Painlevé equations from Nakajima-Yoshioka blow-up relations

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based on ArXiv 1811.04050 with Mikhail Bershtein

13 November 2018

Painlevé III_3 equation and its q-deformation

Object of our interest: Painlevé III_3 equation, its q-deformation and their solutions.

- Toda-like form of these equations is a two bilinear equations on two functions: τ and τ_1. It is symmetric under $\tau \leftrightarrow \tau_1$.
- Painlevé III_3

$$\begin{aligned} D^2_{[\log z]}(\tau,\tau) &= -2z^{1/2}\tau_1^2 \\ D^2_{[\log z]}(\tau_1,\tau_1) &= -2z^{1/2}\tau^2 \end{aligned} \tag{1}$$

  where second Hirota differential $D^2_{[\log z]}(\tau,\tau) = 2\tau''\tau - \tau'^2$, $f' = z\frac{df}{dz}$.
- q-deformation

$$\begin{aligned} \overline{\tau}\underline{\tau} &= \tau^2 - z^{1/2}\tau_1^2 \\ \overline{\tau_1}\underline{\tau_1} &= \tau_1^2 - z^{1/2}\tau^2 \end{aligned} \tag{2}$$

  where $\overline{\tau(z)} = \tau(qz), \underline{\tau(z)} = \tau(q^{-1}z)$.

Gamayun-Iorgov-Lisovyy tau function

Gamayun-Iorgov-Lisovyy in 2012-2013 proposed power series representation for the tau function of the (continuous) Painlevé equations. Tau function — Fourier series of Nekrasov partition functions

$$\tau_j(\sigma, s|z) = \sum_{n\in\mathbb{Z}+j/2} s^n \mathcal{Z}(\sigma+n|z) \tag{3}$$

- $\mathcal{Z}(\sigma|z)$ is a certain Nekrasov partition function which depends on the Painlevé equation we take.
- s and σ play role of the integration constants of the Painlevé equation.
- We take $j=0$ for τ and $j=1$ for τ_1.

Approach to solve bilinear equations on tau functions

- We have some bilinear equation on some tau function, schematically

$$\langle \tau, \tau \rangle = 0 \tag{4}$$

- We take the Gamayun-Iorgov-Lisovyy formula as the ansatz with some function $\mathcal{Z}(\sigma|z)$.
- We collect terms with the power s^m. Roughly, the relations with s^m is equivalent to the relations with s^{m-1}

$$\langle \tau, \tau \rangle = 0 \quad \left\| \tau = \sum_{n \in \mathbb{Z}} s^n \mathcal{Z}(\sigma + n|z) \right\| \quad \text{so} \quad \langle \tau, \tau \rangle|_{s^m} = 0 \Leftrightarrow \langle \tau, \tau \rangle|_{s^0} = 0 \tag{5}$$

- s^0 relations is some bilinear relation on $\mathcal{Z}(\sigma|z)$. In our cases $\mathcal{Z}(\sigma|z)$ is certain Nekrasov function or via the AGT — conformal block.

Nekrasov instanton partition functions

- Nekrasov instanton partition function for pure gauge $U(r)$ YM is defined as the equivariant volume of the instanton moduli space

$$\mathcal{Z}(\epsilon_1, \epsilon_2, \overrightarrow{a}; z) = \sum_{N=0}^{+\infty} z^N \int_{M(r,n)} 1 \tag{6}$$

- This integral localizes on the fixed points of action of the $r+2$-dimensional torus. Fixed points are labeled by r-tuple of the Young diagrams.
- Example for the 4D $SU(2)$ case

$$\begin{aligned} &\mathcal{Z}_{inst}(a_1, a_2; \epsilon_1, \epsilon_2 | z) = \sum_{\lambda^{(1)}, \lambda^{(2)}} \frac{z^{|\lambda^{(1)}|+|\lambda^{(2)}|}}{\prod_{i,j=1}^{2} \mathrm{N}_{\lambda^{(i)}, \lambda^{(j)}}(a_i - a_j; \epsilon_1, \epsilon_2)}, \\ &\mathrm{N}_{\lambda,\mu}(a; \epsilon_1, \epsilon_2) = \\ &= \prod_{s\in\lambda} (a - \epsilon_2(a_\mu(s)+1) + \epsilon_1 l_\lambda(s)) \prod_{s\in\mu} (a + \epsilon_2 a_\lambda(s) - \epsilon_1(l_\mu(s)+1)) \end{aligned} \tag{7}$$

