Hypergeometric integrals, Yang-Baxter equations, and 3D-consistent equations

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Square lattice model of statistical mechanics

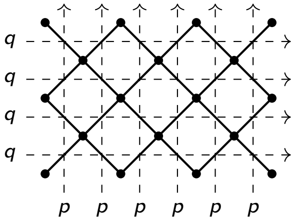
2-d lattice models with nearest-neighbour spin interactions, *e.g.* Ising, Chiral Potts, Fateev-Zamolodchikov models *etc.*

Integer/real valued spins labelled σ_i, located at vertices •

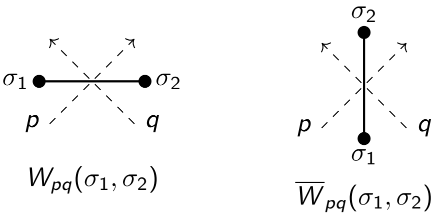
Directed rapidity lines distinguish two types of edges of the square lattice, these two edges have associated Boltzmann weights (BW) W, and $\overline{W}$.

Variables $p, q \in \mathbb{R}$ are associated to the different rapidity lines.

Square lattice

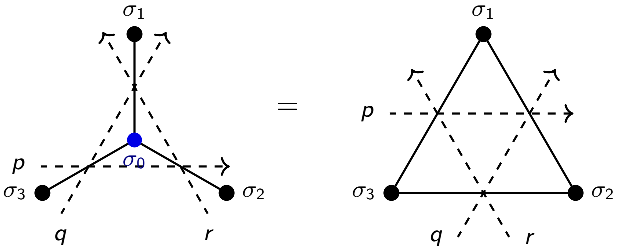


Boltzmann weights (BW)



The star-triangle relation (STR)

This lattice model is integrable if Boltzmann weights (BW) satisfy STR:



STR allows solution of model through commuting transfer matrix method.

Expression of STR for discrete spin model ($\sigma_i \in \mathbb{Z}_N$):

$$\sum_{\sigma_0 \in \mathbb{Z}_N} \overline{W}_{qr}(\sigma_1, \sigma_0) W_{pr}(\sigma_2, \sigma_0) \overline{W}_{pq}(\sigma_3, \sigma_0) = W_{qr}(\sigma_2, \sigma_3) \overline{W}_{pr}(\sigma_1, \sigma_3) W_{pq}(\sigma_2, \sigma_1).$$

STR for continuous spin model ($\sigma_i \in [a, b]$): $\sum_{\sigma_0 \in \mathbb{Z}_N} \rightarrow \int_a^b d\sigma_0$

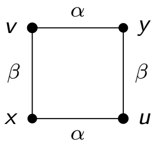
STR for discrete/continuous spin model

($\sigma_i = (x_i, m_i) \in \mathbb{Z}_N \times [a, b]$): $\sum_{\sigma_0 \in \mathbb{Z}_N} \rightarrow \sum_{m_0 \in \mathbb{Z}_N} \int_a^b dx_0$

STR as quantum integrable quad equation

There are essentially 4 different types of quad equations, that may be identified through the YBE/3D-consistency correspondence.

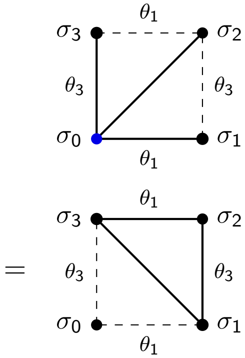
1. Affine-linear quad equation: $Q(\alpha,\beta;x,u,y,v)=0$
2. Three-leg quad equation: $\varphi_{\alpha_1}(x_0,x_1)-\varphi_{\alpha_3}(x_0,x_2)-\varphi_{\alpha_1-\alpha_3}(x_0,x_3)=0$



$= 0$

3. Lagrangian quad equation (cSTR):
$$L_{\alpha_1}(x_0,x_1)-L_{\alpha_3}(x_0,x_2)-L_{\alpha_1-\alpha_3}(x_0,x_3)=$$
$$L_{\alpha_1}(x_2,x_3)-L_{\alpha_3}(x_1,x_3)-L_{\alpha_1-\alpha_3}(x_1,x_2)$$

4. Quantum quad equation (STR):
$$\int_{\mathbb{R}} d\sigma_0 W_{\theta_1}(\sigma_0,\sigma_1)W_{\theta_1+\theta_3}(\sigma_0,\sigma_2)W_{\theta_3}(\sigma_0,\sigma_3)$$
$$=W_{\theta_1}(\sigma_2,\sigma_3)W_{\theta_1+\theta_3}(\sigma_1,\sigma_3)W_{\theta_3}(\sigma_1,\sigma_2)$$

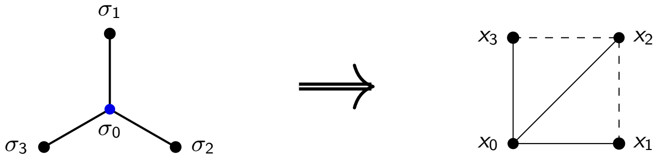


STR as quantum integrable quad equation

Connection between quantum and classical lagrangian forms:

There exists a change of variables $\sigma_i = f_{i,\hbar}(x_i)$, $\theta_i = g_{i,\hbar}(\alpha_i)$, s.t. for $\hbar \to 0$

$$\int_{\mathbb{R}} d\sigma_0 W_{\theta_1}(\sigma_0,\sigma_1) W_{\theta_1+\theta_3}(\sigma_0,\sigma_2) W_{\theta_3}(\sigma_0,\sigma_3)$$
$$= \int_{\mathbb{R}} dx_0 \exp\left\{\hbar^{-1}\left(L_{\alpha_1}(x_0,x_1) - L_{\alpha_3}(x_0,x_2) - L_{\alpha_1-\alpha_3}(x_0,x_3)\right) + O(1)\right\}.$$



Recently have found quantum counterparts for each equation in ABS list.

