Graphic Enumerations and Discrete Painleve Equations via Random Matrix Models

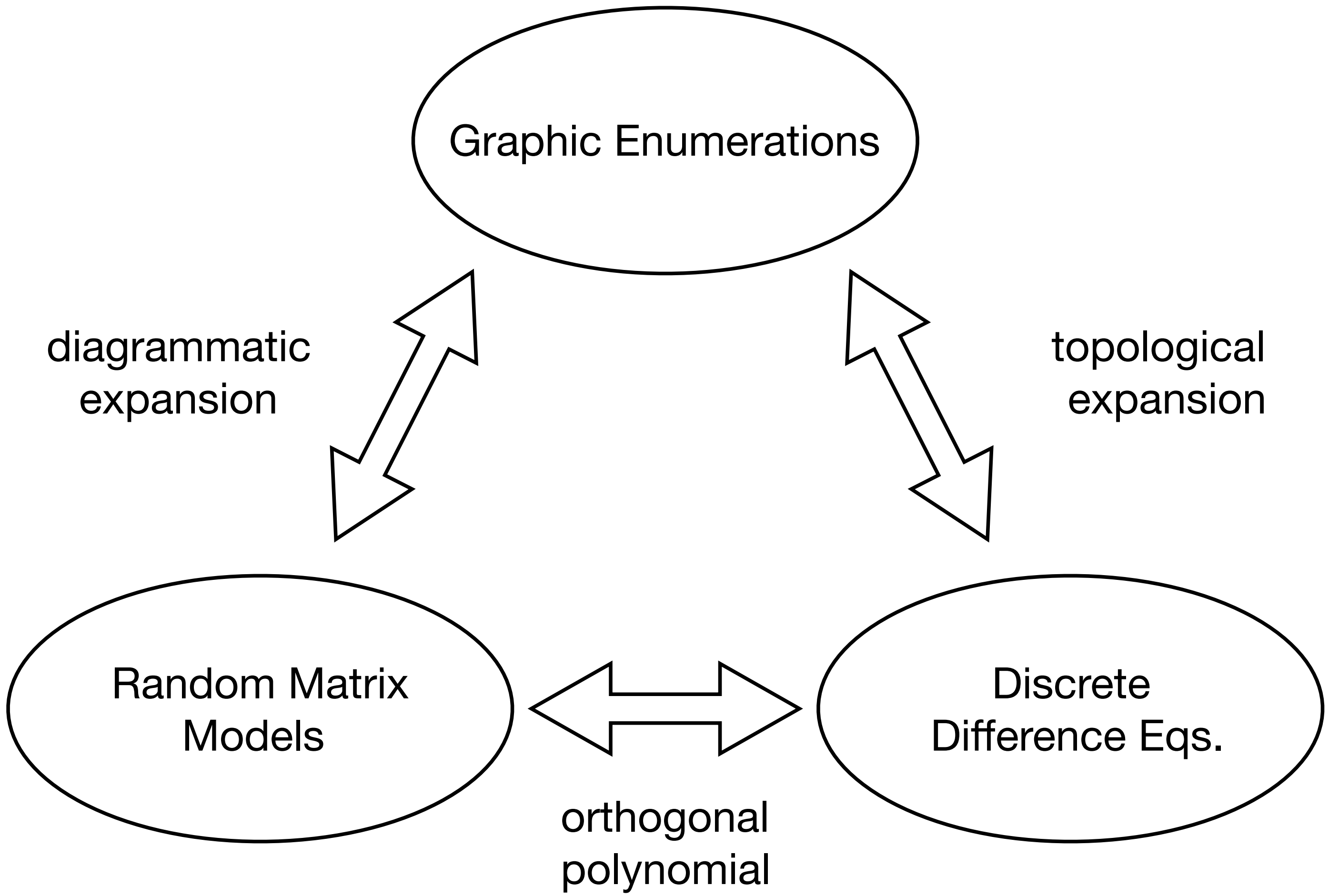
Chuan-Tsung Chan

Department of Applied Physics, Tunghai University, Taiwan

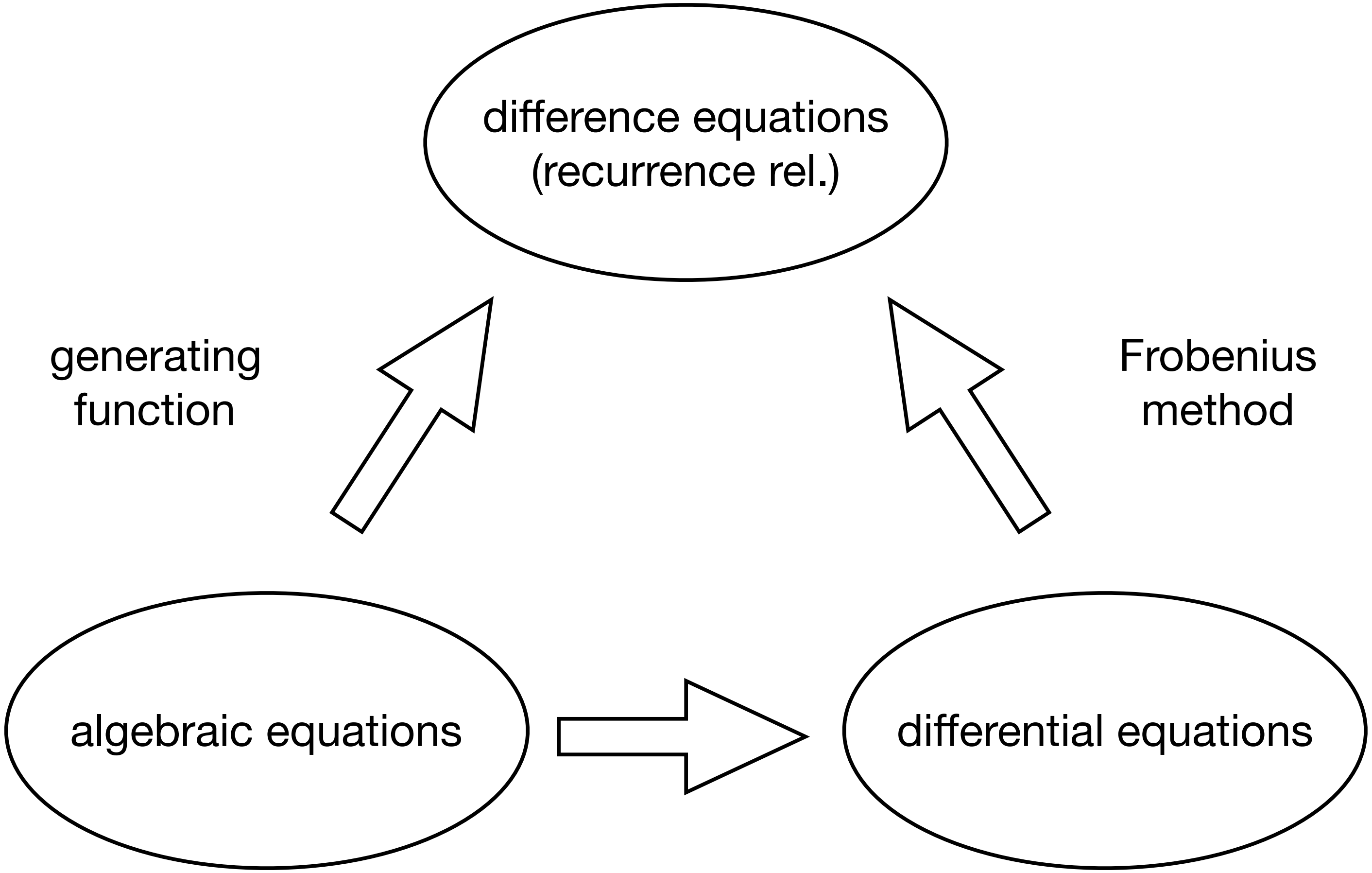
based on the paper arXiv:1712.09231 with Dr. Hsiao-Fan Liu

- The free engergies (FE) of random matrix model (RMM) integrals correspond to the generating functions for the enumeration of discrete random surfaces.
- We devise a new computer-programable scheme for an exact evaluation of the generating function based on a "topological expansion" of the recursive coefficients of the associated orthogonal polynomials.

