Macdonald polynomials of type C_n with one-column diagrams and deformed Catalan numbers

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Plan to talk

1. Explicit formulas for Koornwider polynomials with one-column diagrams
2. Explicit formulas for Macdonald polynomials of type C_n with one-column diagrams
3. Matrix inversion for $P_{(1^r)}(x|a,-a,c,-c|q,t)$
4. Transition matrix between $P_{(1^r)}(x|a,b,c,d|q,t)$ and $m_{(1^r)}(x)$
5. Combinatorial expression for Macdonald polynomials of type C_n with one-column diagrams
6. Matrix inversion for Koornwinder polynomials with one-column diagrams

A. Hoshino and J. Shiraishi,
Macdonald polynomials of type C_n with one-column diagrams and deformed Catalan numbers. SIGMA **14**, 101, 33 pages, (2018).

1. Explicit formulas for Koornwider polynomials with one-column diagrams

- $x = (x_1^{\pm 1}, \cdots, x_n^{\pm 1})$: variables
- a, b, c, d, q, t : parameter
- $\lambda = (\lambda_1, \lambda_2, \cdots)$: partition

<u>Definition and Theorem</u> [Koornwinder] The Koornwinder

polynomial $P_\lambda(x) = P_\lambda(x|a,b,c,d|q,t)$ is uniquely characterized by the conditions

$$(a)\ \ P_\lambda(x) = \sum_{\mu \preceq \lambda} c_{\lambda,\mu} m_\mu(x), \quad (b)\ \ \mathcal{D}_x P_\lambda(x) = d_\lambda P_\lambda(x).$$

- dominance order of partitions

$\lambda = (\lambda_1, \lambda_2, \cdots)$, $\mu = (\mu_1, \mu_2, \cdots)$: partitions

$\lambda \succeq \mu \Leftrightarrow \lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$ for $k = 1, 2, \cdots, n$.

- $$\mathcal{D}_x = \sum_{i=1}^{n} \mathcal{A}_i^+(x)\big(T_{q,x_i} - 1\big) + \sum_{i=1}^{n} \mathcal{A}_i^-(x)\big(T_{q^{-1},x_i} - 1\big)$$

where,

- $T_{q,x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n)$,

- $\mathcal{A}_i^+(x)$
$$= \frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(abcdq^{-1})^{\frac{1}{2}} t^{n-1}(1-x_i^2)(1-qx_i^2)} \prod_{\substack{1\le j\le n,\\ j\neq i}} \frac{1-tx_ix_j}{1-x_ix_j}\frac{1-tx_i/x_j}{1-x_i/x_j},$$

- $\mathcal{A}_i^-(x) = \mathcal{A}_i^+(x^{-1}) \quad (i = 1, \ldots, n).$

<u>Ex.</u> $(n=2)$
$$\begin{aligned} m_{(2)}(x) &= x_1^2 + x_2^2 + 1/x_1^2 + 1/x_2^2, \\ m_{(3)}(x) &= x_1^3 + x_2^3 + 1/x_1^3 + 1/x_2^3, \\ m_{(1,1)}(x) &= x_1x_2 + x_1/x_2 + x_2/x_1 + 1/x_1x_2. \end{aligned}$$

<u>Definition</u> Define the symmetric Laurent polynomial $E_r(x)$'s as follows.

$$\prod_{i=1}^{n}(1-yx_i)(1-y/x_i)=\sum_{r\geq 0}(-1)^r E_r(x)y^r.$$

<u>Remark</u>

$$E_r(x)=\sum_{j=0}^{\lfloor \frac{r}{2}\rfloor}\binom{n-r+2j}{j}m_{(1^{r-2j})}(x),$$

where $\binom{m}{j}$ denotes the binomial coefficient.