V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A: Math. Theor.: 49 (2016) 464001.
A.P. K, arXiv:1803.03219, (2018).

Example: $Q1_{(\delta=0)}$

1. Affine-linear form: $\alpha(x-y)(u-v)-\beta(x-u)(y-v)=0$.
2. Three-leg form: $\varphi_{\alpha_1}(x_1,x_0)-\varphi_{\alpha_1-\alpha_3}(x_2,x_0)-\varphi_{\alpha_3}(x_3,x_0)=0$,

where $\varphi_\alpha(x_i,x_j)=\alpha(x_i-x_j)^{-1}$.

3. Lagrangian form: $L_{\alpha_1}(x_1,x_0)-L_{\alpha_1-\alpha_3}(x_2,x_0)-L_{\alpha_3}(x_3,x_0)$,

where $L_\alpha(x_i,x_j)=\alpha\log|x_i-x_j|-\frac{\alpha}{2}\log|\alpha|$.

Note that three-leg form is equivalent to:

$$\frac{\partial}{\partial x}\left(L_{\alpha_1}(x_1,x)-L_{\alpha_1-\alpha_3}(x_2,x)-L_{\alpha_3}(x_3,x)\right)=0.$$

4. Quantum form: $\int_{\mathbb{R}} d\sigma_0 \overline{W}_{\theta_1}(\sigma_0,\sigma_1)W_{\theta_1+\theta_3}(\sigma_0,\sigma_2)\overline{W}_{\theta_3}(\sigma_0,\sigma_3)$,

where $W_\theta(\sigma_i,\sigma_j)=|x_i-x_j|^{-2\theta}$, $\overline{W}_\theta(\sigma_i,\sigma_j)=W_{\frac{1}{2}-\theta}(\sigma_i,\sigma_j)$, and $\sigma_i,\sigma_j\in\mathbb{R}$, $0<\mathrm{Re}(\theta)<\frac{1}{2}$.

Note that Lagrangian form comes from $O(\hbar^{-1})$ expansion of $\mathrm{Im}(\theta_1)=\alpha_1\hbar^{-1}$, $\mathrm{Im}(\theta_3)=\alpha_3\hbar^{-1}$, as $\hbar\to 0$.

V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A: Math. Theor.: 49 (2016) 464001.
A.P. K, arXiv:1803.03219, (2018).

Example: $Q1_{(\delta=0)}$

Quantum form of $Q1_{(\delta=0)}$ satisfies STR:

$$\int_{\mathbb{R}} d\sigma_0 \overline{W}_{\theta_1}(\sigma_0,\sigma_1) W_{\theta_1+\theta_3}(\sigma_0,\sigma_2) \overline{W}_{\theta_3}(\sigma_0,\sigma_3)$$

$$= R(\theta_1,\theta_3) W_{\theta_1}(\sigma_2,\sigma_3) \overline{W}_{\theta_1+\theta_3}(\sigma_1,\sigma_3) W_{\theta_3}(\sigma_1,\sigma_2)$$

This STR is in fact equivalent to a sum of 3 copies of the $n=1$ Selberg integral formula

$$\int_{[\sigma_1,\sigma_3]} d\sigma_0 \frac{|\sigma_0-\sigma_1|^{1-\theta_1}|\sigma_0-\sigma_3|^{1-\theta_3}}{|\sigma_0-\sigma_2|^{\theta_1+\theta_3}} = \frac{|\sigma_1-\sigma_3|^{1-\theta_1-\theta_3}}{|\sigma_2-\sigma_3|^{\theta_1}|\sigma_2-\sigma_1|^{\theta_3}} \hat{R}(\theta_1,\theta_3)$$

V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A: Math. Theor.: 49 (2016) 464001.
A.P. K, arXiv:1803.03219, (2018).

Example: $H1_{(\varepsilon=0)}$

1. Affine-linear form: $(u-y)(x-v)-2(\beta-\alpha)=0$.

2. Three-leg form: $\phi_{\alpha_1}(x_1,x_0)+\phi_{\alpha_1+\alpha_3}(x_2,x_0)+\varphi_{\alpha_3}(x_3,x_0)=0$,

where $\phi_\alpha(x_i,x_j)=x_i,\ \varphi_\alpha(x_i,x_j)=\dfrac{2\alpha}{x_j-x_i}$.

