Gap probabilities in tiling models and discrete Painlevé equations

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Key Points

1. P. Painlevé motivation — construction of new, purely nonlinear, special functions (similar to the construction of usual special functions, such as Airy, Bessel, and so on, as general solutions of *linear* ODE).
2. *Painlevé transcendent* (general solutions of *Painlevé differential equations*) play an increasingly important role in a wide range of nonlinear problems and applications.
3. The theory of *discrete Painlevé equations* is much more recent, but it is already very clear that the same is true — these equations appear in many important applied problems. So it is essential to understand how to effectively work with them.
4. The classification scheme for discrete Painlevé equations, due to H. Sakai, is much more complicated than the differential case. *There are 22 types of these equations*; in each type there are infinitely many *non-equivalent* equations. Nice expression for some equations in each class are known (e.g., by construction, as in the deautonomization of QRT maps).
5. Even these simple equations only look "nice" when written in *particular coordinates* (that we shall call the *Painlevé coordinates*). When a discrete Painlevé equation appears in application, it is written in *application coordinates* and it can look very complicated. Thus, it is essential to be able to understand the type of a discrete Painlevé equation that appears in an applied problem and whether it is equivalent to a known simple example
6. **Main Problem** that we consider: determine whether a given discrete Painlevé dynamics is equivalent to a known example, and if so, **how to explicitly find the change of coordinates from the application coordinates to the Painlevé coordinates**.
7. **Main point**: since discrete Painlevé equations are essentially algebraic objects, Sakai's theory gives the right set of tools to effectively answer the above question.
8. As an example, we consider the computation of *gap probabilities* in a generalized tiling problem (Alisa Knizel's work).

Main Result

For the applied problem that we consider, the computation of gap probabilities in a particular tiling model, the change of variables from the *application coordinates* (x, y) to the discrete Painlevé coordinates (f, g) is given by

$$f(x,y) = \frac{\sigma_3(xy + u(y-1)) - u^2(x^2 - \sigma_1 x + \sigma_2(y+1)) + u^3(1-y)(\sigma_1 - x) + u^4(1+y)}{\sigma_3 x(xy + u(y-1)) - u^2(\sigma_2 xy + \sigma_3(y+1)) + u^3\sigma_2(1-y) + u^4(\sigma_1(1+y) - x) + u^5(y-1)},$$

$$g(x,y) = \frac{xyz_6 + uz_6(y-1) - u^2(1+y)}{z_6(1+y) - x - u(1+y)}, \qquad \text{where}$$

$$\sigma_1 = z_2 + z_4 + z_6, \qquad \sigma_2 = z_2 z_4 + z_4 z_6 + z_6 z_2, \qquad \sigma_3 = z_2 z_4 z_6.$$

The inverse change of variables is given by

$$x(f,g) = \frac{(\kappa_1 - \kappa_2)g + \nu_6(1 + \kappa_1\kappa_2)(1 - fg) + \nu_6^2(\kappa_1 - \kappa_2)f}{\kappa_1 - \kappa_2 fg},$$

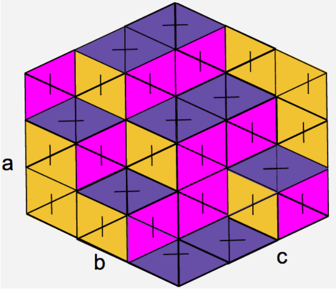
$$y(f,g) = \frac{\nu_1\nu_6(1 - fg)(\nu_6\kappa_1 - (1 + \kappa_1\kappa_2)g) + \kappa_2 fg((\nu_1\nu_6 - 1)g - \nu_6) + \nu_1\kappa_2 g^2 + \kappa_1(1 - \nu_1 g)(g + \nu_6)}{(1 - fg)(\nu_6 - \kappa_2(g - \nu_6\kappa_1)) - \nu_6((g + \nu_6)(\kappa_1\nu_1 + \kappa_2 f(1 - g\nu_1)) - \kappa_1(1 + \nu_6 f))}.$$

Here ρ_i and z_i are *application parameters*, κ_i and ν_i are *Painlevé parameters*, and they are related via $\kappa_1 = \dfrac{u}{z_2}$, $\kappa_2 = \dfrac{z_4}{u}$, and

$$\nu_1 = \frac{1}{z_6},\ \nu_2 = \frac{1}{z_1},\ \nu_3 = \frac{1}{z_3},\ \nu_4 = \frac{1}{z_5},\ \nu_5 = \frac{uz_4}{z_2},\ \nu_6 = u,\ \nu_7 = -\frac{\rho_1 z_4 z_6}{u},\ \nu_8 = -\frac{\rho_2 z_4 z_6}{u}.$$

Probabilistic Model: q-Distributions on Boxed Plane Partitions

- Models of a random surfaces: *boxed plane partition* (*lozenge tiling* of a hexagon).
- Consider tilings of an $a \times b \times c$ hexagon ($a, b, c \geq 1$) by three types of lozenge tilings (obtained by gluing together two adjacent triangles of a regular triangular grid).



Denote the set of all possible such tilings by $\Omega_{a\times b\times c}$. Equip this set with a probability measure, where, for $\mathcal{T} \in \Omega_{a\times b\times c}$,

$$P(\mathcal{T}) = \frac{w(\mathcal{T})}{Z(a,b,c)}, \text{ where } w(\mathcal{T}) = \prod_{\diamond \in \mathcal{T}} w(\diamond),$$

and $Z(a,b,c)$ is the usual normalization constant,

$$Z(a,b,c) = \sum_{\mathcal{T}\in\Omega_{a\times b\times c}} w(\mathcal{T}).$$

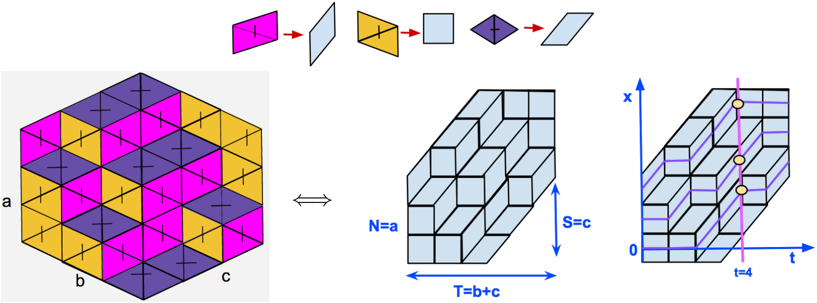
- Originally, the most studied distribution was uniform, $w(\diamond) = 1$. In 2009, A. Borodin, V. Gorin, and E. Rains introduced a far-reaching generalization of this model with a very general *elliptic weight* and (complex) parameters u_1, u_2, p, q:

$$w(\diamond) = w(\diamond_{i,j}) = \frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}, q^{j-3i/2}u_1, q^{j+3i/2-1}, q^{j+3i/2}u_2)},$$

where $\theta_p(x) = \prod_{i=0}^{\infty}(1-p^ix)(1-p^{i+1}/x)$ and $\theta_p(a,b,c,\dots) = \theta_p(a)\theta_p(b)\theta_p(c)\dots$.

