**Discrete Crum’s theorems and applications**

Cheng Zhang, ch.zhang.maths@gmail.com
Department of Mathematics, Shanghai University
Joint work with Linyu Peng (Waseda) & Da-juun Zhang (Shanghai)
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**Discrete Crum’s theorems transformation**

- Darboux’s proposition: covariance of the Sturm-Liouville equations
  \[ y'' + P y' + Q y = 0 \]
- Darboux provided 1-step Darboux transformation (DT) for the Schrödinger equation
  \[ L \psi = (D^2 + U) \psi = \lambda \psi, \quad \varphi \rightarrow \widetilde{\varphi} = (D - \sigma) \varphi, \quad U \rightarrow \tilde{U} = U - 2 \sigma \delta \]
- In terms of factorization of operators
  \[ L = A^I A + A, \quad \tilde{A} = A^I + A, \quad \tilde{\varphi} = A \varphi \]
- Crum derived compact forms of N-step iterated DTs for the Schrödinger equation
  \[ \varphi \rightarrow \psi[N] = \frac{W(\psi_1, \ldots, \psi_N, \psi)}{W(\psi_1, \ldots, \psi_N)}, \quad U \rightarrow \tilde{U}[N] = U - 2D \log W(\psi_1, \ldots, \psi_N) \]
  where \( \psi_j \) are particular solutions associated with \( \lambda_j \).

**Exact discretization and discrete Schrödinger equation**

- Applying iterated DTs implies discretization
  \[ (\varphi, \psi) \rightarrow (\varphi[I], \psi[I]) \rightarrow \cdots \rightarrow (\varphi[N], \psi[N]) \rightarrow \cdots \]
- Chain of compatibility (KdV family): the KdV hierarchy is generated by Lax pairs that can be seen as continuous symmetries; discrete KdV equations can be generated using DTs as discrete symmetries
  \[ \text{Lax pair in continuous symmetry} \quad \text{DTs as discrete symmetry} \quad \text{semi-discrete KdV} \]

**Discrete Crum’s theorem I and lattice KdV equation**

- Construct discrete DT for the discrete Schrödinger equation following the factorisation of the difference operator \( L \).
  \[ L = AB + b, \quad \tilde{L} = BA + b, \quad A = (T - g), \quad B = -(T + f) \]
  one obtains the action of 1-step DT
  \[ \varphi \rightarrow \psi = g \varphi, \quad h \rightarrow \tilde{h} = h + \tilde{g} = g \]
  where \( g \) is a particular solution and a system of difference equations involving \( f, g \) and \( h \)

- By eliminating \( h \), one obtains the discrete dressing chain equation [2] which is also the non-potential form of the lattice KdV equation
  \[ f + \tilde{f} = \tilde{g} + g = 0 \]
  The potential form can be obtained following the natural substitutions \( h - w = \tilde{w}, g = \tilde{w} - w, f = \tilde{w} - w \), then one obtains the non-autonomous lattice potential KdV (HI)

**Discrete Crum’s theorem II (sine-Gordon or mKdV family)**

Consider the spectral problem (related to pmKdV and sine-Gordon)

\[ K \phi = (-D^3 - 2bD) \phi = \lambda \phi \]

This equation is related to the Schrödinger equation via a gauge transformation \( \varphi \rightarrow \psi \phi \)

- Exact discretization of the spectral problem: based on the factorisation of operators
  \[ K = (-F \partial_x - 2) + Y \]
  one obtains the 1-step DT (\( 1 \) or \( 2 \) set)

**Concluding remarks**

- Solutions of the non-autonomous lattice equations can be obtained using the discrete Crum’s theorems. It also allows to search for discrete “quantum potentials”.
- Along the Darboux discretisation processes, two families of integrable equations (KdV and sG-mKdV family), including their continuous, semi-discrete and lattice versions, are explicitly demonstrated.
- One can on longer exactly discretize the difference equations along the DT discretization chain. This reveals the multi-dimensional consistency of underlying differential equations.
- Some results are available in certain forms in the KP theory and its reductions. However, the approach is directly based on Lax pairs of 2D KdV-type equations, and the \( \tau \)-function formalism is not a priori required.

**Reference**

[2] C.A. Evdokimov, P.H. van der Kamp, CZ, Dressing the dressing chain, SIGMA, 14, 059, 2018