Discrete Crum's theorems and applications

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ArXiv:1802.10044v2 [nlin.SI]

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Darboux-Crum transformation

- Darboux's proposition: covariance of the Sturm-Liouville equations

$$y'' + P\,y' + Q\,y = 0$$

- Darboux provided 1-step Darboux transformation (DT) for the Schrödinger equation

$$\mathcal{L}\varphi = (-D^2 + U)\,\varphi = \lambda\,\varphi\,, \quad \varphi \mapsto \widetilde{\varphi} = (D-\sigma)\varphi, \quad U \mapsto \widetilde{U} = U - 2\sigma_x$$

- In terms of factorization of operators

$$\mathcal{L} = A^{\dagger}A + a\,, \quad \widetilde{\mathcal{L}} = A\,A^{\dagger} + a\,, \quad \widetilde{\varphi} = A\,\varphi$$

- Crum derived compact forms of N-step iterated DTs for the Schrödinger equation

$$\varphi \mapsto \varphi[N] = \frac{W(\psi_1,\dots,\psi_N,\varphi)}{W(\psi_1,\dots,\psi_N)}\,, \quad U \mapsto U[N] = U - 2D^2\log W(\psi_1,\dots,\psi_N)$$

  where ψ_j are particular solutions associated with λ_j

Exact discretization and discrete Schrödinger equation

- Applying iterated DTs implies discretization

$$(\varphi, u) \mapsto (\varphi[1], u[1]) \mapsto \dots \mapsto (\varphi[N], u[N]) \mapsto \dots$$

- Chain of compatibility (KdV family): the KdV hierarchy is generated by Lax pairs that can be seen as continuous symmetries; discrete KdV equations can be generated using DTs as discrete symmetries

$$\varphi(x,t_1,\dots) \xleftarrow[\text{KdV hierarchy}]{\text{Lax pair as continuous symmetry}} \varphi(x) \xrightarrow[\text{semi-discrete KdV}]{\text{DTs as discrete symmetry}} \varphi(x,n) \to \varphi(x,n,m,\dots)$$

- Exact discretization [1]: Eliminating φ_x and φ_{xx} in the 1 and 2-step DTs:

$$\varphi = \widetilde{\varphi} = (D-\sigma)\varphi\,, \quad \widetilde{\widetilde{\varphi}} = (D-\widetilde{\sigma})\,(D-\sigma)\,\varphi$$

  one obtains a discrete non-autonomous the Schrödinger equation

$$\boxed{L\varphi = -\widetilde{\widetilde{\varphi}} - h\,\widetilde{\varphi} + a\,\varphi = \lambda\,\varphi\,, \quad L = -T^2 - hT + a}$$

  with the discrete potential $h = \widetilde{\sigma} + \sigma$.

Discrete Crum's theorem I and lattice KdV equation

- Construct *discrete DT* for the discrete Schrödinger equation following the factorisation of the difference operator L

$$L = A\,B + b, \quad \widehat{L} = B\,A + b, \quad A = (T-g)\,, \quad B = -(T+f)$$

  one obtains the action of 1-step DT

$$\varphi \mapsto \widehat{\varphi} = \widetilde{\varphi} - g\,\varphi\,, \quad h \mapsto \widehat{h} = \widetilde{h} + \widetilde{\widetilde{g}} - g$$

  where $g = \widetilde{\psi}/\psi$ (ψ is a particular solution) and a system of difference equations involving f, g and h

$$-f + \widetilde{g} + h = 0\,, \quad -\widetilde{f} + g + \widehat{h} = 0\,, \quad f\,g = a - b$$