From Plane Partitions to Orthogonal Polynomial Ensembles

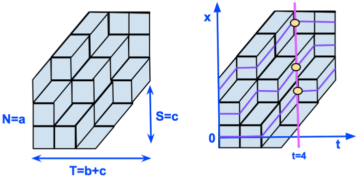
Change variables to $N = a$, $T = b + c$, $S = c$ and interpreting plane partitions as nonintersecting paths via an affine transformation



- The most general elliptic weight $w(\diamond) = \dfrac{(u_1 u_2)^{1/2} q^{j-1/2} \theta_p(q^{2j-1} u_1 u_2)}{\theta_p(q^{j-3i/2-1}, q^{j-3i/2} u_1, q^{j+3i/2-1}, q^{j+3i/2} u_2)}$ corresponds to certain biorthogonal functions (not polynomials).
- The most general orthogonal polynomial case is the limit $p \to 0$, $u_1 = O(\sqrt{p})$, $u_2 = O(\sqrt{p})$, $u_1 u_2 = p\kappa^2 q^{-S}$ with the *q-Racah weight* $w(\diamond) = \kappa q^{j-(S+1)/2} - \dfrac{1}{\kappa q^{j-(S+1)/2}}$.
- Taking the limit with $\kappa \to 0$ (with appropriate rescaling) gives q-Hahn weights $w(\diamond) = q^{-j}$.

Gap Probabilities (the q-Hahn case)

View the boxed partition model as the non-intersecting paths model; equip it with the q-Hahn weight $w(\diamond) = q^{-j}$. Fix a section t. Let the coordinates of the nodes be $C(t) = (x_1, \ldots, x_N)$.



Theorem (Borodin, Gorin, Rains (2009))

$$\mathrm{Prob}\{C(t) = (x_1, \ldots, x_N)\} = const \cdot \prod_{0 \le i < j \le M} (q^{-x_i} - q^{-x_j})^2 \prod_{i=1}^{N} w(x_i),$$

where $w(x)$ is the weight function of the q-Hahn polynomial ensemble up to a factor not depending on x.

Gap probability

The one-interval gap probability function D_s^N is

$$D_s^N = \mathrm{Prob}[\max\{x_i\} < s].$$

Theorem (Knizel (2016), q-Volume case)

The gap probability D_s^N for the q-Hahn ensemble can be computed recursively

$$D_s^N = \frac{(D_{s-2}^N)^2}{D_{s-1}^N} \frac{(r_{s-1}w - qvz_1z_2)(r_sw - quz_1z_2)(t_{s-1} - qz_1)(t_{s-1} - qz_2)}{uvz_1z_2(qz_1 - z_3)(qz_1 - z_5)(qz_2 - z_4)(qz_2 - z_6)},$$

where the sequence (r_s, t_s) satisfies the recursion (equivalent to after some change of parameters)

$$(r_st_{s-1} + 1)(r_{s-1}t_{s-1} + 1) = \frac{z_1z_2(t_{s-1} - z_3)(t_{s-1} - z_4)(t_{s-1} - z_5)(t_{s-1} - z_6)}{z_3z_4z_5z_6(qt_{s-1} - z_1)(qt_{s-1} - z_2)},$$

$$(r_st_s + 1)(r_st_{s-1} + 1) = \frac{uv(z_1z_2)^2(r_sz_3 + 1)(r_sz_4 + 1)(r_sz_5 + 1)(r_sz_6 + 1)}{(r_sw_s - vz_1z_2)(qr_sw_s - uz_1z_2)}.$$

The parameters $u, v, w, z_1, \ldots, z_6$ and the initial conditions are explicitly computed in terms of α, β, q, s. The above recursion coincides with the q-P $\left(A_2^{(1)}/E_6^{(1)}\right)$ of (**KNY**) *after some change of parameters.*

- This relation is obtained through the DRHP approach, that can be interpreted as describing isomonodromy deformations of a q-connection.
- Moduli spaces of such connections turn out to coincide with Sakai's q-Painlevé surfaces.
- Thus, the isomonodromy deformations of connections are maps in this q-Painlevé family, and hence should be given by q-P equations.

How to identify them?

Theorem (Borodin-Boyarchenko (2002))

Fix $\operatorname{card}(\mathfrak{X}) > k > 0$ *and set* $w(\psi) = \begin{bmatrix} 0 & w(\psi) \\ 0 & 0 \end{bmatrix}$. *For any* $s \geq k$ *there exists unique analytic function* $m_s(\psi) : \mathbb{C} \setminus \mathfrak{N}_s \to \operatorname{Mat}(\mathbb{C}, 2)$ *with simple poles at points in* $\mathfrak{N}_s = \{x_0, \ldots, x_{s-1}\}$ *such that*

$$\operatorname{Res}_{\psi = x} m_s(\psi) = \lim_{\psi \to x} m_s(\psi) w(\psi), \quad x \in \mathfrak{N}_s;$$

$$m_s(\psi) \cdot \begin{bmatrix} \psi^{-k} & 0 \\ 0 & \psi^{k} \end{bmatrix} = \mathbb{I} + O\left(\frac{1}{\psi}\right) \text{ as } \psi \to \infty.$$

Introduce matrix

$$A_s(z) = m_s(q^{-1}z) A_0(z) m_s^{-1}(z), \qquad \text{where} \quad A_0(z) = \begin{bmatrix} \dfrac{q w(x+1)}{w(x)} & 0 \\ 0 & 1 \end{bmatrix} \text{ with } z = q^{-x}.$$

In the q-Hahn case,

$$\frac{q w(x+1)}{w(x)} = \frac{(z - \alpha q) \cdot (z - q^{-M})}{\alpha \beta (z - q) \cdot (z - \beta^{-1} q^{-M})}.$$

Then the following holds:

- Gap probabilities D_s^N can be computed in terms of the matrix elements of $A_s = \begin{bmatrix} a_{11}^s & a_{12}^s \\ a_{21}^s & a_{22}^s \end{bmatrix}$;
- For any s matrix element a_{21}^s has a unique zero. Denote it by t_s and $a_{11}^s(t_s)$ by p_s.
- Evolution $(t_s, p_s) \to (t_{s+1}, p_{s+1})$ is described by (**after some adjustments**) q-P$(A_2^{(1)})$.