Structure of the Nekrasov function

Structure of the Nekrasov function

$$\mathcal{Z} = \mathcal{Z}_{cl}\mathcal{Z}_{1-loop}\mathcal{Z}_{inst}, \tag{8}$$

- $\mathcal{Z}_{inst}$ is given by the Nekrasov formula. Case 5D differs from the case 4D in the Nekrasov formula by $a \mapsto 1 - q^a$.
- Classical part $\mathcal{Z}_{cl} = z^{\Delta(\sigma)}$.
- $\mathcal{Z}_{1-loop}$ is given by the double gamma functions with periods $\epsilon_{1,2}$.

Nekrasov functions and solutions of Painlevé equations

Different choice of $\mathcal{Z}$:

- 4D $SU(2)$ Nekrasov function with $\epsilon_1 + \epsilon_2 = 0$ (or Vir $c = 1$ conformal block) give the solution of continuous Painlevé equations $PVI, V, III's$ (Gamayun-Iorgov-Lisovyy, 2012-2013) — proved by different methods.
- 5D $SU(2)$ Nekrasov function with $\epsilon_1 + \epsilon_2 = 0$ — q-deformed Painlevé III_3 equation (Bershtein-S.,2016) and q-deformed Painlevé VI, V, III_1, III_2 equations (Jimbo-Nagoya-Sakai, 2017, Matsuhira-Nagoya, 2018)
- Quiver gauge theories and $SU(N), N > 2$ Nekrasov functions: generalization to the isomonodromic problems of rank > 2, larger number of punctures, generalization of Toda-like equations to the number of nodes > 2, quantization of the τ function... (Bershtein, Gavrylenko, Marshakov, Iorgov, Lisovyy 2015-2018)

Bilinear relations on 5D $SU(2)$ Nekrasov function

- q-deformed Painlevé III_3: conjectural bilinear relation on 5D Nekrasov functions

$$\sum_{2n\in\mathbb{Z}} \mathcal{Z}(\sigma+n|q^{-1}z)\mathcal{Z}(\sigma-n|qz) = (1-z^{1/2}) \sum_{2n\in\mathbb{Z}} \mathcal{Z}(\sigma+n|z), \mathcal{Z}(\sigma-n|z) \quad (9)$$

- One of the aims of the talk: present the proof of this relation. Before it was checked up to the power z^{12}.
- The continuous limit $z \mapsto R^4 z, q = e^R, R \to 0$ give us

$$\sum_{2n\in\mathbb{Z}} D^2(\mathcal{Z}(\sigma+n|z), \mathcal{Z}(\sigma-n|z)) = -z^{1/2} \sum_{2n\in\mathbb{Z}} \mathcal{Z}(\sigma+n|z), \mathcal{Z}(\sigma-n|z) \quad (10)$$

We proved these relations by the representation theory of Super Virasoro algebra when we proved the Gamayun-Iorgov-Lisovyy conjecture .

Nakajima-Yoshioka relations

There are blow-up relations on 4D and 5D partition functions [Nakajima, Yoshioka 2003,2005]. They express instanton partition function on $\widehat{\mathbb{C}^2} = (\mathbb{C}^2$ blowed up in the point) as a bilinear relation on $\mathbb{C}^2$ instanton partition function

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{\mathbb{C}^2}(a+\epsilon_1 n|\epsilon_1,\epsilon_2-\epsilon_1|z)\mathcal{Z}_{\mathbb{C}^2}(a+\epsilon_2 n|\epsilon_1-\epsilon_2,\epsilon_2|z) \quad (11)$$

and

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|z) = \mathcal{Z}_{\mathbb{C}^2}(a|\epsilon_1,\epsilon_2|z) \quad (12)$$

There are also differential (for 4D) and q-difference (for 5D) Nakajima-Yoshioka relations.