- Hereafter, we consider the case $\lambda=(1^r)$.
  (Ex. $(1^3)=(1,1,1)=$ )

Theorem

$$P_{(1^r)}(x|a,b,c,d|q,t) = \sum_{k,l,i,j\geq 0} (-1)^{i+j} c_o(i,j;t^{n-r+1}) c_e(k,l;t^{n-r+1+i+j}) E_{r-i-j-2k-2l}(x),$$

where

$$c_o(i,j;s) = \frac{(-a/b;t)_i (scd/t;t)_i}{(t;t)_i(-sac/t;t)_i} \frac{(s;t)_{i+j}(-sac/t;t)_{i+j}(s^2a^2c^2/t^3;t)_{i+j}}{(s^2abcd/t^2;t)_{i+j}(sac/t^{3/2};t)_{i+j}(-sac/t^{3/2};t)_{i+j}} \times \frac{(-c/d;t)_j(sab/t;t)_j}{(t;t)_j(-sac/t;t)_j} b^i d^j,$$

$$c_e(k,l;s) = \frac{(1/c^2;t)_l(s/t;t)_{2k+l}}{(t;t)_l(sc^2;t)_{2k+l}} \frac{1-st^{2k+2l-1}}{1-st^{-1}} c^{2l} \times \frac{(tc^2/a^2;t^2)_k(sc^2t;t^2)_k(s^2c^4/t^2;t^2)_k}{(t^2;t^2)_k(sc^2/t;t^2)_k(s^2a^2c^2/t;t^2)_k} a^{2k}.$$

2. Explicit formulas for Macdonald polynomials of type C_n with one-column diagrams

Corollary Set the parameters as $(a, b, c, d) \to (a, -a, c, -c)$.

$$P_{(1^r)}(x|a, -a, c, -c|q, t)$$
$$= \sum_{\substack{k,l \geq 0 \\ 2k+2l \leq r}} E_{r-2k-2l}(x) \frac{(1/c^2; t)_l (s/t; t)_{2k+l}}{(t; t)_l (sc^2; t)_{2k+l}} \frac{1 - st^{2k+2l-1}}{1 - st^{-1}} c^{2l}$$
$$\times \frac{(tc^2/a^2; t^2)_k (sc^2 t; t^2)_k (s^2 c^4/t^2; t^2)_k}{(t^2; t^2)_k (sc^2/t; t^2)_k (s^2 a^2 c^2/t; t^2)_k} a^{2k},$$

where $s = t^{n-r+1}$.

Remark

$$P^{(C_n)}_{(1^r)}(x|b; q, t) = P_{(1^r)}(x|b^{1/2}, -b^{1/2}, q^{1/2} b^{1/2}, -q^{1/2} b^{1/2}|q, t).$$

Definition

$$\mathrm{M}(s,l) =(-1)^l s^{-l}\frac{(s^2/t^2;t^2)_l}{(t^2;t^2)_l}\frac{1-s^2t^{4l-2}}{1-s^2t^{-2}} \\ \times {}_4\phi_3\left[\begin{matrix}-sa^2,-sc^2,s^2t^{2l-2},t^{-2l}\\ -s,-st,s^2a^2c^2/t\end{matrix};t^2,t^2\right].$$

Theorem

$$P_{(1^r)}(x|a,-a,c,-c|q,t)=\sum_{l=0}^{\lfloor \frac{r}{2}\rfloor}\mathrm{M}(t^{n-r+1},l)E_{r-2l}(x).$$

3. Matrix inversion for $P_{(1^r)}(x|a,-a,c,-c|q,t)$

- Matrtix inversion

Thoerem ([B], p.1, Theorem, [L], p.5, Corollary)

$$\mathcal{M}_{r,r-2i}(u,v;x,y;q) = y^i v^i \frac{(x/y;q)_i}{(q;q)_i} \frac{(uq^{r-2i};q)_{2i}}{(uxq^{r-i};q)_i (uyq^{r-2i+1};q)_i}$$
$$\left(r, i \in \mathbb{Z}_{\geq 0},\ i \leq \left[\frac{r}{2}\right] \right).$$

We have $\mathcal{M}(u,v;x,y;q)\mathcal{M}(u,v;y,z;q) = \mathcal{M}(u,v;x,z;q)$. In particular, $\mathcal{M}(u,v;x,y;q)$ and $\mathcal{M}(u,v;y,x;q)$ are mutually inverse.