Structure of a generic $A_s(z)$ of type $\lambda = (z_1, \ldots, z_6; u, v, w, w; 3)$

$$A(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad A(0) = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix},$$

where $\deg(a_{11}) \leq 3$, $\deg(a_{12}) \leq 2$, $\deg(a_{21}) \leq 2$, $\deg(a_{22}) \leq 3$ and

$$\det A(z) = uv(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)$$

We also impose asymptotic conditions

$$\det A(z) = uvz^6 + \mathcal{O}(z^5) \qquad \operatorname{tr} A(z) = (u + v)z^3 + \mathcal{O}(z^2).$$

Parameter evolution

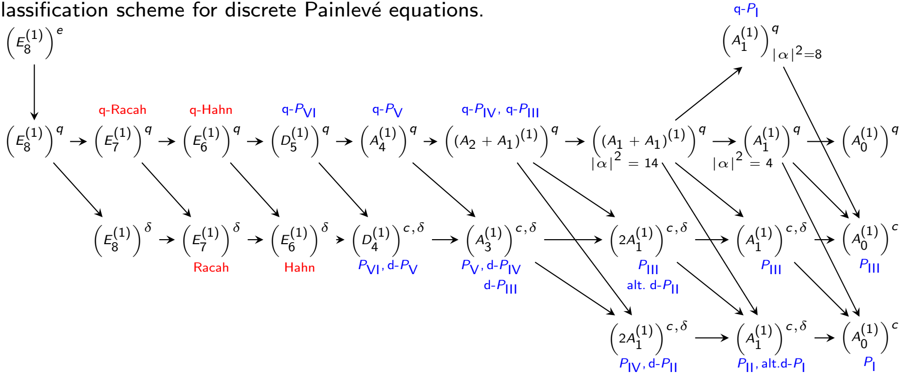
When $A_s(z) \to A_{s+1}(z)$, the parameters evolve as

$$(z_1^s, z_2^s, \ldots, z_6^s, u_s, v_s, w_s) \to (z_1^{s+1}, z_2^{s+1}, \ldots, z_6^{s+1}, u_{s+1}, v_{s+1}, w_{s+1})$$

with $z_2^{s+1} = qz_2^s$, $z_4^{s+1} = qz_4^s$, $w_{s+1} = qw_s$, and $z_i^{s+1} = z_i^s$ for $i \neq 2, 4$.

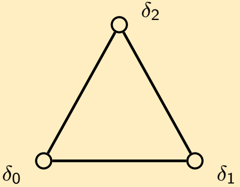
Weight degenerations and Sakais Classification scheme for Discrete Painlevé equations

The main goal of this project is to both find a way to extend the results from the q-Hahn case to a more general q-Racah case, and also to see how it fits the degeneration cascade in Sakai's classification scheme for discrete Painlevé equations.

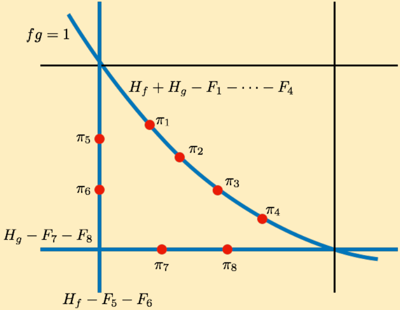


Every discrete Painlevé equation is a discrete dynamical system given by a non-homogeneous birational automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. It is resolved by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points, and becomes a flow on a family of such surfaces. Configuration of blowup points is encoded by an affine Dynkin diagram. Its "dual" affine Dynkin diagram encodes the affine Weyl symmetry group of the family (above) and Discrete Painlevé equation is equivalent to a *translation* in its lattice.

$A_2^{(1)}$ surface model



$$\delta_0 = H_f + H_g - F_1 - F_2 - F_3 - F_4$$
$$\delta_1 = H_F - F_5 - F_6$$
$$\delta_2 = H_g - F_7 - F_8$$

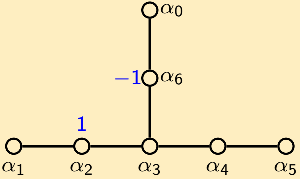


Here the configuration of the blowup points is the following:

- Four points $\pi_i(b_i = \nu_i, \nu_i^{-1})$, $i = 1, \ldots, 4$ on the $(1,1)$ curve $fg = 1$;
- Points $\pi_5(0, b_5^{-1} = \nu_5 k_2^{-1})$, $\pi_6(0, b_6^{-1} = \nu_6 k_2^{-1})$ on the line $f = 0$ and $\pi_7(0, b_7 = k_1 \nu_7^{-1})$, $\pi_8(0, b_8 = k_1 \nu_8^{-1})$ on the line $g = 0$.

The points π_i lie on the $(2,2)$-curve that is the pole divisor of the symplectic form $\omega = \dfrac{df \wedge dg}{fg(1-fg)} = \dfrac{df \wedge ds}{fs(1-s)} = \dfrac{ds \wedge dg}{gs(1-s)}$, $s = fg$, that is used to define the *period map*.

