A panoramic view of discrete Painlevé equations: from $\mathbf{E}_8$ to $\mathbf{A}_1$ and back

B. Grammaticos and A. Ramani

with

R. Willox, J. Satsuma, K.M. & T. Tamizhmani

and many others

What is the oldest known discrete Painlevé equation?

Laguerre (1885): integrable non-autonomous recursion relations

Shohat (1939) derived the recursion relation

$$x_{n+1} + x_n + x_{n-1} = 1 + \frac{\alpha n + \beta + \gamma(-1)^n}{x_n}$$

(No relation to Painlevé equations was established at the time)

Jimbo and Miwa (1981) obtained the contingiuty relation for $\mathrm{P_{II}}$

$$\frac{\alpha(n + {}^1\!/_2) + \beta}{x_{n+1} + x_n} + \frac{\alpha(n - {}^1\!/_2) + \beta}{x_{n-1} + x_n} = -2x_n^2 + 1$$

Again no continuum limit was given

Things changed with Brezin & Kazakov (1991)

They obtained the same equation as Shohat (with $\gamma = 0$)
and found that the continuum limit was $w'' = 6w^2 + t$ (Painlevé I)

Nijhoff and Papageorgiou: non-autonomous form of McMillan
(also Periwal & Shevitz)

$$x_{n+1} + x_{n-1} = \frac{(\alpha n + \beta)x_n + \gamma}{1 - x_n^2}$$

Then

together with J. Hietarinta we proposed a systematic method
for the derivation of discrete Painlevé equations

How does one derive such equations?

Our preferred method: deautonomisation

In a nutshell:

Start from a QRT mapping (why?)

Assume that the parameters are functions of n

Fix their form through integrability criterion:

singularity confinement

Discovery of q (multiplicative) discrete Painlevé equations

q-P$_{\mathrm{III}}$ $(q_n = q_0\lambda^n)$

$$x_{n+1}x_{n-1} = \frac{(x_n - aq_n)(x_n - q_n/a)}{(1 - bx_n)(1 - x_n/b)}$$

d-P$_{\mathrm{IV}}$ $(z_n = \alpha n + \beta)$

$$(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n + z_n)^2 - c^2}$$

q-P$_{\mathrm{V}}$

$$(x_{n+1}x_n - 1)(x_nx_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - cx_nq_n)(1 - x_nq_n/c)}$$

q-P$_{\mathrm{VI}}$

$$\frac{(x_nx_{n+1} - q_nq_{n+1})(x_nx_{n-1} - q_nq_{n-1})}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)}$$
$$= \frac{(x_n - aq_n)(x_n - q_n/a)(x_n - bq_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}$$

How to put some order?