3. Lagrangian form: $\Lambda_{\alpha_1}(x_1,x_0)+\Lambda_{\alpha_1+\alpha_3}(x_2,x_0)+L_{\alpha_3}(x_0,x_3)$,

where $\Lambda_\alpha(x_i,x_j)=x_ix_j,\ L_\alpha(x_i,x_j)=2\alpha\log|x_i-x_j|$,

such that 3-leg form is: $\dfrac{\partial}{\partial x}\left(\Lambda_\alpha(x_1,x)-\Lambda_{\alpha+\beta}(x_2,x)+L_\beta(x_3,x)\right)=0$.

4. Quantum form: $\displaystyle\int_{\mathbb{R}} d\sigma_0 V_{\theta_1}(\sigma_0,\sigma_1)V_{\theta_1+\theta_3}(\sigma_0,\sigma_2)\overline{W}_{\theta_3}(\sigma_0,\sigma_3)$

where $V_\theta(\sigma_i,\sigma_j)=e^{\mathrm{i}\sigma_i\sigma_j}$, $\overline{W}_\theta(\sigma_i,\sigma_j)=\left|2\sinh\frac{\sigma_i-\sigma_j}{2}\right|^{2\theta-1}$, and $\sigma_i,\sigma_j\in\mathbb{R}$, $0<\mathrm{Re}(\theta)<\frac{1}{2}$.

Lagrangian form is $O(\hbar^{-1})$ expansion of $(\sigma_0,\sigma_3)=(x_0,x_3)\hbar$, $(\sigma_1,\sigma_2)=(x_1,x_2)\hbar^{-2}$, $\mathrm{Im}(\theta_3)=\beta\hbar^{-1}$, as $\hbar\to 0$.

A.P. K, arXiv:1803.03219, (2018).

Example: $H1_{(\varepsilon=0)}$

A.P. K, arXiv:1803.03219, (2018).

Quantum form satisfies: $\displaystyle\int_{\mathbb{R}} d\sigma_0 V_\alpha(\sigma_0,\sigma_1)V_{\alpha+\beta}(\sigma_0,\sigma_2)\overline{W}_\beta(\sigma_0,\sigma_3)$
$= R(\beta)V_\alpha(\sigma_2,\sigma_3)V_{\alpha+\beta}(\sigma_1,\sigma_3)W_\beta(\sigma_1,\sigma_2)$

where $W_\theta(\sigma_i,\sigma_j) = \dfrac{\Gamma(\frac{1}{2}-\theta+\mathrm{i}(\sigma_i+\sigma_j))\Gamma(\frac{1}{2}-\theta-\mathrm{i}(\sigma_i+\sigma_j))}{\Gamma(\frac{1}{2}+\mathrm{i}(\sigma_i+\sigma_j))\Gamma(\frac{1}{2}-\mathrm{i}(\sigma_i+\sigma_j))\Gamma(\frac{1}{2}+\theta)\Gamma(\frac{1}{2}-\theta)}$.

This STR over $(-\infty,\sigma_3]$ may be written as (with $\mathrm{e}^{\sigma_0}=x$, $\mathrm{e}^{\sigma_3}=x_3$, $\sigma_1=x_1$, $\sigma_2=x_2$)

$$\int_{[0,x_3]} dx\,(xx_3^{-1})^{\mathrm{i}(x_1+x_2)}(xx_3)^{\frac{1}{2}-\theta_3}\,|x-x_3|^{\theta_3-\frac{1}{2}} = \Gamma(2\theta_3)\frac{\Gamma(\frac{1}{2}+\mathrm{i}(x_1+x_2)-\theta_3)}{\Gamma(\frac{1}{2}+\mathrm{i}(x_1+x_2)+\theta_3)}.$$

Setting $x\to x_3\sqrt{x}$, gives the Euler beta function

$$\int_{[0,1]} dx\,x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\,, \qquad \mathrm{Re}(a)\,,\mathrm{Re}(b)>0.$$

where $a=\mathrm{i}(x_1+x_2)-\theta_3+\frac{1}{2}$, and $b=2\theta_3$.

Overview of complete YBE/3D-consistency correspondence

	Quantum		Classical	
	Elliptic: $\Phi(x,\mathrm{p},\mathrm{q})$	$\longrightarrow$	$Q4$	
	$\downarrow$		$\downarrow$	
	Hyperbolic: $\varphi(x,\mathrm{q})$	$\longrightarrow$	$Q3_{(\delta=0,1)}, H3_{(\delta,\varepsilon=0,1)}$	ABS
STRs	$\downarrow$		$\downarrow$	Equations
	Rational: $\Gamma(x)$	$\longrightarrow$	$Q2, Q1_{(\delta=1)}, H2_{(\varepsilon=0,1)}$	
	$\downarrow$		$\downarrow$	
	Algebraic: x	$\longrightarrow$	$Q1_{(\delta=0)}, H1_{(\varepsilon=0,1)}$	

V.V. Bazhanov and S.M. Sergeev, Adv. Theor. Math. Phys.: 16, (2012).
V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A: Math. Theor.: 49 (2016) 464001.
A.P. K, arXiv:1803.03219, (2018).