$E_6^{(1)}$ symmetry sub-lattice $Q = \mathrm{Span}_{\mathbb{Z}}\{\alpha_i | \alpha_i \bullet \delta_j = 0\}$



$$\alpha_0 = \mathcal{F}_7 - \mathcal{F}_8 \qquad \alpha_1 = \mathcal{F}_6 - \mathcal{F}_5$$

$$\alpha_2 = \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_6 \qquad \alpha_3 = \mathcal{F}_1 - \mathcal{F}_2$$

$$\alpha_4 = \mathcal{F}_2 - \mathcal{F}_3 \qquad \alpha_5 = \mathcal{F}_3 - \mathcal{F}_4$$

$$\alpha_6 = \mathcal{H}_f - \mathcal{F}_1 - \mathcal{F}_7$$

$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$$

$A_2^{(1)}/E_6^{(1)}$ period map

The period map $\chi : Q \to \mathbb{C}$, $\chi(\alpha_i) = a_i$, is used to pass from the original parameters ν_i and k_j that still have some Möbius gauge freedom to the invariant *root variables* $a_i = \exp(\alpha_i)$. Moreover, the evolution of the root variables is also canonical. We get

$$a_0 = \frac{\nu_7}{\nu_8}, \quad a_1 = \frac{\nu_6}{\nu_5}, \quad a_2 = \frac{k_2}{\nu_1\nu_6}, \quad a_3 = \frac{\nu_1}{\nu_2}, \quad a_4 = \frac{\nu_2}{\nu_3}, \quad a_5 = \frac{\nu_3}{\nu_4}, \quad a_6 = \frac{k_1}{\nu_1\nu_7}.$$

The dynamic on parameters $\bar{\nu}_i = \nu_i$, $\bar{k}_1 = q^{-1}k_1$, $\bar{k}_2 = qk_2$ results in $\bar{a}_2 = qa_2$, $\bar{a}_6 = q^{-1}a_6$, and $\bar{a}_i = a_i$ otherwise; here $q = \exp(\chi(\delta)) = a_0a_1a_2^2a_3^3a_4^2a_5a_6^2 = \dfrac{k_1k_2}{\nu_1\cdots\nu_8}$.

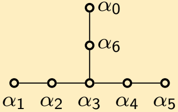
The structure of difference Painlevé equations is encoded by the extended affine Weyl symmetry group, which in our case is $\widetilde{W}\left(E_6^{(1)}\right)$.

$\widetilde{W}\left(E_6^{(1)}\right) = \mathrm{Aut}(E_6^{(1)}) \ltimes W(E_6^{(1)})$

The full extended Weyl symmetry group $\widetilde{W}\left(E_6^{(1)}\right)$ is a *semi-direct product* of

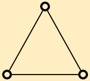
- The affine Weyl symmetry group of reflections $w_i = w_{\alpha_i}$ acting on $\mathrm{Pic}(\mathcal{X})$ as reflections in simple roots, $w_{\alpha_i}(\mathcal{C}) = \mathcal{C} + (\alpha_i \bullet \mathcal{C})\alpha_i$.

$$W(E_6^{(1)}) = \left\langle w_0, \dots, w_6 \;\middle|\; \begin{array}{ll} w_i^2 = e & \\ w_i \circ w_j = w_j \circ w_i & \text{when } \alpha_i \;\; \alpha_j \text{ (not connected)} \\ w_i \circ w_j \circ w_i = w_j \circ w_i \circ w_j & \text{when } \alpha_i - \alpha_j \text{ (connected)} \end{array} \right\rangle$$



- The finite group of Dynkin diagram automorphisms

$$\mathrm{Aut}\left(E_6^{(1)}\right) \simeq \mathrm{Aut}\left(A_2^{(1)}\right) \simeq \mathbb{D}_3,$$



where $\mathbb{D}_3 = \{e, m_0, m_1, m_2, r, r^2\} = \langle m_0, r \mid m_0^2 = r^3 = e, m_0 r = r^2 m_0 \rangle$ is the usual *dihedral group* of the symmetries of a triangle.

The action of $\widetilde{W}\left(E_6^{(1)}\right)$ on $\mathrm{Pic}(\mathcal{X})$ can be extended to the action on the space of initial conditions, giving us the birational representation of $\widetilde{W}\left(E_6^{(1)}\right)$.

For the standard example, knowing the action on the *root variables*, $\bar{a}_2 = qa_2$, $\bar{a}_6 = q^{-1}a_6$, and $\bar{a}_i = a_i$ otherwise, we see that mapping φ_* induces the translation

$$\langle \bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6 \rangle = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle + \langle 0, 0, -1, 0, 0, 0, 1 \rangle \delta$$

and then, using some standard techniques, we can represent this translation as a word in the generators:

$$\varphi_* = r w_2 w_3\, w_1 w_2 w_6\, w_3 w_4 w_0\, w_6 w_3 w_5\, w_4 w_2 w_3\, w_1 w_2.$$

This allows us to compute the action of φ_* on $\mathrm{Pic}(\mathcal{X})$ and also, using the standard birational representation of $\widetilde{W}\left(E_6^{(1)}\right)$, to compute the actual birational automorphism φ of $\mathbb{P}^1 \times \mathbb{P}^1$ whose lifting to the resolved surface $\mathcal{X}$ induces the mapping φ_*; in our case it is given by equation (8.8) of **KNY**:

$$\left(q\text{-P}\left(A_2^1/E_6^{(1)}\right)\right): \quad \begin{cases} \dfrac{(fg-1)(\bar{f}g-1)}{f\bar{f}} = \dfrac{\left(g - \frac{1}{\nu_1}\right)\left(g - \frac{1}{\nu_2}\right)\left(g - \frac{1}{\nu_3}\right)\left(g - \frac{1}{\nu_4}\right)}{\left(g - \frac{\nu_5}{k_2}\right)\left(g - \frac{\nu_6}{k_2}\right)} \\[2ex] \dfrac{(fg-1)(f\underline{g}-1)}{g\underline{g}} = \dfrac{(f-\nu_1)(f-\nu_2)(f-\nu_3)(f-\nu_4)}{\left(f - \frac{k_1}{\nu_7}\right)\left(f - \frac{k_1}{\nu_8}\right)} \end{cases}.$$

Note that a more traditional approach is to start with the equation and then obtain the corresponding translation vector.

Structure of a generic $A(z)$ of type $\lambda = (z_1, \dots, z_6; u, v, w, w; 3)$

$$A(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad A(0) = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix},$$

where $\deg(a_{11}) \leq 3$, $\deg(a_{12}) \leq 2$, $\deg(a_{21}) \leq 2$, $\deg(a_{22}) \leq 3$ and

$$\det A(z) = uv(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)$$

We also impose asymptotic conditions

$$\det A(z) = uvz^6 + \mathcal{O}(z^5) \qquad \operatorname{tr} A(z) = (u + v)z^3 + \mathcal{O}(z^2).$$

Parameter evolution

When $A(z) \to \overline{A}(z)$, the parameters evolve as

$$(z_1, z_2, \dots, z_6, u_s, v_s, w_s) \to (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_6, \overline{u}, \overline{v}, \overline{w})$$

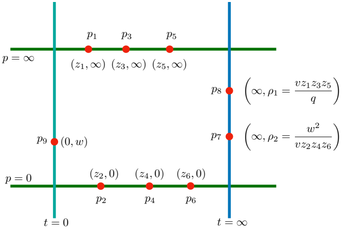
with $\overline{z}_2 = qz_2$, $\overline{z}_4 = qz_4$, $\overline{w} = qw$, and $\overline{z}_i = z_i$ for $i \neq 2, 4$.