Our (bottom-up) approach based on the observation that

Most dPs possess the property of self-duality

Same equation for evolution in n and in parameters

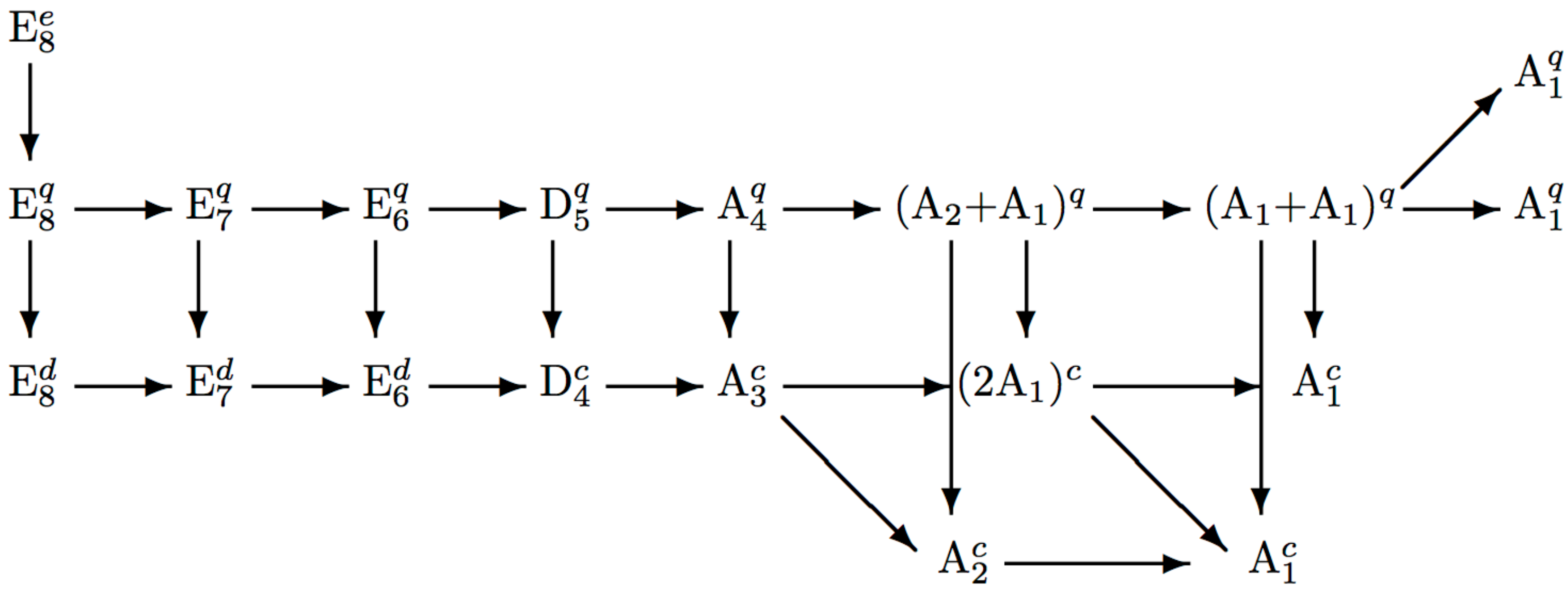
Description by affine Weyl groups

Main result: definition of dP

Mapping obtained by periodic repetition of non-closed pattern on a lattice associated to an affine Weyl group

Consequence: the number of dPS is infinite

Sakai's approach (top-down) from classification of surfaces



Upper index e: a discrete equation involving elliptic functions

Upper index q: equation of q-type

Upper index δ: difference equation not related continuous Painlevé

Upper index c: difference equation, contiguity of continuous Painlevé

One important result of Sakai: existence of elliptic dPs

AR, BG and Ohta: description of 8-parameter dP

Elementary Miura

$$\frac{X-\mathrm{sn}^2(\alpha-\beta+2s)}{X-\mathrm{sn}^2(\alpha-\beta-2s)}\frac{Y-\mathrm{sn}^2(\beta-\gamma+2s)}{Y-\mathrm{sn}^2(\beta-\gamma-2s)}\frac{W-\mathrm{sn}^2(\gamma-\alpha+2s)}{W-\mathrm{sn}^2(\gamma-\alpha-2s)}=$$
$$\left(\frac{1-m^2\mathrm{sn}^2(\alpha-s)\mathrm{sn}^2(\beta+s)}{1-m^2\mathrm{sn}^2(\alpha+s)\mathrm{sn}^2(\beta-s)}\times(\alpha\to\beta,\beta\to\gamma)\times(\beta\to\gamma,\gamma\to\alpha)\right)^2$$

Canonical form of generic $\mathrm{E}_8^{(1)}$ equations

Explicit construction of examples of elliptic dPs

based on the trihomographic form $(X\to x_{n+1},Y\to x_n,W\to x_{n-1})$

The trihomographic form

$$\frac{x_{n+1}-a_n}{x_{n+1}-b_n}\frac{x_{n-1}-c_n}{x_{n-1}-d_n}\frac{x_n-e_n}{x_n-f_n}=1$$

with

$$a_n=(z_n+k_n)^2,\ b_n=(z_n-k_n)^2,\ c_n=(z_{n+1}+k_n)^2,\ d_n=(z_{n+1}-k_n)^2,$$
$$e_n=(z_n+z_{n+1}-k_n)^2,\ \ f_n=(z_n+z_{n+1}+k_n)^2$$

is equivalent to

$$\frac{(x_n-x_{n+1}+z_{n+1}^2)(x_n-x_{n-1}+z_n^2)+4x_nz_nz_{n+1}}{z_n(x_n-x_{n+1}+z_{n+1}^2)+z_{n+1}(x_n-x_{n-1}+z_n^2)}=R(x_n)$$

with

$$R(x_n)=\frac{x_n-k_n^2}{z_n+z_{n+1}}+z_n+z_{n+1}$$

This is a formal equivalence

The linear RHS is generic

i.e. can be obtained from the most general (quartic/cubic) by a simplification process