- By eliminating h, one obtains the discrete dressing chain equation [2] which is also the non-potential form of the lattice KdV equation

$$-\widetilde{f} + \widehat{f} - \widehat{\widetilde{g}} + g = 0$$

  The potential form can be obtained following the natural substitutions $h = w - \widetilde{\widetilde{w}}$, $g = \widehat{w} - \widetilde{w}$, $f = \widehat{\widetilde{w}} - w$, then one obtains the non-automonous lattice potential KdV (H1)

$$(\widehat{w} - \widetilde{w})(\widehat{\widetilde{w}} - w) = a - b$$

  equipped with the Lax pair (the discrete Schrödinger equation and its DT)

$$-\widetilde{\widetilde{\varphi}} - (\widetilde{\widetilde{w}} - w)\,\widetilde{\varphi} + a\,\varphi = \lambda\,\varphi, \quad \widehat{\varphi} = \widetilde{\varphi} - (\widehat{w} - \widetilde{w})\,\varphi$$

- Discrete Crum's theorem as action of an N-step DT

$$\varphi \mapsto \varphi[N] = \frac{C(\psi_1,\psi_2,\dots,\psi_N,\varphi)}{C(\psi_1,\psi_2,\dots,\psi_N)}\,, \quad h \mapsto h[N] = h^{(N)} - s_1^{(2)} + s_1$$

  where $C(\varphi_1,\varphi_2,\dots,\varphi_l)$ is the Casorati determinant with ψ_j, $j = 1,\dots,N$, be particular solutions associated with λ_j, and

$$s_1 = -\frac{\begin{vmatrix} \psi_1 & \psi_1^{(1)} & \dots & \psi_1^{(N-2)} & \psi_1^{(N)} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \psi_N & \psi_N^{(1)} & \dots & \psi_N^{(N-2)} & \psi_N^{(N)} \end{vmatrix}}{C(\psi_1,\psi_2,\dots,\psi_N)}\,.$$

  In viewing $C(\varphi_1,\varphi_2,\dots,\varphi_l)$ as the τ function (in terms of discrete plane wave factors), one can express $s_1 = -(log\,\tau)_x$, where a continuous variable x can be "artificially" introduced [3].

Discrete Crum's theorem II (sine-Gordon or mKdV family)

Consider the spectral problem (related to pmKdV and sine-Gordon)

$$\mathcal{K}\,\phi = (-D^2 - 2vD)\phi = \lambda\,\phi$$

This equation is related to the Schrödinger equation via a gauge transformation $\varphi \mapsto e^{\int v}\phi$

- Exact discretization of the spectral problem: based on the factorisation of operators

$$\mathcal{K} = -(F\,\partial_x - 1)\,(F^{-1}\,\partial_x - c) + \eta\,, \quad \widetilde{\mathcal{K}} = -\,(F^{-1}\,\partial_x - c)(F\,\partial_x - 1) + \eta\,.$$

  one obtains the 1-step DT ($b = 1, \eta = a$)

$$\widetilde{\phi} = (F^{-1}\,\partial_x - c)\phi\,, \quad \widetilde{v} - v = (\log F)_x$$

  and a semi-discrete sine-Gordon equation (taking $F = -\frac{ie^{i(v-\widetilde{v})}}{\sqrt{c}}$)

$$(\widetilde{v} + v)_x = 2\sqrt{c}\,\sin(v - \widetilde{v})$$

  Combining 1 and 2-step DTs leads to a second-order difference equation ($h = -F\widetilde{F}$)

$$\boxed{K\,\phi = \left(h\,T^2 + (1+\widetilde{a}\,h)\,T + a\right)\,\phi = \lambda\,\phi\,, \quad K = hT^2 + (1+\tilde{a}h)T + a}$$

- Construct *DT* following the factorisation of K

$$K = (A\,T + f + c)(B\,T - g - b) + \sigma\,, \quad \widehat{K} = (B\,T - g - b)(A\,T + f + c) + \sigma$$

  this leads to a complicated system of difference equations involving A, B, f, g and h. One can simplify the system using the reduction $g = -a\,B$, $\sigma = b$, then the DT is

$$\phi \mapsto \widehat{\phi} = B\,\widetilde{\phi} + (a\,B - b)\,\phi, \quad h \mapsto \widehat{h} = B\,\widetilde{h}/\widetilde{\widetilde{B}}, \quad B = b\,\psi/(\widetilde{\psi} + a\,\psi)$$

  where ψ is a particular soltuion, and the remaining system is

$$g = -a\,B\,, \quad A = \frac{B-1}{b - a\,B}\,, \quad f + c = \frac{b-a}{b - a\,B}\,, \quad \widehat{h}\,\widetilde{\widetilde{B}} = \widetilde{h}\,B$$