Moduli space of q-Hahn connections

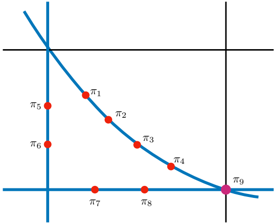
Let us now explicitly describe the moduli space of q-Hahn connections of type $\lambda = (z_1, \ldots, z_6; u, qv, w, w, ; 3)$. After gauging we can put $a_{21}(z) = z(z-t)$, where $t = t_1/t_2$ is our first spectral coordinate. The second spectral coordinate we adjust slightly and put

$$p = \frac{p_1}{p_2} = \frac{z_1 z_3 z_5\, a_{11}(t)}{(t-z_1)(t-z_3)(t-z_5)}.$$

If we just use $p = a_{11}(t)$, we get singular points $(z_i, 0)$ that results in a -6 curve that appears after we resolve the singularities of the parameterization using blowup, the above change of variables results in two -3-curves that are easier to handle. Then we get the following singularities picture:



need to blow-up at π_9



Note that the q-Hahn surface is *not minimal* and requires *blowing down* the -1-curve $t = 0$. It is easier to match it with the standard example by blowing up the point $\pi_9(\infty, 0)$ in the standard (f, g)-coordinates and establishing the identification on the level of Picard lattices, and then extending it to the birational change of coordinates.

Matching the two dynamics

After some minor trial and error, we see that the following identification works:

$$\begin{aligned}
&\mathcal{H}_f = \mathcal{H}_t && \mathcal{F}_1 = \mathcal{E}_1, && \mathcal{F}_3 = \mathcal{E}_3, && \mathcal{F}_5 = \mathcal{E}_7, && \mathcal{F}_7 = \mathcal{E}_2, && \mathcal{F}_9 = \mathcal{H}_t - \mathcal{E}_9,\\
&\mathcal{H}_g = \mathcal{H}_t + \mathcal{H}_p - \mathcal{E}_6 - \mathcal{E}_9, && \mathcal{F}_2 = \mathcal{H}_t - \mathcal{E}_6, && \mathcal{F}_4 = \mathcal{E}_5, && \mathcal{F}_6 = \mathcal{E}_8, && \mathcal{F}_8 = \mathcal{E}_4.
\end{aligned}$$

The standard techniques then give us the explicit change of variables from the *application coordinates* (or the spectral coordinates t and p) to the *Painlevé coordinates* f and g:

$$f = \frac{1}{t}, \qquad g = \frac{twz_6}{z_6(p-w)+tw}.$$

We also get the parameter matching;

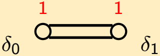
$$\begin{aligned}
&k_1 = \frac{1}{w}, && \nu_1 = \frac{1}{z_1}, && \nu_3 = \frac{1}{z_3}, && \nu_5 = \rho_1 z_6, && \nu_7 = \frac{z_2}{w},\\
&k_2 = w, && \nu_2 = \frac{1}{z_6}, && \nu_4 = \frac{1}{z_5}, && \nu_6 = \rho_2 z_6, && \nu_8 = \frac{z_4}{w},
\end{aligned}$$

(note that there is a parameter constraint in q-Hahn, $w^2 = uvz_1\cdots z_6$).
With this identification the spectral coordinates evolution under isomonodromic transformations coincides with q-P$\left(A_2^1/E_6^{(1)}\right)$ of (**KNY**).

The q-Racah orthogonal ensemble and q-$P(A_1^{(1)}/E_7^{(1)})$

Consider now the example of a q-Racah orthogonal polynomial ensemble and q-$P\left(A_1^{(1)}/E_7^{(1)}\right)$.

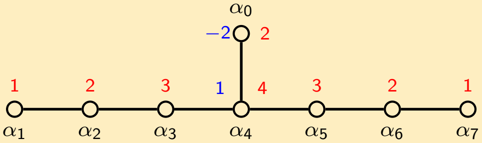
q-$P\left(A_1^{(1)}/E_7^{(1)}\right)$ surface and reference dynamic



Dynkin diagram $A_1^{(1)}$

$$\delta_0 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 - \mathcal{F}_4$$
$$\delta_1 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7 - \mathcal{F}_8$$

The surface data



Dynkin diagram $E_7^{(1)}$

$$\alpha_0 = \mathcal{H}_f - \mathcal{H}_g \qquad \alpha_4 = \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_5$$
$$\alpha_1 = \mathcal{F}_3 - \mathcal{F}_4 \qquad \alpha_5 = \mathcal{F}_5 - \mathcal{F}_6$$
$$\alpha_2 = \mathcal{F}_2 - \mathcal{F}_3 \qquad \alpha_6 = \mathcal{F}_6 - \mathcal{F}_7$$
$$\alpha_3 = \mathcal{F}_1 - \mathcal{F}_2 \qquad \alpha_7 = \mathcal{F}_7 - \mathcal{F}_8$$

The symmetry data

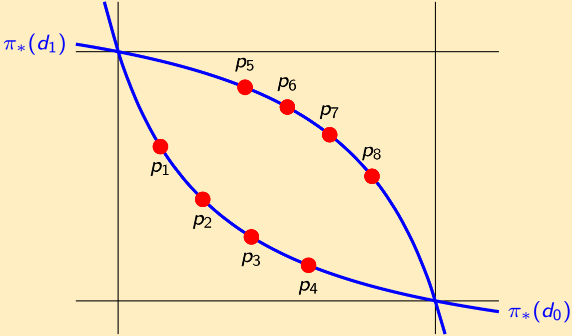
$A_1^{(1)}$ point configuration and the surface model

$$p_i\left(\nu_i, \frac{1}{\nu_i}\right), \quad i = 1, \ldots, 4;$$

$$p_i\left(\frac{\kappa_1}{\nu_i}, \frac{\nu_i}{\kappa_2}\right), \quad i = 5, \ldots, 8.$$

$$\pi_*(d_0): \ fg = 1$$

$$\pi_*(d_1): \ fg = \kappa = \frac{\kappa_1}{\kappa_2}$$



The points π_i lie on the (reducible) $(2,2)$-curve that is the pole divisor of the symplectic form $\omega = (k-1)\dfrac{df \wedge dg}{(fg-1)(fg-k)} = (k-1)\dfrac{df \wedge ds}{f(s-1)(s-k)} = (k-1)\dfrac{ds \wedge dg}{g(s-1)(s-k)}$, where again we put $s = fg$. Degeneration to $\left(A_1^{(1)}/E_7^{(1)}\right)$ case is very straightforward, just put $\kappa \to 0$.