From additive go to multiplicative case

$$\frac{x_{n+1}-\sinh^2 a_n}{x_{n+1}-\sinh^2 b_n}\,\frac{x_{n-1}-\sinh^2 c_n}{x_{n-1}-\sinh^2 d_n}\,\frac{x_n-\sinh^2 e_n}{x_n-\sinh^2 f_n}=1$$

and to elliptic

$$\frac{x_{n+1}-\mathrm{sn}^2 a_n}{x_{n+1}-\mathrm{sn}^2 b_n}\,\frac{x_{n-1}-\mathrm{sn}^2 c_n}{x_{n-1}-\mathrm{sn}^2 d_n}\,\frac{x_n-\mathrm{sn}^2 e_n}{x_n-\mathrm{sn}^2 f_n}=E_n$$

where E_n is given in terms of θ functions

An advantage of the trihomographic form:
Singularity confinement application is straightforward ($z_n = \alpha n+\beta$)

$$\frac{x_{n+1}-(4z_n-\alpha)^2}{x_{n+1}-\alpha^2}\,\frac{x_{n-1}-(4z_n+\alpha)^2}{x_{n-1}-\alpha^2}\,\frac{x_n-4z_n^2}{x_n-36z_n^2}=1$$

Singularity patterns: $\{4z_{n-1}^2, \alpha^2, 4z_{n+1}^2\}$ and
$\{36z_{n-3}^2, (4z_{n-3}-\alpha)^2, (2z_{n-3}-4\alpha)^2, 81\alpha^2, (2z_{n+3}+4\alpha)^2, (2z_{n+3}+\alpha)^2, 36z_{n+3}^2\}$
Find periodicity: introduce 6 functions $a_n, \ldots, f_n$

$$\frac{x_{n+1}-(4z_n-\alpha+a_n)^2}{x_{n+1}-(\alpha+b_n)^2}\,\frac{x_{n-1}-(4z_n+\alpha+c_n)^2}{x_{n-1}-(\alpha+d_n)^2}\,\frac{x_n-(2z_n+e_n)^2}{x_n-(6z_n+f_n)^2}=1$$

Require same singularity patterns as before

We find

$$a_n = 2g_{n-1}+g_{n+1}+g_{n+2},\ b_n = g_{n+2}-g_{n+1},\ c_n = g_{n-1}+g_n+2g_{n+2},$$
$$d_n = g_n - g_{n-1},\ e_n = g_n + g_{n+1},\ f_n = 2g_{n-1} + g_n + g_{n+1} + 2g_{n+2}$$

where g_n is a function of period 4 times 5

$$g(n) = \phi_4(n) + \phi_5(n)$$

Standard, E_8, form for $R(x_n)$

Convention: standard-form upper-case, trihomographic lower-case

$$Z_n+Z_{n-1} = 2z_n-\alpha+g_{n+1}+g_{n-1}, \quad Z_n+Z_{n+1} = 2z_n+\alpha+g_{n+2}+g_n$$

$$K_n = 2z_n + g_{n+2} + g_{n-1}$$

The overall period of the mapping is $20 = 4 \times 5$

Singularity pattern: two patterns both of lengths 3 and 7

Another mapping with period 3×4 is already known

Its singularity patterns have both length 5

So the natural question is:

Do mappings with singularity patterns of lengths 8,2 and 6,4 exist?

The answer is, yes!

Drawbacks of the trihomographic form:

To go from the linear RHS to the most general we must couple 2 to 4 trihomographic

Taking limits is not very convenient

A different approach is necessary

Introduction of an ancillary variable

In the additive case

$$x_n = \xi_n^2$$

Multiplicative, elliptic

$$x_n = \xi_n + 1/\xi_n, \qquad x_n = \theta_1^2(\xi_n)/\theta_0^2(\xi_n)$$

Start from additive form $z_n = \alpha n + \beta$

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = R(x_n)$$

with

$$R(x_n) = 2\frac{x_n^4 + S_2 x_n^3 + S_4 x_n^2 + S_6 x_n + S_8}{S_1 x_n^3 + S_3 x_n^2 + S_5 x_n + S_7}$$

where S_k are the elementary symmetric functions of $z_n + \kappa_n^i$

Introduce $\Pi(\xi_n) = \prod_{i=1}^{8}(z_n + \kappa_n^i + \xi_n)$ and find

$$R(x_n) = 2\xi_n \frac{\Pi(\xi_n) + \Pi(-\xi_n)}{\Pi(\xi_n) - \Pi(-\xi_n)}$$

Finally

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2} \frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^{8}(\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^{8}(\kappa_n^i + z_n + \xi_n)}$$