Definition

$$d(u,v)_r := \frac{(t^2 v^{1/2};t)_r}{(u^{1/2};t)_r}(u^{1/4}/v^{3/4})^r,$$

$$\widetilde{\mathcal{M}}_{r,r-2i}(u,v;x,y;t)$$

$$:= \mathcal{M}_{r,r-2i}(u,v;x,y;t^2)\times d(u,v)_r/d(u,v)_{r-2i}$$

$$= \frac{(x/y;t^2)_i}{(t^2;t^2)_i}\frac{(v^{1/2}t^{r-2i+2};t)_{2i}}{(u^{1/2}t^{r-2i};t)_{2i}}\frac{(ut^{2r-4i};t^2)_{2i}}{(uxt^{2r-2i};t^2)_i(uyt^{2r-4i+2};t^2)_i}\left(\frac{yu^{1/2}}{v^{1/2}}\right)^i.$$

Remark $\widetilde{\mathcal{M}}(u,v;x,y;t)$ and $\widetilde{\mathcal{M}}(u,v;y,x;t)$ are mutually inverse.

Proposition For $s=t^{n-r+1}$

$$\mathrm{M}(s,l)=\sum_{j=0}^{l}\widetilde{\mathcal{M}}_{r,r-2j}(t^{-2n+2}/c^4,t^{-2n-4},c^2/ta^2,1/t^2;t)$$
$$\times\,\mathcal{M}_{r-2j,r-2l}(t^{-n},t,1/c^2,1;t).$$

Inverse of $\mathrm{M}(s,l)$

Theorem For $s = t^{n-r+1}$

$$\begin{aligned}\widetilde{\mathrm{M}}(s,l) &:= \sum_{j=0}^{l} \mathcal{M}_{r,r-2j}(t^{-n}, t, 1, 1/c^2; t) \\ &\qquad \times \widetilde{\mathcal{M}}_{r-2j,r-2l}(t^{-2n+2}/c^4, t^{-2n-4}, 1/t^2, c^2/ta^2; t) \\ &= (st^{l-1})^{-l} \frac{(t^{2l}s^2; t^2)_l}{(t^2; t^2)_l} \\ &\qquad \times {}_4\phi_3 \left[\begin{matrix} -t^{-2l+2}/sa^2, -t^{-2l+2}/sc^2, t^{-2l+2}/s^2, t^{-2l} \\ -t^{-2l+1}/s, -t^{-2l+2}/s, t^{-4l+5}/s^2a^2c^2 \end{matrix} ; t^2, t^2 \right],\end{aligned}$$

$$E_r(x) = \sum_{l=0}^{\lfloor \frac{r}{2} \rfloor} \widetilde{\mathrm{M}}(t^{n-r+1}, l) P_{(1^{r-2l})}(x|a, -a, c, -c|q, t).$$

Kostka polynomials, deformed Catalan number

- $P^{(C_n)}_{(1^r)}(x|t;q,t)\,(= P_{(1^r)}(x|t^{1/2},-t^{1/2},t^{1/2}q^{1/2},-t^{1/2}q^{1/2}|q,t))$
$$=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{(1/qt;t^2)_j(t^{2n-2r},t^2)_j}{(t^2;t^2)_j(qt^{2n-2r+3};t^2)_j}\frac{1-t^{2n-2r+4j}}{1-t^{2n-2r}}(qt)^j E_{r-2j}(x),$$
- $E_r(x)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\frac{(qt;t^2)_j(t^{2n-2r+2j+2},t^2)_j}{(t^2;t^2)_j(qt^{2n-2r+2j+1};t^2)_j}P^{(C_n)}_{(1^{r-2j})}(x|t;q,t),$
- $P^{(C_n)}_{(1^r)}(x|q;q,q)=s^{(C_n)}_{(1^r)}(x)=E_r(x)-E_{r-2}(x),$
- $E_r(x)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}s^{(C_n)}_{(1^{r-2j})}(x).$

Recall
$$E_r(x)=\sum_{j=0}^{\lfloor\frac{r}{2}\rfloor}\binom{n-r+2j}{j}m_{(1^{r-2j})}(x).$$

Corollary

$$
\begin{aligned}
s^{(C_n)}_{(1^r)}(x) =& P^{(C_n)}_{(1^r)}(x|q;q,q) \\
=& \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \left(\binom{n-r+2k}{k} - \binom{n-r+2k}{k-1} \right) m_{(1^{r-2k})}(x) \\
=& \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{n-r+1}{n-r+k+1} \binom{n-r+2k}{k} m_{(1^{r-2k})}(x).
\end{aligned}
$$