$A_1^{(1)}/E_1^{(1)}$ period map

The period map $\chi : Q \to \mathbb{C}$, $\chi(\alpha_i) = a_i$, in this case gives us the root variables $a_i = \exp(\chi(\alpha_i))$:

$$a_0 = \frac{\kappa_1}{\kappa_2}, \quad a_1 = \frac{\nu_3}{\nu_4}, \quad a_2 = \frac{\nu_2}{\nu_3}, \quad a_3 = \frac{\nu_1}{\nu_2}, \quad a_4 = \frac{\kappa_2}{\nu_1\nu_5}, \quad a_5 = \frac{\nu_5}{\nu_6}, \quad a_6 = \frac{\nu_6}{\nu_7}, \quad a_7 = \frac{\nu_7}{\nu_8}.$$

The dynamic on parameters $\bar{\nu}_i = \nu_i$, $\bar{k}_1 = q^{-1}k_1$, $\bar{k}_2 = qk_2$ results in $\bar{a}_0 = q^{-2}a_0$, $\bar{a}_4 = qa_4$, and $\bar{a}_i = a_i$ otherwise.

For the standard example, we can represent the mapping φ_* that induces the translation

$$\varphi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle \mapsto \overline{\alpha} = \alpha + \langle 2, 0, 0, 0, -1, 0, 0, 0 \rangle \delta.$$

as

$$\varphi_* : w_0 w_4 w_5 w_3 w_4 w_6 w_5 w_2 w_3 w_4 w_1 w_2 w_3 w_0 w_4 w_7 w_6 w_5 w_4 w_3 w_0 w_4 w_6 w_5 w_2 w_3 w_4 w_7 w_6 w_5 w_1 w_2 w_3 w_4,$$

This allows us to compute the action of φ_* on $\operatorname{Pic}(\mathcal{X})$ and also, using the standard birational representation of $\widetilde{W}\left(E_7^{(1)}\right)$, to compute the actual birational automorphism φ of $\mathbb{P}^1 \times \mathbb{P}^1$ whose lifting to the resolved surface $\mathcal{X}$ induces the mapping φ_*; in our case it is given by equation (8.7) of **KNY** (first obtained by B. Grammaticos and A. Ramani, who called it asymmetric q-P_{VI}):

$$\begin{cases} \dfrac{\left(fg - \frac{\kappa_1}{\kappa_2}\right)\left(\overline{f}g - \frac{\kappa_1}{q\kappa_2}\right)}{(fg-1)(\overline{f}g-1)} = \dfrac{\left(g - \frac{\nu_5}{\kappa_2}\right)\left(g - \frac{\nu_6}{\kappa_2}\right)\left(g - \frac{\nu_7}{\kappa_2}\right)\left(g - \frac{\nu_8}{\kappa_2}\right)}{\left(g - \frac{1}{\nu_1}\right)\left(g - \frac{1}{\nu_2}\right)\left(g - \frac{1}{\nu_3}\right)\left(g - \frac{1}{\nu_4}\right)}, \\ \dfrac{\left(fg - \frac{\kappa_1}{\kappa_2}\right)\left(f\underline{g} - \frac{q\kappa_1}{\kappa_2}\right)}{(fg-1)(f\underline{g}-1)} = \dfrac{\left(f - \frac{\kappa_1}{\nu_5}\right)\left(f - \frac{\kappa_1}{\nu_6}\right)\left(f - \frac{\kappa_1}{\nu_7}\right)\left(f - \frac{\kappa_1}{\nu_8}\right)}{(f-\nu_1)(f-\nu_2)(f-\nu_3)(f-\nu_4)}. \end{cases}$$

Moduli space for q-Racah connections

In this case, we look at moduli spaces $\lambda = (z_1, \ldots, z_6; u, d = d_1, d_2; 6)$ of 2×2 matrices satisfying the following conditions:

$$A(z) = \frac{1}{P(z)} \begin{bmatrix} a_{11} & \dfrac{a_{12}}{z} \\ a_{21} & a_{22} \end{bmatrix}, \quad a_{21}(0) = 0,$$

where $\deg(a_{11}) \leq 6$, $\deg(a_{12}) \leq 8$, $\deg(a_{21}) \leq 5$, $\deg(a_{22}) \leq 6$ and

$$\det A(z) = \frac{P(z)}{Q(z)}, \qquad \begin{aligned} P(z) &= (z - z_1)(z - u^2/z_2)(z - z_3)(z - u^2/z_4)(z - z_5)(z - u^2/z_6) \\ Q(z) &= \frac{z_1 z_3 z_5}{z_2 z_4 z_6}(z - u^2/z_1)(z - z_2)(z - u^2/z_3)(z - z_4)(z - u^2/z_5)(z - z_6), \end{aligned}$$

with some asymptotic conditions and modulo gauge transformations of the form

$$\hat{A}(z) = R(z/q + u^2/z)A(z)R^{-1}(z + u^2/(qz)), \qquad R(z) = \begin{bmatrix} r_{11}(z) & r_{12}(z) \\ 0 & r_{22}(z) \end{bmatrix},$$

where $\deg(r_{11}) = \deg(r_{22}) = 0$ and $\deg(r_{12}) \leq 1$ and the very important involution condition $A(u^2/z) = A^{-1}(z)$.