Nakajima-Yoshioka relations: $c = -2$ tau function

Take particular case $\epsilon_1 + \epsilon_2 = 0$ in Nakajima-Yoshioka relations . Then in CFT terms $c = 1$ partition function is a bilinear combination of $c = -2$

$$\mathcal{Z}_{c=1}(\sigma|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}^{+}_{c=-2}(\sigma + n|z/4)\mathcal{Z}^{-}_{c=-2}(\sigma - n|z/4), \qquad (13)$$

where $\mathcal{Z}^{\pm}$ correspond to $\pm\epsilon_1, \mp 2\epsilon_1$ partition functions.
Then it is natural to compose Fourier series

$$\tau(\sigma, s|z) = \tau^{-}(\sigma, s|z)\tau^{+}(\sigma, s|z) \qquad (14)$$

where

$$\tau^{\pm}(\sigma, s|z) = \sum_{n\in\mathbb{Z}} s^{n/2}\mathcal{Z}^{\pm}_{c=-2}(\sigma + n|z/4) \qquad (15)$$

$c = -2$ tau function: differential relations

Differential Nakajima-Yoshioka relations

$$D^1_{[\log z]}(\tau^-, \tau^+) = z^{1/4}\tau_1, \qquad D^2_{[\log z]}(\tau^-, \tau^+) = 0,$$
$$D^3_{[\log z]}(\tau^-, \tau^+) = z^{1/4}\tau_1', \qquad D^4_{[\log z]}(\tau^-, \tau^+) = -2z.\tau \tag{16}$$

The PIII_3 equation follow from these equations. Namely, second equation implies

$$z\frac{d}{dz}\tau^{\pm} = \frac{1}{2}(\zeta \pm \sqrt{\zeta'})\tau^{\pm}, \tag{17}$$

where $\zeta = \tau'/\tau$ is a Hamiltonian of the PIII_3 equation. From the fourth equation

$$(\zeta'' - \zeta')^2 = 4\zeta'^2(\zeta - \zeta') - 4z\zeta', \tag{18}$$

which is Hamiltonian form of PIII_3 equation.

Toda-like form from NY relations: q-case

q-difference Nakajima-Yoshioka relations

$$\begin{aligned} \overline{\mathrm{\tau}^+}\underline{\mathrm{\tau}^-} - \underline{\mathrm{\tau}^+}\overline{\mathrm{\tau}^-} &= -2z^{1/4}\tau_1, \\ \overline{\mathrm{\tau}^+}\underline{\mathrm{\tau}^-} + \underline{\mathrm{\tau}^+}\overline{\mathrm{\tau}^-} &= 2\tau. \end{aligned} \tag{19}$$

Proposition

Take (19) and $\tau = \mathrm{\tau}^+\mathrm{\tau}^-$. Then τ and τ_1 satisfy Toda-like equation

$$\overline{\tau}\underline{\tau} = \tau^2 - z^{1/2}\tau_1^2. \tag{20}$$

Proof and consequences

Proof.

Proof is extremely elementary. We substitute τ_1 and τ in different ways

$$\overline{\tau^+}\,\overline{\tau^-}\,\underline{\tau^+}\,\underline{\tau^-} = \frac{1}{4}(\overline{\tau^+}\,\underline{\tau^-} + \underline{\tau^+}\,\overline{\tau^-})^2 - \frac{1}{4}(\overline{\tau^+}\,\underline{\tau^-} - \underline{\tau^+}\,\overline{\tau^-})^2 \tag{21}$$

□

- Nakajima-Yoshioka relations are proven, so we obtain the proof of the conjectured bilinear relation (9) in q-case automatically.
- There is, of course, differential analogue of this Proposition.

Toda-like equations: Chern-Simons generalization

- Bershtein, Gavrylenko, Marshakov, 2018: (m, N)-generalizations of Toda-like equations, $N \in \mathbb{N}, 0 \leq m \leq N$

$$\tau_{m;j}(qz)\tau_{m;j}(q^{-1}z) = \tau_{m;j}(z)^2 - z^{1/N}\tau_{m;j+1}(q^{m/N}z)\tau_{m;j-1}(q^{-m/N}z),\ j \in \mathbb{Z}/N\mathbb{Z}. \quad (22)$$

- Corresponding to the Toda family of spectral curves (see soon in Pasha Gavrylenko talk).
- Solutions: Fourier series of $SU(N)$ Nekrasov instanton partition function modified by level m Chern-Simons term.
- Modification: each summand in Nekrasov formula multiplying on

$$\mathsf{T}_\lambda^m = \prod_{(i,j)\in\lambda} u^{-1} q_1^{1-i} q_2^{1-j}. \quad (23)$$