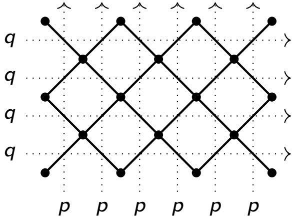
Summary of connection to hypergeometric integrals

(Quantum) Hypergeometric Integral	(Classical) Quad Equation
Elliptic beta integral	$Q4$
Hyperbolic beta integral	$Q3_{(\delta=1)}$
Hyperbolic Saalschütz integral	$Q3_{(\delta=0)}$
Hyperbolic Askey-Wilson integral	$H3_{(\delta=1;\,\varepsilon=1)}$
Hyperbolic Saalschütz integral	$H3_{(\delta=1;\,\varepsilon=1)}$ (alt.)
Hyperbolic Barnes's first lemma	$H3_{(\delta=0,1;\,\varepsilon=1-\delta)}$
Hyperbolic Barnes's ${}_2F_1$ integral	$H3_{(\delta=0;\,\varepsilon=0)}$
Askey integral	$Q2$
Barnes's second lemma	$Q1_{(\delta=1)}$
de Branges-Wilson integral	$H2_{(\varepsilon=1)}$
Barnes's second lemma	$H2_{(\varepsilon=1)}$ (alt.)
Barnes's first lemma	$H2_{(\varepsilon=0)}$
Selberg integral	$Q1_{(\delta=0)}$
Euler beta integral	$H1_{(\varepsilon=1)}$
Barnes's ${}_2F_1$ integral	$H1_{(\varepsilon=1)}$ (alt.)
Euler beta integral	$H1_{(\varepsilon=0)}$

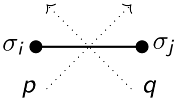
A.P. K, arXiv:1803.03219, (2018).

Recall the square lattice model of statistical mechanics

Square lattice

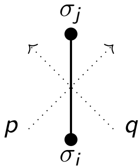


Boltzmann weights (BW)



edge $(ij) \in E^{(1)}$

$W_{\theta_{ij}}(\sigma_i, \sigma_j)$



edge $(ij) \in E^{(2)}$

$\overline{W}_{\theta_{ij}}(\sigma_i, \sigma_j)$

The partition function is defined by

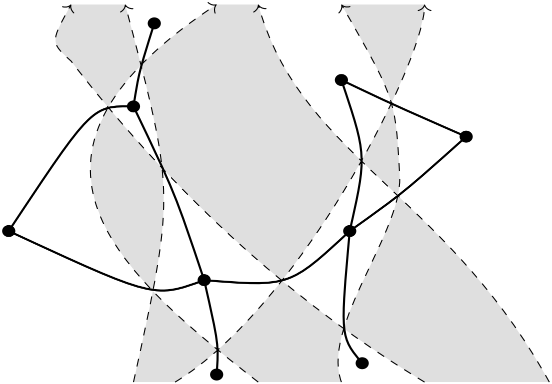
$$Z = \int_{\mathbb{R}^n} \prod_{(ij)\in E^{(1)}} W_{\theta_{ij}}(\sigma_i, \sigma_j) \prod_{(ij)\in E^{(2)}} \overline{W}_{\theta_{ij}}(\sigma_i, \sigma_j) \prod_{i \in V_{int}} d\sigma_i$$

"Exactly solved" means determining closed form expression for Z.

Property of Z-invariance: Z is invariant under continuous deformations of the rapidity lines which preserve their orientation, as long as no closed directed paths are formed.

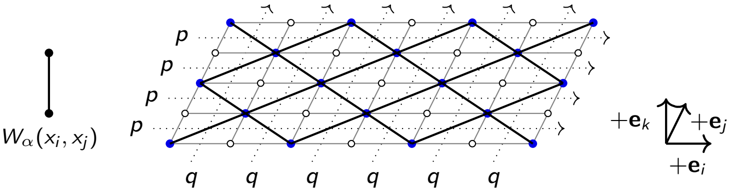
R.J. Baxter, J. Phil.Trans.Roy.Soc.Lond. A289 (1978) 315-346

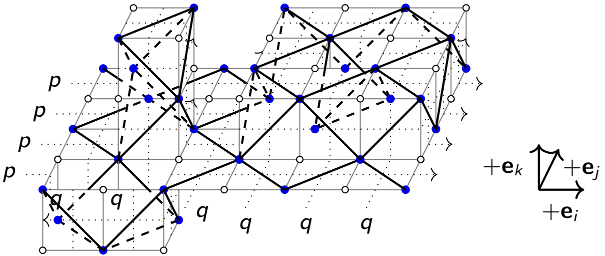
Z-invariance



A generalisation of Z-invariance

A.P. K, J. Phys. A: Math. Theor. 50 (2017) 495202.



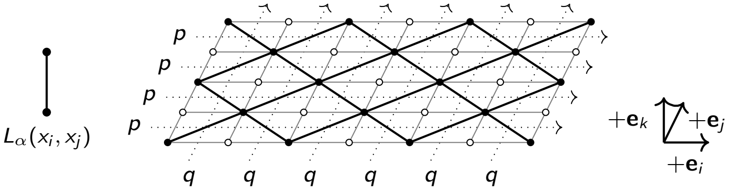


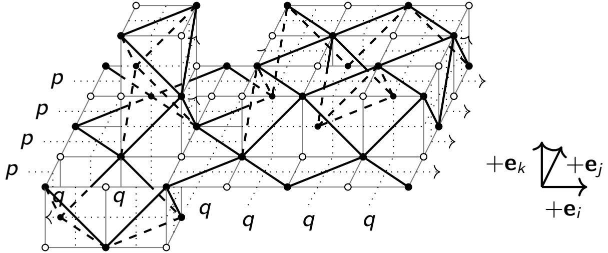
Inspiration from Lagrangian multiform closure property for ABS equations.