Parameter evolution

In this case, the parameters evolve as

$$\bar{z}_2 = qz_2, \quad \bar{z}_4 = qz_4, \quad \bar{d} = q^{-1}d, \quad \bar{z}_i = z_i \text{ otherwise.}$$

Moduli space of q-Racah connections

Let us now explicitly describe the moduli space of q-Racah connections. After gauging we can put $a_{21}(z) = z(z-t)(z-u^2)(z^2-u^2)$, where $t = t_1/t_2$ is our first spectral coordinate, and the second spectral coordinate p is again the *adjusted* value of $a_{11}(t)$.
In the coordinates (t,p) we get more than 8 points because of the involution $t \leftrightarrow u^2/t$ and $p \leftrightarrow 1/p$, e.g., we get the following six pairs of points:

$$\left(\frac{u^2}{z_1},0\right),(z_1,\infty), \quad \left(\frac{u^2}{z_3},0\right),(z_3,\infty), \quad \left(\frac{u^2}{z_5},0\right),(z_5,\infty),$$
$$(z_2,0),\left(\frac{u^2}{z_2},\infty\right), \quad (z_4,0),\left(\frac{u^2}{z_4},\infty\right), \quad (z_6,0),\left(\frac{u^2}{z_6},\infty\right),$$

points $(u,1)$ and $(-u,-1)$, and points $(\infty,-\rho_1=d)$ and $\left(\infty,-\rho_2=\dfrac{z_1z_3z_5}{z_2z_4z_6qd}\right)$.

To fix this, we need to introduce the *involution-invariant* coordinates $x = t + \dfrac{u^2}{t}$ and $y = \dfrac{pt-u}{pu-t}$ gluing these pairs of points together.

Then, in the (x,y)-coordinates we get the correct picture.

But can it be matched with the standard example?

$$\pi_i\left(z_i + \frac{u^2}{z_i}, \frac{z_i}{u}\right), \quad i = 1, 3, 5;$$

$$\pi_i\left(z_i + \frac{u^2}{z_i}, \frac{u}{z_i}\right), \quad i = 2, 4, 6;$$

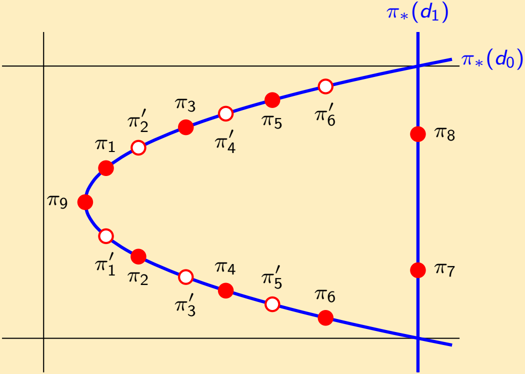
$$\pi_7\left(\infty, \rho_1 = -d\right),$$

$$\pi_8\left(\infty, \rho_2 = -\frac{z_1 z_3 z_5}{z_2 z_4 z_6 q d}\right).$$

We also get conjugated points

$$\pi_i'\left(z_i + \frac{u^2}{z_i}, \frac{u}{z_i}\right), \quad i = 1, 3, 5;$$

$$\pi_i'\left(z_i + \frac{u^2}{z_i}, \frac{z_i}{u}\right), \quad i = 2, 4, 6.$$



Note that the points π_7 and π_8 lie on the $(1,0)$-curve $\pi_*(d_1) = V(X = 1/x)$ and $\pi_1, \ldots, \pi_6$ lie on the $(1,2)$-curve $\pi_*(d_0) = V(u(y^2+1) - xy)$; note also that when $x = z_i + \dfrac{u^2}{z_i}$, the equation $u(y^2+1) - xy$ factors as $u(y^2+1) - xy = u(y - y(\pi_i))(y - y(\pi_i'))$. Finally, there is an additional blowup point $\pi_9(-2u, -1)$, similar to the q-Hahn case.

Matching the q-Racah dynamics with the standard dynamics

Looking at the decomposition of the anti-canonical divisor class,

$$\delta_0 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_9 = \mathcal{H}_x + 2\mathcal{H}_y - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_9,$$
$$\delta_1 = \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7 - \mathcal{F}_8 = \mathcal{H}_x - \mathcal{E}_7 - \mathcal{E}_8,$$

we see that it makes sense to preliminary take

$$\begin{aligned}
\mathcal{H}_f &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_2 - \mathcal{E}_9, & \mathcal{H}_x &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_7 - \mathcal{F}_8,\\
\mathcal{H}_g &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_4 - \mathcal{E}_9, & \mathcal{H}_y &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_7 - \mathcal{F}_9,\\
\mathcal{F}_1 &= \mathcal{E}_1, & \mathcal{E}_1 &= \mathcal{F}_2,\\
\mathcal{F}_2 &= \mathcal{E}_6, & \mathcal{E}_2 &= \mathcal{H}_g - \mathcal{F}_7,\\
\mathcal{F}_3 &= \mathcal{E}_3, & \mathcal{E}_3 &= \mathcal{F}_3,\\
\mathcal{F}_4 &= \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{H}_f - \mathcal{F}_7,\\
\mathcal{F}_5 &= \mathcal{E}_7, & \mathcal{E}_5 &= \mathcal{F}_4,\\
\mathcal{F}_6 &= \mathcal{E}_8, & \mathcal{E}_6 &= \mathcal{F}_1,\\
\mathcal{F}_7 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_9, & \mathcal{E}_7 &= \mathcal{F}_5,\\
\mathcal{F}_8 &= \mathcal{H}_y - \mathcal{E}_9, & \mathcal{E}_8 &= \mathcal{F}_6,\\
\mathcal{F}_9 &= \mathcal{H}_x - \mathcal{E}_9, & \mathcal{E}_9 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_7 - \mathcal{F}_8 - \mathcal{F}_9.
\end{aligned}$$

Matching the q-Racah dynamics with the standard dynamics

Computing the root variables,

$$a_0 = \frac{z_4}{z_2},\ a_1 = \frac{z_5}{z_3},\ a_2 = \frac{z_3}{z_1},\ a_3 = \frac{z_1 z_6}{u^2},\ a_4 = -\frac{u^2}{\rho_1 z_4 z_6},\ a_5 = \frac{\rho_1}{\rho_2},\ a_6 = -\frac{\rho_2 z_2 z_4}{u^2},\ a_7 = \frac{u^2}{z_2 z_4}.$$

and using our parameter dynamics, we get the following translation element:

$$\psi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle \mapsto \overline{\alpha} = \alpha + \langle 0, 0, 0, 0, 0, 0, -1, 2 \rangle \delta,$$

which is **different** from the standard translation vector

$$\varphi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle \mapsto \overline{\alpha} = \alpha + \langle 2, 0, 0, 0, -1, 0, 0, 0 \rangle \delta.$$