CS generalization: $c = -2$ tau function

- We obtained by the same method Toda-like equations from Nakajima-Yoshioka relations only for $N = 2$.
- For $m = 2$ Toda-like equation is equivalent to the case $m = 0$ and moreover, generally

$$\mathcal{Z}_2(u; q_1, q_2|z) = (z; q_1, q_2)_\infty \mathcal{Z}_0(u; q_1, q_2|z) \tag{24}$$

- For $m = 1$ Toda-like equation is equivalent to the Painlevé $A_7^{(1)}$ equation

$$\overline{\overline{\tau}}\underline{\underline{\tau}} = \tau^2 - z^{1/2}\overline{\tau}\underline{\tau}, \tag{25}$$

so we have proved that Fourier series of CS $m = 1$ Nekrasov functions give a solution of this equation.

q-Painleve VI and q-Painlevé III_3 $c=-2$ tau functions

- q-Painlevé VI equation: system of eight first order q-difference bilinear equations on 8 tau functions $\tau_1 \dots \tau_8$.
- In the case $q^{\theta_0} = q^{\theta_z} = q^{\theta_1} = q^{\theta_\infty} = i$ it is equivalent to system of four second order q-difference bilinear equations ($j \in \mathbb{Z}/2\mathbb{Z}$)

$$\overline{\tau_j^+} = \frac{\tau_j^+ \tau_j^- - z^{1/4} \tau_{j+1}^+ \tau_{j-1}^-}{\underline{\tau_j^-}}, \quad \overline{\tau_j^-} = \frac{\tau_j^+ \tau_j^- + z^{1/4} \tau_{j+1}^+ \tau_{j-1}^-}{\underline{\tau_j^+}}. \qquad (26)$$

  which coincide with NY equations on $c=-2$ tau functions.
- According to the solution of the q-Painlevé VI (Jimbo-Nagoya-Sakai, 2017)

$$(-qz^{1/2}; q, q)_\infty^2 \mathcal{Z}_{inst}(i, i, i, iq^{\pm 1/2}, u|z^{1/2}) = \mathcal{Z}_{inst}(u; q^{-1}, q^2|z) \qquad (27)$$

  and we checked this statement analytically up to z^5.

q-Painleve VI and q-Painlevé III_3: cluster dynamics

$c=1$ tau functions of q-Painleve VI and q-Painlevé III_3 is A-cluster coordinates.

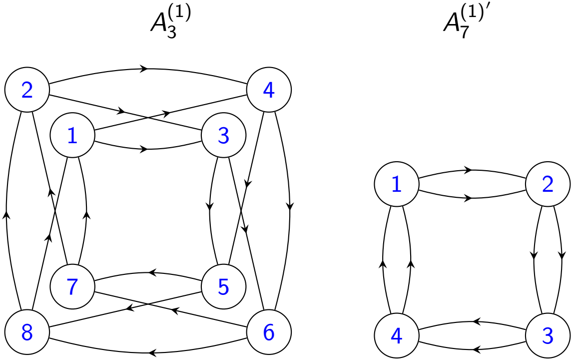

Figure: Quiver of q-Painlevé VI and q-Painlevé III'$_3$

Elementary q-shift $z \mapsto qz$ of q-Painlevé III'$_3$ equation corresponds to the the q-shift $t \mapsto q^{1/2}t$ for q-Painlevé VI equation — half of the elementary shift. Such generator in Weyl group $W(D_5^{(1)})$ exists and for $\theta_0=\theta_t$, $\theta_1=\theta_\infty$ it is a q-shift.

$c = -2$ tau function: ABJ

- q-Painlevé III_3 tau function for $|q| = 1$, $s = 1$ [Bonelli-Grassi-Tanzini, 2017] :

$$\tau(\kappa, \hbar, \xi) = Z_{CS}(\hbar, \xi) \det(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}), \tag{28}$$

where ρ is inverse operator to the Hamiltonian of the relativistic Toda chain

$$\rho_{\mathbb{P}^1 \times \mathbb{P}^1} = (e^p + e^{-p} + e^x + m_{\mathbb{P}^1 \times \mathbb{P}^1} e^{-x})^{-1}. \tag{29}$$

- Spectral determinant for $z = q^M, M \in \mathbb{Z}$ is ABJ grand canonical partition function. It factorizes according to the parity of eigenvalues of ρ: $\tau = \tau^+ \tau^-$.
- "Quantum Wronskian" relations in ABJ th. [Grassi-Hatsuda-Marino, 2014]

$$i z^{1/4} \overline{\tau_1^+} \underline{\tau_1^-} - \overline{\tau^+} \underline{\tau^-} = -(1 + z^{1/2}) \tau^+ \tau^-, \tag{30}$$

$$i z^{1/4} \underline{\tau_1^+} \overline{\tau_1^-} + \underline{\tau^+} \overline{\tau^-} = (1 + z^{1/2}) \tau^+ \tau^- \tag{31}$$

which is equivalent to the q-difference NY relations.