S. Lobb and F. Nijhoff, J. Phys. A: Math. Theor. 42 (2009) 454013.

Closure property for system of discrete Laplace equations

A.P. K, J. Phys. A: Math. Theor. 50 (2017) 495202.





Discrete Laplace equations from the partition function

The partition function:

$$Z = \int_{\mathbb{R}^n} \prod_{(ij)\in E^{(1)}} W_{\theta_{ij}}(\sigma_i,\sigma_j) \prod_{(ij)\in E^{(2)}} \overline{W}_{\theta_{ij}}(\sigma_i,\sigma_j) \prod_{i\in V_{int}} d\sigma_i$$

for $\hbar \to 0$ has quasi-classical expansion of the form $Z =$

$$\int_{\mathbb{R}^n} \exp\left\{ \hbar^{-1} \left(\sum_{(ij)\in E^{(1)}} L_{\alpha_{ij}}(x_i,x_j) + \sum_{(ij)\in E^{(2)}} \overline{L}_{\alpha_{ij}}(x_i,x_j) \right) + O(1) \right\} \prod_{i\in V_{int}} dx_i .$$

The quantity in the exponential is an action functional for system of discrete Laplace equations.

Saddle point equation gives system of discrete Laplace equations

$$\sum_j \varphi_{\alpha_{ij}}(x_i,x_j) = 0, \quad i \in V_{int},\ (j) \in neighbour(i).$$

Closure property: The action functional is invariant under "cubic" deformations of the quad surface.

Follows from skew symmetry $L_\alpha(x_i,x_j) = L_{-\alpha}(x_i,x_j)$, and classical STR:

$$L_\alpha(x_0,x_1) + L_{\alpha+\beta}(x_0,x_2) + L_\beta(x_0,x_3) =$$
$$L_\alpha(x_2,x_3) + L_{\alpha+\beta}(x_1,x_3) + L_\beta(x_1,x_2).$$

Dictionary of YBE/3D-consistency correspondence

Statistical Mechanics	ABS Equations
Spins at vertices	Fields at vertices
Rapidity difference variables	Parameters on edges
Boltzmann weight (BW)	2-point Lagrangian
Inversion relations	Anti-symmetry of Lagrangians
Star-triangle relation (STR)	Classical STR
STR saddle point equation	Three-leg equation
Partition function (PF)	Action functional
PF saddle point equation	Discrete Laplace equations
Z-invariance	Closure of action functional
Universal R-matrix	Yang-Baxter map
Evaluation of PF	?
?	Non-ABS equations
Others	Others

V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A: Math. Theor. 49 (2016) 464001.
A.P. K, J. Phys. A: Math. Theor. 50 (2017) 495202.
V.V. Bazhanov, S.M. Sergeev, Nucl. Phys B:926, 509-543 (2018).
A.P. K, arXiv:1803.03219 (2018).

Connection to supersymmetric gauge theory

A supersymmetric gauge theory depends on a gauge group G.
Elliptic beta integral (quantum $Q4$/master solution) represents duality between dual $S^1 \times S^3$ theories.

$$\lambda \int_0^{2\pi} \frac{dz}{4\pi} \frac{\prod_{i=1}^6 \Phi(t_i + z; \mathrm{p}, \mathrm{q}) \Phi(t_i - z; \mathrm{p}, \mathrm{q})}{\Phi(2z; \mathrm{p}, \mathrm{q}) \Phi(-2z; \mathrm{p}, \mathrm{q})} = \prod_{1 \leq i < j \leq 6} \Phi(t_i + t_j; \mathrm{p}, \mathrm{q}) \,,$$

"Electric" SUSY index $G = SU(2)$ Dual "magnetic" index $G = 1$

V.P. Spiridonov. Uspekhi Mat. Nauk, 56:181-182, (2001)
F. Dolan and H. Osborn, Nucl. Phys. B 818:137-178, (2009)

This equation also can be put in the form of STR which is quantum counterpart of $Q4$.

V.V. Bazhanov and S.M. Sergeev, Adv. Theor. Math. Phys.: 16, (2012).
A.P. K and M. Yamazaki, J. Stat. Mech., 023108 (2018).

The elliptic beta sum/integral

Theorem. A.P. K, J. Phys. A: Math. Theor. 48 (2015) 435201 ($r=1$ is Spiridonov's formula)

$$\lambda \sum_{y=0}^{r-1} \int_0^{2\pi} \frac{dz}{4\pi}\, \rho(z, y, t_i, u_i; \mathrm{p}, \mathrm{q}) = \prod_{1\le i<j\le 6} \Phi(t_i + t_j, u_i + u_j; \mathrm{p}, \mathrm{q}),$$

where

$$\rho(z, y, t_i, u_i; \mathrm{p}, \mathrm{q}) = \frac{\prod_{i=1}^{6} \Phi(t_i + z, u_i + y; \mathrm{p}, \mathrm{q}) \Phi(t_i - z, u_i - y; \mathrm{p}, \mathrm{q})}{\Phi(2z, 2y; \mathrm{p}, \mathrm{q}) \Phi(-2z, -2y; \mathrm{p}, \mathrm{q})},$$