However, these elements are **conjugated**. This can be observed, for example, by looking at the corresponding words in the affine Weyl symmetry group:

$$\psi_* : w_7 w_6 w_5 w_4 w_3 w_0 w_4 w_5 w_2 w_3 w_4 w_1 w_2 w_3 w_0 w_4 w_6 w_5 w_4 w_3 w_0 w_4 w_6 w_5 w_2 w_3 w_4 w_1 w_2 w_3 w_0 w_4 w_5 w_6,$$

$$\varphi_* : w_0 w_4 w_5 w_3 w_4 w_6 w_5 w_2 w_3 w_4 w_1 w_2 w_3 w_0 w_4 w_7 w_6 w_5 w_4 w_3 w_0 w_4 w_6 w_5 w_2 w_3 w_4 w_7 w_6 w_5 w_1 w_2 w_3 w_4.$$

Using the far commutativity and the braid relations in $W\left(E_7^{(1)}\right)$, we get

$$\psi_* = (w_6 w_5 w_4 w_0 w_7 w_6 w_5 w_4) \varphi_* (w_6 w_5 w_4 w_0 w_7 w_6 w_5 w_4)^{-1}.$$

This adjusts the divisor matching as

$$\begin{aligned}
\mathcal{H}_f &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_9, & \mathcal{H}_x &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6,\\
\mathcal{H}_g &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_6 - \mathcal{E}_9, & \mathcal{H}_y &= \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_9,\\
\mathcal{F}_1 &= \mathcal{H}_x - \mathcal{E}_6, & \mathcal{E}_1 &= \mathcal{F}_2,\\
\mathcal{F}_2 &= \mathcal{E}_1, & \mathcal{E}_2 &= \mathcal{H}_g - \mathcal{F}_5,\\
\mathcal{F}_3 &= \mathcal{E}_3, & \mathcal{E}_3 &= \mathcal{F}_3,\\
\mathcal{F}_4 &= \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{H}_g - \mathcal{F}_6,\\
\mathcal{F}_5 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_2 - \mathcal{E}_6 - \mathcal{E}_9, & \mathcal{E}_5 &= \mathcal{F}_4,\\
\mathcal{F}_6 &= \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_9, & \mathcal{E}_6 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_5 - \mathcal{F}_6,\\
\mathcal{F}_7 &= \mathcal{E}_7, & \mathcal{E}_7 &= \mathcal{F}_7,\\
\mathcal{F}_8 &= \mathcal{E}_8, & \mathcal{E}_8 &= \mathcal{F}_8,\\
\mathcal{F}_9 &= \mathcal{H}_x - \mathcal{E}_9, & \mathcal{E}_9 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_9.
\end{aligned}$$

The new root variables are

$$a_0 = \frac{u^2}{z_2 z_4},\ a_1 = \frac{z_5}{z_3},\ a_2 = \frac{z_3}{z_1},\ a_3 = \frac{z_1}{z_6},\ a_4 = \frac{z_2 z_6}{u^2},\ a_5 = \frac{z_4}{z_2},\ a_6 = -\frac{u^2}{\rho_1 z_4 z_6},\ a_7 = \frac{\rho_1}{\rho_2},$$

which, given the parameter dynamic $\bar{z}_2 = q z_2$, $\bar{z}_4 = q z_4$, $\bar{d} = q^{-1} d$ (and so $\bar{\rho}_i = q^{-1}\rho_i$), immediately gives us the correct translation element:

$$\psi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle \mapsto \overline{\alpha} = \alpha + \langle 2, 0, 0, 0, -1, 0, 0, 0 \rangle \delta.$$

Matching the q-Racah dynamics with the standard dynamics

We then get the following parameter matching: $\kappa_1 = \dfrac{u}{z_2}$, $\kappa_2 = \dfrac{z_4}{u}$, and

$$\nu_1 = \frac{1}{z_6},\ \nu_2 = \frac{1}{z_1},\ \nu_3 = \frac{1}{z_3},\ \nu_4 = \frac{1}{z_5},\ \nu_5 = \frac{uz_4}{z_2},\ \nu_6 = u,\ \nu_7 = -\frac{\rho_1 z_4 z_6}{u},\ \nu_8 = -\frac{\rho_2 z_4 z_6}{u}.$$

Main Result

The change of variables from the spectral coordinates to the discrete Painlevé coordinates matching the q-Racah isomonodromic dynamics to the standard dynamics is given by

$$f(x,y) = \frac{\sigma_3(xy + u(y-1)) - u^2(x^2 - \sigma_1 x + \sigma_2(y+1)) + u^3(1-y)(\sigma_1 - x) + u^4(1+y)}{\sigma_3 x(xy + u(y-1)) - u^2(\sigma_2 xy + \sigma_3(y+1)) + u^3\sigma_2(1-y) + u^4(\sigma_1(1+y) - x) + u^5(y-1)} \to \frac{1}{x},$$

$$g(x,y) = \frac{xyz_6 + uz_6(y-1) - u^2(1+y)}{z_6(1+y) - x - u(1+y)} \to \frac{xyz_6}{z_6(1+y) - x}, \qquad \text{where}$$

$$\sigma_1 = z_2 + z_4 + z_6, \qquad \sigma_2 = z_2 z_4 + z_4 z_6 + z_6 z_2, \qquad \sigma_3 = z_2 z_4 z_6.$$

The inverse change of variables is given by

$$x(f,g) = \frac{(\kappa_1 - \kappa_2)g + \nu_6(1 + \kappa_1\kappa_2)(1 - fg) + \nu_6^2(\kappa_1 - \kappa_2)f}{\kappa_1 - \kappa_2 fg},$$

$$y(f,g) = \frac{\nu_1\nu_6(1-fg)(\nu_6\kappa_1 - (1+\kappa_1\kappa_2)g) + \kappa_2 fg((\nu_1\nu_6 - 1)g - \nu_6) + \nu_1\kappa_2 g^2 + \kappa_1(1 - \nu_1 g)(g + \nu_6)}{(1-fg)(\nu_6 - \kappa_2(g - \nu_6\kappa_1)) - \nu_6((g+\nu_6)(\kappa_1\nu_1 + \kappa_2 f(1 - g\nu_1)) - \kappa_1(1 + \nu_6 f))}.$$

Everything does have the correct limit to the q-Hahn case as $u \to 0$.

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