Ex. The transition matrix is described as

$$
\begin{pmatrix} s^{(C_n)}_{(1^n)} \\ s^{(C_n)}_{(1^{n-1})} \\ s^{(C_n)}_{(1^{n-2})} \\ s^{(C_n)}_{(1^{n-3})} \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
1 & & 1 & & 2 & & 5 & & \cdots \\
& 1 & & 2 & & 5 & & 14 & \\
& & 1 & & 3 & & 9 & & \cdots \\
& & & 1 & & 4 & & 14 & \\
& & & & \ddots & & \ddots & &
\end{pmatrix}
\begin{pmatrix} m_{(1^n)} \\ m_{(1^{n-1})} \\ m_{(1^{n-2})} \\ m_{(1^{n-3})} \\ \vdots \end{pmatrix}.
$$

Definition (Kostka polynomials) Define $K^{(C_n)}_{(1^r)(1^{r-2j})}(t)$ as

$$s^{(C_n)}_{(1^r)}(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} K^{(C_n)}_{(1^r)(1^{r-2j})}(t) P^{(C_n)}_{(1^{r-2j})}(x|t;0,t).$$

Theorem $K^{(C_n)}_{(1^r)(1^{r-2j})}(t)$ are polynomials in t with nonnegative integral coefficients.

$$\begin{aligned} K^{(C_n)}_{(1^r)(1^{r-2j})}(t) &= t^{2j} \frac{[n-r+1]_{t^2}}{[n-r+j+1]_{t^2}} \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} \\ &= \begin{bmatrix} n-r+2j \\ j \end{bmatrix}_{t^2} - \begin{bmatrix} n-r+2j \\ j-1 \end{bmatrix}_{t^2}, \end{aligned}$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q,$$

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \prod_{k=1}^{j} \frac{[m-k+1]_q}{[k]_q} = \frac{[m]_q!}{[j]_q![m-j]_q!}.$$

Define $\mathcal{K}^{(C_n)}_{i,j}(t) = K^{(C_n)}_{(1^{n-i})(1^{n-2j})}$. First few entries of $\mathcal{K}^{(C_n)}(t)$ read

$$
\begin{pmatrix}
1 & t^2 & t^4 + t^8 & t^6 + t^{10} + t^{12} + t^{14} + t^{18} & \cdots \\
 & 1 & t^2 + t^4 & t^4 + t^6 + t^8 + t^{10} + t^{12} & \cdots \\
 & & 1 & t^2 + t^4 + t^6 & t^4 + t^6 + 2t^8 + t^{10} + 2t^{12} + t^{14} + t^{16} \\
 & & & 1 & t^2 + t^4 + t^6 + t^8 & \cdots \\
 & & & & \ddots & \ddots
\end{pmatrix}
$$

4.Transition matrix between $P_{(1^r)}(x|a,-a,c,-c|q,t)$ and $m_{(1^r)}(x)$

Definition For $s = t^{m+1}$, define $C(s,j)$ as follows:

$$C(s,j) := \sum_{l=0}^{j} \mathrm{M}(s,l)\binom{m+2j}{j-l}.$$

Remark

$$P_{(1^r)}(x|a,-a,c,-c|q,t) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} C(t^{n-r+1},j)\, m_{(1^{r-2j})}(x).$$

Definition Define the upper triangular matrix $\mathcal{C} = (\mathcal{C}_{ij})_{i,j\in\mathbb{Z}_{\geq 0}}$ as follows:

$$\mathcal{C}_{r,r+2i} = C(t^{r+1}, i) \qquad (r, i \geq 0).$$

Theorem The transition matrix $\mathcal{C}$ satisfies following recursion ((a, q, t)-deformed Catalan triangle):

$$\begin{aligned}
&\mathcal{C}_{0,0} = 1, \\
&\mathcal{C}_{i-1,i-1} = \mathcal{C}_{i,i} && (i = 1, 2, 3, \ldots), \\
&F(1, -1)\mathcal{C}_{1,j-1} = \mathcal{C}_{0,j} && (j = 2, 4, 6, \ldots), \\
&\mathcal{C}_{i-1,j-1} + F(t^i, -1)\mathcal{C}_{i+1,j-1} = \mathcal{C}_{i,j} && (i + j : \text{even},\ 0 < i < j),
\end{aligned}$$

where

$$F(s, l) = \frac{(1 - t^l/s)(1 - t^{l+2}/sa^2c^2)(1 + t^{l+1}/sa^2)(1 + t^{l+1}/sc^2)}{(1 - t^{2l+1}/s^2a^2c^2)(1 - t^{2l+3}/s^2a^2c^2)}.$$

Remark $\mathcal{C}_{ij}$'s do not depend on rank n (stability condition).