$$\Phi(z, m; \mathrm{p}, \mathrm{q}) = \mathrm{e}^{R(z,m)} \prod_{j,k=0}^{\infty} \frac{1 - \mathrm{e}^{-\mathrm{i}z} \mathrm{p}^{-m} (\mathrm{pq})^{j+1}\, \mathrm{p}^{r(k+1)}}{1 - \mathrm{e}^{\mathrm{i}z} \mathrm{p}^{m} (\mathrm{pq})^{j}\, \mathrm{p}^{rk}} \frac{1 - \mathrm{e}^{-\mathrm{i}z} \mathrm{q}^{m} (\mathrm{pq})^{j+1}\, \mathrm{q}^{rk}}{1 - \mathrm{e}^{\mathrm{i}z} \mathrm{q}^{-m} (\mathrm{pq})^{j}\, \mathrm{q}^{r(k+1)}},$$

$$\mathrm{p}, \mathrm{q}, t_i \in \mathbb{C}, \quad u_i \in \mathbb{Z}, \quad |\mathrm{p}|, |\mathrm{q}| < 1, \quad \mathrm{Im}(t_i) > 0\,, \quad i = 1, \ldots, 6\,.$$

Balancing condition: $\displaystyle\sum_{i=1}^{6} t_i = 2\mathrm{i}\eta, \quad \sum_{i=1}^{6} u_i = 0\,.$

Corresponds to duality for $\mathcal{N}=1$ $S^1 \times S^3/\mathbb{Z}_r$ $SU(2)$, and integrable lattice model with spins $\sigma \in \mathbb{R} \times \mathbb{Z}_r$.

M. Yamazaki, J. Stat. Phys. 154:895 (2014).

A_n hypergeometric sum/integral transformation formula

The elliptic beta sum/integral is the building block for sum/integral generalisations of elliptic hypergeometric functions, *e.g.*

Theorem. A.P. K and M. Yamazaki, SIGMA, 14, 013 (2018) ($r=1$ is Rains's formula)

For: $a_i, b_i \in \mathbb{Z}$, $|\mathrm{p}|, |\mathrm{q}| < 1$, $\mathrm{Im}(t_i), \mathrm{Im}(u_i) > 0\,, \quad i = 1, \ldots, m+n+2\,,$

$$I^m_{A_n}(t_i, u_i; a_i, b_i) = I^n_{A_m}(t'_i, u'_i; a'_i, b'_i) \prod_{i,j=1}^{m+n+2} \Phi(t_i + u_j, a_i + b_j; \mathrm{p}, \mathrm{q}),$$

where $I^m_{A_n}(t_i, u_i; a_i, b_i) = \dfrac{\lambda^n}{n!} \displaystyle\sum_{y_1,\ldots,y_n=0}^{r-1} \int_0^{2\pi} \cdots \int_0^{2\pi} \rho(z_j, y_j; t_i, u_i, a_i, b_i) \prod_{j=1}^{n} \frac{dz_j}{4\pi}$,

$$\text{and } \rho = \frac{\prod_{i=1}^{n+1} \prod_{j=1}^{m+n+2} \Phi(t_j + z_i, a_j + y_i; \mathrm{p}, \mathrm{q}) \Phi(u_j - z_i, b_j - y_i; \mathrm{p}, \mathrm{q})}{\prod_{1 \leq i < j \leq n+1} \Phi(z_i - z_j, y_i - y_j; \mathrm{p}, \mathrm{q}) \Phi(z_j - z_i, y_j - y_i; \mathrm{p}, \mathrm{q})}.$$

Duality of $S^1 \times S^3/\mathbb{Z}_r$ $SU(n)$, and integrable model with $\sigma \in \mathbb{R}^n \times (\mathbb{Z}_r)^n$.
M. Yamazaki, J. Stat. Phys. 154:895 (2014).

Also have transformation formulas for BC_n hypergeometric sum/integrals.
V.P. Spiridonov, Adv. Math. 331 (2018) 830-873
A.P. K and M. Yamazaki, SIGMA, 14, 013 (2018)

Lens generalisation of τ-functions for elliptic Painlevé

Two parameter extension of elliptic beta/sum integral ($n=1, m=1$ case)

$$I(x,\tilde{x}) = \sum_{\tilde{z}=0}^{r-1} \int_{[0,1]} dz \, \frac{\prod_{j=0}^{7} \Gamma(x_j \pm z, \tilde{x}_j \pm \tilde{z})}{\Gamma(\pm 2z, \pm 2\tilde{z})}, \tag{1}$$

($\tilde{x} \in \mathbb{Z}$ or $\tilde{x} \in \mathbb{Z} + \frac{1}{2}$) satisfies three-term relation

$$\begin{aligned}&\Big(\mathrm{e}^{\frac{2\pi \mathrm{i}}{r}(x_k - \tilde{x}_k)} \theta_\sigma(x_2 \pm x_3, \tilde{x}_2 \pm \tilde{x}_3) \, T_{\tau,i} \\ &+ \mathrm{e}^{\frac{2\pi \mathrm{i}}{r}(x_i - \tilde{x}_i)} \theta_\sigma(x_3 \pm x_1, \tilde{x}_3 \pm \tilde{x}_1) \, T_{\tau,j} \\ &+ \mathrm{e}^{\frac{2\pi \mathrm{i}}{r}(x_j - \tilde{x}_j)} \theta_\sigma(x_1 \pm x_2, \tilde{x}_1 \pm \tilde{x}_2) \, T_{\tau,k}\Big) \, I(x,\tilde{x}) = 0,\end{aligned} \tag{2}$$

and has nice transformation under $W(E_7)$ reflection.