This is a form perfect for the application of singularity confinement

Results

Exhaustive study of E_8 associated dPs

Singularity analysis for difference Painlevé equations associated with the affine Weyl group E8
A. Ramani, B. Grammaticos, J. Phys. A 50 (2017) 055204

Generic case: enter singularity $\xi_n = \kappa_n^i + z_n$, exit at $\xi_{n+1} = \kappa_n^i - z_{n-1}$
Condition:

$$\kappa_{n+1}^i + \kappa_n^i = 0$$

Another singularity when ξ_n becomes infinite
Condition:

$$z_{n+1} - 2z_n + z_{n-1} = \frac{1}{2}\sum_{i=1}^{8} \kappa_n^i$$

33 different cases of additive dPs were identified

Extension to multiplicative is straightforward

Another most interesting result: generic elliptic dP

$$\frac{\theta_0^2(\xi_n - z_n - z_{n+1})x_{n+1} - \theta_1^2(\xi_n - z_n - z_{n+1})}{\theta_0^2(\xi_n + z_n + z_{n+1})x_{n+1} - \theta_1^2(\xi_n + z_n + z_{n+1})} \times \frac{\theta_0^2(\xi_n - z_n - z_{n-1})x_{n-1} - \theta_1^2(\xi_n - z_n - z_{n-1})}{\theta_0^2(\xi_n + z_n + z_{n-1})x_{n-1} - \theta_1^2(\xi_n + z_n + z_{n-1})} = \frac{\prod_{i=1}^{8} \theta_1(\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^{8} \theta_1(\kappa_n^i + z_n + \xi_n)}$$

Previous results by Murata, Sakai et al, Noumi-Yamada-Kajiwara

The deconstruction and restoration process

Start from additive Painlevé equation related to $E_8^{(1)}$

$$\frac{x_{n+1}-a_n}{x_{n+1}-b_n}\frac{x_{n-1}-c_n}{x_{n-1}-d_n}\frac{x_n-e_n}{x_n-f_n}=1$$

Coefficients have periods 2 and 7

Autonomous limit (and take $z=1$)

$$\frac{x_{n+1}-9}{x_{n+1}-1}\frac{x_{n-1}-9}{x_{n-1}-1}\frac{x_n-9}{x_n-25}=1$$

Define $X_n=3(x_n-9)/(2x_n-2)$ and obtain what we call the remnant equation

$$X_{n+1}X_{n-1}=A\frac{X_n-1}{X_n}$$

Conserved quantity

$$K = \frac{(X_n X_{n-1} + A)(X_n + X_{n-1} - 1)}{X_n X_{n-1}}$$

Deautonomisation

Singularity patterns

Confined pattern $\{1, 0, \infty, \infty, 0, 1\}$
Also length 7 pattern $\{f, 0, \infty, \infty, 0, f', \infty\}$ repeating cyclically

Confinement constraint

$$A_{n+2} A_{n-1} = A_{n+1} A_n$$

integrated to $\log A_n = \alpha n + \beta + \gamma(-1)^n$

Discrete Painlevé equation is associated to $(\mathrm{A}_1{+}\mathrm{A}_1)^{(1)}$

Restoration
The invariant made to coincide with a QRT canonical form
Homographic transformation

$$X_n = \frac{1}{z+1/z}\frac{1-zy_n}{z-y_n}$$

with $A = -1/(z+1/z)^2$ leads to

$$K = \frac{(y_n - z)(y_{n-1} - z)(1 - zy_n)(1 - zy_{n-1})}{(y_n y_{n-1} - z^2)(y_n y_{n-1} - 1)}$$

and the mapping

$$\left(\frac{y_n y_{n+1} - z^2}{y_n y_{n+1} - 1}\right)\left(\frac{y_n y_{n-1} - z^2}{y_n y_{n-1} - 1}\right) = \frac{y_n - z^3}{y_n - 1/z}$$