Remark If $b = t = q$, we have $F(t^m, d) = 1$.

• Proof of Theorem

Theorem (Contiguity relation) For generic parameter s, we have

$$\mathrm{M}(s,l)+F(s,-1)\mathrm{M}(st^2,l-1)=\mathrm{M}(st,l)+\mathrm{M}(st,l-1).$$

Define $P^{(C_n)}_{(1^r)}:=P_{(1^r)}(x|a,-a,c,-c|q,t)$, we have the following Theorem:

Theorem   For $n\in\mathbb{Z}_{>0}$, define $\mathbf{P}^{(n)}$, $\mathbf{m}^{(n)}$ as follows:

$$\mathbf{P}^{(n)}={}^t(P^{(C_n)}_{(1^n)},P^{(C_n)}_{(1^{n-1})},\ldots,P^{(C_n)}_{(1)},P^{(C_n)}_{\emptyset},0,0,0,\ldots),$$
$$\mathbf{m}^{(n)}={}^t(m_{(1^n)},m_{(1^{n-1})},\ldots,m_{(1)},m_{\emptyset},0,0,0,\ldots),$$
$$\mathbf{P}^{(n)}=\mathcal{C}\mathbf{m}^{(n)}.$$

Then the transition matrix $\mathcal{C}=(\mathcal{C})_{i,j\in\mathbb{Z}_{\geq 0}}$ satisfies above recursion and stability condition.

5. Combinatorial expression for Macdonald polynomials of type C_n with one-column diagrams

Theorem

$$\mathcal{C}_{r,r+2i} = \sum_{(d_1,\dots,d_i)\in\mathcal{P}(r,i)} F(t^{r+1}, d_1)F(t^{r+1}, d_2)\cdots F(t^{r+1}, d_i),$$

where

$$\mathcal{P}(r,i) \\ = \left\{(d_1, d_2, \dots, d_i) \in \mathbb{Z}^i | 0 \leq d_1 \leq r, d_k - 1 \leq d_{k+1} \leq r \text{ for } 1 \leq k < i\right\}.$$

Remark

$$P_{(1^r)}(x|a,-a,c,-c|q,t) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \mathcal{C}_{n-r,n-r+2j}\, m_{(1^{r-2j})}(x).$$

6. Matrix inversion for Koornwinder polynomials with one-column diagrams

- Modifying the definition of c_o for $P_{(1^r)}(x|a,b,c,d|q,t)$, the summation of i, j (c_o part) is written by $\sum_{i=0}^{j} {}_4\phi_3$.
- ${}_4\phi_3$ part ($:= \mathrm{N}(s,j)$) satisfies certain contiguity relation.
- $\mathrm{N}(s,j)$ is described by matrix inversion.
- There is a following degenerating sequence. The transition matrices on every degenerating steps are described matrix inversions and satisfies the stability condition (not depend on n).

$$
\begin{aligned}
&P_{(1^r)}(x|a,b,c,d|q,t) \to P_{(1^r)}(x|a,-a,c,d|q,t) \\
&\quad \to P_{(1^r)}(x|a,-a,c,-c|q,t) \to P_{(1^r)}(x|t^{1/2}c,-t^{1/2}c,c,-c|q,t) \\
&\quad \to P_{(1^r)}(x|t^{1/2},-t^{1/2},1,-1|q,t) = E_r(x).
\end{aligned}
$$

- The entries in the transition matrix between $P_{(1^r)}(x|a,b,c,d|q,t)$ and $m_{(1^r)}(x)$ satisfy similar recursion of type C_n.

D. M. Bressoud, A matrix inverse, Proc. Amer. Math. Soc. **88** (1983), 446–448.

M. Lassalle, Some conjectures for Macdonald polynomials of type B, C, D, Sem. Lothar. Combin. 52, (2004), Art. B52h, 24 pp. (electronic).