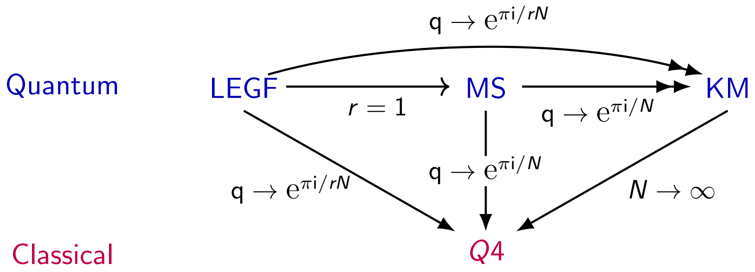
This are two of the key properties used to construct generalisation of hypergeometric τ-function for elliptic discrete Painlevé equation.

A.P. Kels, M. Yamazaki, arXiv:1810.12103 (2018).

(Based on structure of Ohta-Ramani-Grammaticos (ORG) τ-functions.)

M. Noumi, Adv. Stud. Pure Math. (2018)

YBE/3D-consistency correspondence for the elliptic case

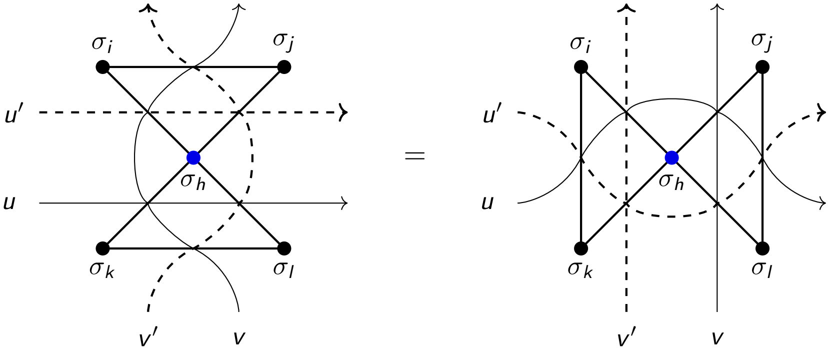


LEGF = Elliptic beta sum/integral model ($\sigma \in \mathbb{R} \times \mathbb{Z}_r$)
MS = Master solution model ($\sigma \in \mathbb{R}$)
KM = Kashiwara-Miwa model ($\sigma \in \mathbb{Z}_N$)

V.V. Bazhanov and S.M. Sergeev, Adv. Theor. Math. Phys.: 16, (2012).
V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A: Math. Theor. 49 (2016) 464001.
A.P. K and M. Yamazaki, J. Stat. Mech., 023108 (2018).

Multi-component YBE

$A_n \leftrightarrow A_n$ hypergeometric transformation

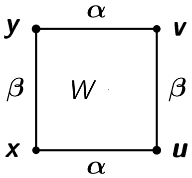


Can be considered a multi-component generalisation of STR with $\sigma_i \in \mathbb{R}^n$.

V.V. Bazhanov and S.M. Sergeev, Nucl. Phys. B 856:475-496, (2012)
V.V. Bazhanov, A.P. K, S.M. Sergeev, J. Phys. A:Math. Theor. 46: 152001, (2013)

New multi-component quad equations

Quad equation $\mathbf{Q}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{y}; \boldsymbol{\alpha}, \boldsymbol{\beta}; W)$ defined as a function of multi-component variables on the face, edges, vertices of a quadrilateral.



There are n-component vector variables at corners

$$\boldsymbol{x} = (x_1, \ldots, x_n)\,, \quad \boldsymbol{u} = (u_1, \ldots, u_n)\,, \quad \boldsymbol{y} = (y_1, \ldots, y_n)\,, \quad \boldsymbol{v} = (v_1, \ldots, v_n). \tag{3}$$

Each component of the vector variables is independent.
There are 2-component parameters on edges

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2)\,, \qquad \boldsymbol{\beta} = (\beta_1, \beta_2). \tag{4}$$

There is an n-component parameter W on the face.