$c = -2$ τ function: Painlevé VI generalization

- [Iorgov, Lisovyy, Teschner, 2014] give naturally relation of $\tau_{c=1}$ to the Riemann-Hilbert problem on $\mathbb{CP}^1$ with 4 punctures. Their arguments work for the case $b^2 \in \mathbb{Z} \setminus 0$, in particular for $c = -2$. Precise formulation of RH for this case?
- Central charge of symplectic fermions is $c = -2$. We expect that $\tau_{c=-2}$ is a conformal block for the symplectic fermions.
- For special resonance values of parameters $\tau_{c=1}$ is expressed by the determinant of hypergeometric ${}_2F_1$ functions [Morozov, Mironov, 2017]. Similarly $\tau_{c=-2}$ is equal to the Pfaffian. Nakajima-Yoshioka relations in this terms?

Thank you for the attention!

Conformal blocks

- We have graded Lie algebra (for example, Virasoro algebra) $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$.
- We take highest weight vector $|v_\lambda\rangle$, s.t. $U(\mathfrak{n}^+)$ act on it by zero and $U(\mathfrak{h})$ act as on eigenvector.
- Verma module $U(\mathfrak{n})^-|v_\lambda\rangle$
- Whittaker vector in the Verma module

$$|W(z)\rangle = \sum_{N=0}^{+\infty} z^N |N\rangle, \quad deg(|N\rangle) = N \tag{32}$$

s.t.

$$g|W(z)\rangle = \beta_g z^{deg(g)} |W(z)\rangle \tag{33}$$

- Conformal block

$$\mathcal{Z}(z) = \langle W(1)|W(z)\rangle \tag{34}$$

Representation-theoretic interpretation

Nakajima-Yoshioka blow-up relations on 4D Nekrasov functions have representation-theoretic interpretation (Bershtein, Feigin, Litvinov, 2013).

- 4D Nekrasov function with $\epsilon_1 + \epsilon_2 = 0$ correspond via AGT correspondence to the $c = 1$ 4-point Virasoro conformal block.
- Introduce vertex operator algebra *Urod* as a sum of Heisenberg algebra Fock modules $\bigoplus_{k\in\mathbb{Z}} F_{k\sqrt{2}}$ but with modified stress-energy tensor.
- There are $\mathrm{Vir} \oplus \mathrm{Vir}$ subalgebra with $b_1^2 + b_2^{-2} = -1$ in the $U(Urod \otimes \mathrm{Vir})$, moreover there is a decomposition of the Verma module

$$U_1 \otimes \mathbb{L}_{P,b} = \bigoplus_{k\in\mathbb{Z}} \mathbb{L}_{P_1+kb_1,b_1} \otimes \mathbb{L}_{P_2+kb_2^{-1},b_2} \tag{35}$$

- Take Whittaker vector $v_{1/\sqrt{2}} \otimes |W(z)\rangle$ in l.h.s. Then it decompose into sum of $\mathrm{Vir} \oplus \mathrm{Vir}$ Whittaker vectors in r.h.s. Squaring this relation we obtain Nakajima-Yoshioka blow-up relation.

Algebraic solutions and log limit of $c = -2$ τ function

As it is for $c = 1$ τ function there exist limit $\sigma \to 0$ and "algebraic" solution for the $\sigma = 1/4, s = \pm 1$

- It is known that $\tau(1/4, \pm 1|z) = z^{1/16} e^{\mp\sqrt{z}}$. One can find that

$$\tau^+ = e^{2iz^{1/4}} z^{1/32} e^{2\sqrt{z}}, \quad \tau^- = e^{2iz^{1/4}} z^{1/32} e^{2\sqrt{z}} \tag{36}$$

- Logarithmic limit for $c = 1$ is $s = e^{2\Omega\sigma}, \sigma \to 0$. It also could be applied for the $c = -2$ τ function