w.l.o.g.

$$A = X^2 + f, \quad R = v(X - w), \quad P_0 = A + 2d_0, \quad Q_0 = u_0(X - v_0).$$

Continued fraction recursion, or discrete Lax, gives

$$\begin{aligned} (X + v_n)(P_{n+1} - P_n) &= \tfrac{1}{2} u_n (Q_{n-1} - Q_{n+1}) \\ (X + v_n) Q_n &= \tfrac{1}{2} u_n (P_{n+1} + P_n) \\ P_n^2 + Q_{n-1} Q_n &= A^2 + 4R. \end{aligned}$$

Eliminate u_n from leading order $\implies$ dynamical system for d_n, v_n.
Eliminate v_n and set $\alpha = u^2$, $\beta = u^2(f + v^2)$ to find

$$d_{n+1} d_{n-1} = \frac{\alpha d_n + \beta}{d_n^2} \quad \text{(QRT)}$$

with first integral $H = 2uv$ as above. Set $d_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{(\tau_n)^2} \implies$ Somos-4.

Hankel determinants for tau-functions

Hankel formula: Conversely, from series $Y_0 - a_0 = \sum_{j\geq 1} s_j X^{-j}$, construct

$$\tau_n = \begin{vmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & & \vdots \\ \vdots & & \ddots & \vdots \\ s_n & \cdots & \cdots & s_{2n-1} \end{vmatrix} = \det(s_{i+j-1})_{1\leq i,j\leq n}.$$

Then, by a standard result on the J-fraction of series (Wall),

$$\tau_n = s_1^n d_2^{n-1} d_3^{n-2} \cdots d_n.$$

Hence $d_{n+1} = \frac{\tau_{n+1}\tau_{n-1}}{(\tau_n)^2}$, and up to gauge freedom $\tau_n \to AB^n\tau_n$ any solution of Somos-4 can be written in Hankel form.
Note: This formula for d_n is independent of genus g.

Original conjecture of Somos/Barry

In Somos-4, take $\alpha = \beta = 1$ and all initial values 1 to obtain

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots \quad (A006720 \text{ in OEIS}).$$

With $f = -3$, $u = -1$, $v = -2$ this comes from the quartic curve

$$Y^2 = (X^2 - 3)^2 - 4(X + 2) \text{ with } u_0 = -2,\ d_0 = d_1 = 1,\ w_0 = -1$$

$$\implies Y_0 - a_0 = \frac{1 + \frac{1}{2}(X^2 - 3 - Y)}{X + 1} \begin{array}{l} = X^{-1} + 2X^{-3} + X^{-4} + 6X^{-5} + \cdots \\ = z + z^2 + 3z^3 + 8z^4 + 23z^5 + \cdots \end{array}$$

where $z = 1/(X + 1)$ (cf. Chang & Hu). Then $\tau_0 = \tau_1 = 1$,

$$\tau_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2, \quad \tau_3 = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 8 \\ 3 & 8 & 23 \end{vmatrix} = 3.$$

Also cubic model: $y^2 = 4x^3 - 4x + 1$ (cf. Xin)
$\implies \frac{1}{2}(1 - y)/x - 1 = x + x^2 + 3x^2 + 8x^4 + 23x^5 + \cdots !$

4D map in genus two

When $\mathrm{g} = 2$ w.l.o.g. set $A = X^3 + fX + g$, $R = u(X^2 - vX + w)$

$$\implies Y^2 = (X^3 + fX + g)^2 + 4u(X^2 - vX + w),$$

with $\quad P_n = A + 2d_n(X + e_n), \quad Q_n = u_n(X^2 - v_nX + w_n).$
Then continued fraction/Lax defines a birational map

$$(v_{n-1}, w_{n-1}, d_n, e_n) \mapsto (v_n, w_n, d_{n+1}, e_{n+1})$$

given by

$$\begin{aligned} v_n &= -v_{n-1} - e_n \\ w_n &= -w_{n-1} - v_{n-1}v_n + d_n - u/d_n + f \\ d_{n+1} &= -d_n + w_n - v_n^2 - f \\ e_{n+1} &= (-d_ne_n + v_nw_n - g)/d_{n+1} \end{aligned}$$

and first integrals

$$\begin{aligned} H_1 = uv &= d_n(2d_ne_n - v_nw_{n-1} - v_{n-1}w_n - fe_n - g), \\ H_2 = uw &= d_n(d_ne_n^2 - w_{n-1}w_n + ge_n). \end{aligned}$$

Symplectic structure and Liouville integrability

The 4D map is conveniently rewritten as a coupled 2nd order system:

$$\begin{array}{rcl} d_{n+1}+d_n+d_{n-1}+u/d_n+v_n^2+v_nv_{n-1}+v_{n-1}^2+f & = & 0, \\ (2v_n+v_{n-1})d_n+(2v_n+v_{n+1})d_{n+1}+v_n^3+fv_n-g & = & 0, \end{array}$$

and in this form it can be verified that it is symplectic with

$$\omega = \mathrm{d}d_{n-1}\wedge\mathrm{d}d_n+\mathrm{d}v_{n-1}\wedge\mathrm{d}v_n+(2v_{n-1}+v_n)\,\mathrm{d}v_{n-1}\wedge\mathrm{d}d_n.$$

Then $\{\,H_1,H_2\,\}=0$, hence the map is integrable in the Liouville sense. The general J-fraction argument above implies the Hankel formula

$$d_n=\frac{\tau_n\tau_{n-2}}{\tau_{n-1}^2}, \qquad \tau_n=\det(s_{i+j-1})_{1\le i,j\le n}$$

General algebro-geometric arguments suggest the analytic formula

$$\tau_n=AB^nC^{n^2}\sigma(\mathbf{v}_0+n\mathbf{v}), \qquad \mathbf{v}_0,\mathbf{v}\in\mathbb{C}^2/\Lambda,$$

which implies that τ_n satisfies a 5-term Somos-8. Empirically we find

$$v_n=\frac{F_n}{\tau_n\tau_{n\text{ -}1}}, \qquad F_n=\rho_n\,\rho_n^*=\text{product of Hankel/sigma functions?}$$

Closing remark: another example in $\mathbb{C}((Z))$

Famous continued fraction formulae for π and $\frac{4}{\pi}$ due to Wallis and Brouncker is a special case of a result of Euler/Stieltjes/Ramanujan:

$$\frac{4\Gamma(\frac{Z+3}{4})^2}{\Gamma(\frac{Z+1}{4})^2} = Z + \cfrac{1^2}{2Z + \cfrac{3^2}{2Z + \cfrac{5^2}{2Z + \cfrac{7^2}{2Z + \cdots}}}}$$