New multi-component quad equations

$$
\begin{aligned}
\mathbf{Q}(\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_{12};\boldsymbol{\alpha},\boldsymbol{\beta};W)&=0\,,\\
\mathbf{Q}(\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}_3,\boldsymbol{x}_{13};\boldsymbol{\alpha},\boldsymbol{\gamma};W)&=0\,,\\
\mathbf{Q}(\boldsymbol{x}_0,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_{23};\boldsymbol{\beta},\boldsymbol{\gamma};W)&=0\,,\\
\mathbf{Q}(\boldsymbol{x}_3,\boldsymbol{x}_{13},\boldsymbol{x}_{23},\boldsymbol{x}_{123};\boldsymbol{\alpha},\boldsymbol{\beta};W)&=0\,,\\
\mathbf{Q}(\boldsymbol{x}_2,\boldsymbol{x}_{12},\boldsymbol{x}_{23},\boldsymbol{x}_{123};\boldsymbol{\alpha},\boldsymbol{\gamma};W)&=0\,,\\
\mathbf{Q}(\boldsymbol{x}_1,\boldsymbol{x}_{12},\boldsymbol{x}_{13},\boldsymbol{x}_{123};\boldsymbol{\beta},\boldsymbol{\gamma};W)&=0\,,
\end{aligned}
\tag{5}
$$

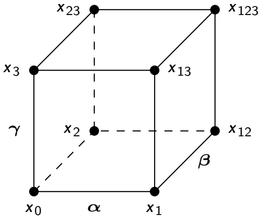


Figure: The vertices are associated to n-component variables $\boldsymbol{x}_i$, two-component parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, are associated to the edges, and an n-component parameter W (not shown) is associated to the faces.

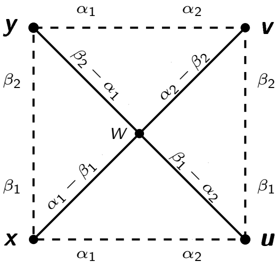
New multi-component quad equations

Consider quad equations $\mathbf{Q}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta};W)$ that have a four-leg form

$$Q_k(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v},\boldsymbol{y};\boldsymbol{\alpha},\boldsymbol{\beta};W)=$$
$$\sum_{i=1}^{n}\left(\varphi(x_i,w_k;\alpha_1,\beta_1)+\varphi(u_i,w_k;\beta_1,\alpha_2)+\varphi(v_i,w_k;\alpha_2,\beta_2)+\varphi(y_i,w_k;\beta_2,\alpha_1)\right)=0\,, \tag{6}$$

$k=1,\ldots,n$, where $\varphi(x,y;\alpha,\beta)$ is the leg function, given here by

$$\varphi(x,y;\alpha,\beta)=\frac{f(\alpha,\beta)}{x-y}+g(\alpha,\beta)\,. \tag{7}$$



New multi-component quad equations

The following is an example of a choice of $f(\alpha,\beta)$, and $g(\alpha,\beta)$, that leads to an equation (6) that is 3D-consistent:

$$\mathbf{Q}1a: \quad f(\alpha,\beta)=\alpha-\beta\,, \quad g(\alpha,\beta)=\delta\alpha\beta(\alpha-\beta)\,, \tag{8}$$

where $\delta=0,1$.

$\mathbf{Q}1a$, with $\delta=0$, and $n=1$, is a discrete Laplace equation for $Q1_{\delta=0}$, and also equivalent to a linear equation of Atkinson.

There are also several other solutions of $f(\alpha,\beta)$, $g(\alpha,\beta)$ that are 3D-consistent.

New multi-component quad equations

The affine linear form is given by

$$
\begin{aligned}
&Q_k(\boldsymbol{x},\boldsymbol{u},\boldsymbol{y},\boldsymbol{v};\boldsymbol{\alpha},\boldsymbol{\beta};W) \\
&= \sum_{i=1}^{n} \Big\{ \big[(f(\alpha_1,\beta_1)(w_k-v_i)+f(\alpha_2,\beta_2)(w_k-x_i))\,(w_k-u_i)(w_k-y_i) \\
&\quad + (f(\beta_1,\alpha_2)(w_k-y_i)+f(\beta_2,\alpha_1)(w_k-u_i))\,(w_k-v_i)(w_k-x_i)\big] \\
&\quad \times \prod_{\substack{j=1\\ j\neq i}}^{n}(w_k-x_j)(w_k-u_j)(w_k-y_j)(w_k-v_j)\Big\} \\
&\quad + h(\boldsymbol{\alpha},\boldsymbol{\beta})\prod_{i=1}^{n}\{(w_k-x_i)(w_k-u_i)(w_k-y_i)(w_k-v_i)\}\,.
\end{aligned}
\tag{9}
$$

where

$$
h(\boldsymbol{\alpha},\boldsymbol{\beta}) = \delta(\alpha_1-\alpha_2)(\beta_1-\beta_2)(\alpha_1+\alpha_2-\beta_1-\beta_2)\,,
\tag{10}
$$

New multi-component quad equations

Still have important things to work out, such as:

- Proving 3D-consistency for $n \geq 2$.
- Lax pair, Bäcklund transformations for $n \geq 2$.
- Quantum/classical Yang-Baxter equations

These are not the type of equations that are expected from the form of multi-component YBE.

On the other hand, it is unknown if the classical equations coming from multi-component YBE are 3D-consistent.

Summary

Main points for Yang-Baxter/3D-consistency correspondence:

- The star-triangle relation is a quantum ABS equation.
- An ABS equation is equivalent to the equation of the critical point of a hypergeometric integral.
- The partition function describes a quantum system of discrete Laplace equations.
- Z-invariance is a quantum closure property for the action functional of the system of discrete Laplace equations.

Correspondence motivated form of new multi-component quad equations, but these are not the same equations that come from YBE.

Connections of integrability to hypergeometric integrals and supersymmetric gauge theories also very useful.

Thanks for your attention!