- Reduction to lattice equations:

  1) the non-potential lattice potential mKdV equation (eliminate h and A)

$$(a - \widehat{b})(1 - \widetilde{B})\widehat{B}B = (\widetilde{a} - b)(1 - \widehat{B})\widetilde{B}\widehat{\widetilde{B}}$$

  2) let $h = \frac{1}{\alpha\widetilde{\alpha}}\frac{w}{\widetilde{\widetilde{w}}}$, $B = \frac{\beta}{\alpha}\frac{\widehat{w}}{\widetilde{w}}$ leads to the non-autonomous lattice potential mKdV equation

$$\alpha\left(w\,\widehat{w} - \widetilde{w}\,\widehat{\widetilde{w}}\right) - \beta\left(w\,\widetilde{w} - \widehat{w}\,\widehat{\widetilde{w}}\right) = 0$$

  3) let $h = \frac{\widetilde{P}}{a\,P}$, $B = \frac{P}{Q}$, then one has constraints on P and Q

$$\widetilde{a}\,\widehat{\widetilde{P}}\,\widetilde{P}\,Q = a\,\widehat{P}\,P\,\widetilde{\widetilde{Q}}\,, \quad Q - \widetilde{Q} = P - \widehat{P},$$

  further reductions lead to $a\,\widehat{P}\,P = b\,\widetilde{Q}\,Q$, $P = \widetilde{z} - z$, $Q = \widehat{z} - z$, which is the non-autonomous lattice Schwarzian KdV (cross-ratio) equation ($Q0^{\delta=0}$)

$$a(\widehat{\widetilde{z}} - \widehat{z})(\widetilde{z} - z) = b\,(\widehat{\widetilde{z}} - \widetilde{z})(\widehat{z} - z)$$

- Discrete Crum's theorem (N-step DT $(\phi, h) \mapsto (\phi[N], h[N])$):

$$\phi[N] = \Big(\prod_{j=1}^{N} b_j\Big)\,\frac{C(\psi_1,\dots,\psi_N,\phi)}{\det\mathcal{M}_N}, \; h[N] = h^{(N)}\,\frac{C(\psi_1,\dots,\psi_N)}{\det\mathcal{M}_N}\,T^2\Big(\frac{\det\mathcal{M}_N}{C(\psi_1,\dots,\psi_N)}\Big)$$

$$\mathcal{M}_N = \begin{pmatrix} 1 & -a & \dots & \prod_{j=1}^{N}(-a^{(j-1)}) \\ \psi_1 & \psi_1^{(1)} & \dots & \psi_1^{(N)} \\ \psi_2 & \psi_2^{(1)} & \dots & \psi_2^{(N)} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_N & \psi_N^{(1)} & \dots & \psi_N^{(N)} \end{pmatrix}$$

Concluding remarks

- Solutions of the non-autonomous lattice equations can be obtained using the discrete Crum's theorems. It also allow to search for discrete "quantum potentials".
- Along the Darboux discretisation processes, two families of integrable equations (KdV and sG-mKdV family), including their continuous, semi-discrete and lattice versions, are explicitly demonstrated.
- One can on longer *exactly discretize* the difference equations along the DT discretization chain. This reveals the multi-dimensional consistency of underlying lattice equations.
- Some results are available in certain forms in the KP theory and its reductions. However, the approach is directly based on Lax pairs of 2D KdV-type equations, and the τ-function formalism is not, *a priori*, required

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