Deautonomisation already known: $E_7^{(1)}$-associated dP

$$\left(\frac{y_n y_{n+1} - z_n z_{n+1}}{y_n y_{n+1} - 1}\right)\left(\frac{y_n y_{n-1} - z_n z_{n-1}}{y_n y_{n-1} - 1}\right) = \frac{y_n - z_{n+3} z_n z_{n-3}}{y_n - z_{n+3} z_{n-3} / z_{n+1} z_n z_{n-1}}$$

with $z_n = \alpha n + \phi_7(n)$ and ϕ_7 periodic: $\phi_7(n+7) = \phi_7(n)$

The period 2 of the original mapping has been lost

Singularities: two confined patterns of lengths 6 and 4

The length-7 cyclic pattern has disappeared

but

a periodic term of period 7 has made its appearance

Algebrogeometric interpretation
and relation to work of Carstea, Dzhamay and Takenawa

Double step evolution

Better illustrated through an example

$$X_{n+1}X_{n-1} = A\frac{X_n - 1}{X_n}$$

with invariant

$$K = \frac{(X_nX_{n-1} + A)(X_n + X_{n-1} - 1)}{X_nX_{n-1}}$$

Double step evolution

$$K = \frac{(X_nX_{n-2} - X_n - A)(X_nX_{n-2} - X_{n-2} - A)}{X_nX_{n-2} - A}$$

Putting $A = a^2$ and rescaling X

$$(X_nX_{n+2} - 1)(X_nX_{n-2} - 1) = \frac{X_n}{a^2X_n - a}$$

Starting from this double-step mapping
and applying the restoration process we find the invariant

$$K = \frac{(y_{n-2} - z)(y_n - z)\big((z + 1/z)(y_{n-2}y_n + 1) - 2(y_{n-2} + y_n)\big)}{(z^2 y_n - y_{n-2})(z^2 y_{n-2} - y_n)}$$

The corresponding mapping is

$$\left(\frac{z^2 y_{n+2} - y_n}{y_{n+2} - z^2 y_n}\right)\left(\frac{z^2 y_{n-2} - y_n}{y_{n-2} - z^2 y_n}\right) = \frac{(y_n - z)^2 (y_n - z^3)}{z(z y_n - 1)^3}.$$

Its deautonomisation is known, leading to an $\mathrm{E}_7^{(1)}$-associated dP
The parameters of the equation can be expressed in terms of a variable of the form $\alpha n + \phi_7(n)$
We remark again the appearance of the period-7 term

Is that all? No! Single step

$$x_{n+1}x_{n-1} = z_n(1 - x_n)$$

where $\log z_n = \alpha n + \beta + \phi_2(n) + \phi_3(n)$ and double-step equations

$$(y_n y_{n+2} - 1)(y_n y_{n-2} - 1) = \frac{(1 - z_{n+1}y_n/c_n)(1 - z_{n-2}y_n/c_n)}{1 - y_n/c_n}$$

$\log z_n = 2\alpha n + \beta + \phi_2(n) + \phi_3(n)$ and $c(n+2)c(n) = z(n+1)z(n)$ can be related through **Miura transformations**

Auxiliary quantity

$$R_n = \left(\frac{1 - x_n}{x_{n-1}}\right)\left(\frac{1 - x_{n-1}}{x_n}\right)$$

Form is dictated by the singularity structure of the x equation

The Miura transformation is expressed as homography of R

The guide is the singularity structure of the two equations

We find

$$y_n = c_n(1 - R_n)$$

First half of the Miura

$$(y_n x_n - c_n)(y_n x_{n-1} - c_n) = c_n(c_n - y_n)$$

The second one can be easily obtained using the equations

$$z_n(y_n x_n - c_n)(y_{n+1} x_n - c_{n+1}) = (1 - x_n)c_n c_{n+1}$$

Note that the Miura does not allow the construction of a single-step equation for y_n but only a double-step one

Summing-up

Exploration of the $E_8^{(1)}$-associated dPs

Two approaches:
trihomographic representation and use of ancillary variable

Both are adapted to application of singularity confinement

Derived simplest elliptic dP

From E8 to A1 and back: deconstruction-restoration

Multi-step evolutions

Equations related through Miuras

What we did not present: from E8 to E7-E6 through limits