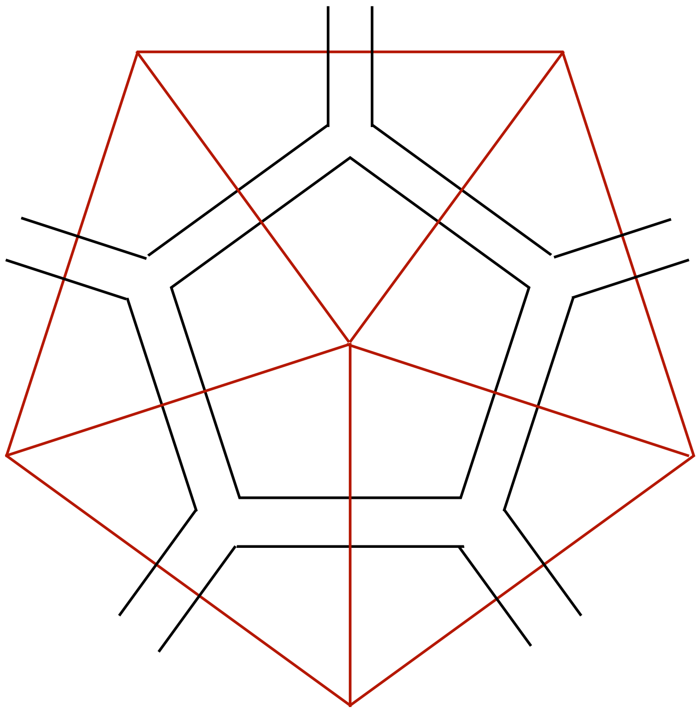
Grand Scheme of this Work



Tactics of Solutions



Feynman Rules and Topological Expansion



N dependence for a fat-graph

$\Rightarrow$ topological expansion

weighting for a junction $\Rightarrow \; Ng$,
weighting for a propogator $\Rightarrow \; 1N$,
weighting for a loop $\Rightarrow \; N$,

dual transformation:

$n \Rightarrow F, P \Rightarrow E, L \Rightarrow V.$

total weighting $\Rightarrow$

$(Ng)^n N^{-P} N^L = g^n N^{n-P+L}$

$= g^n N^{F-E+V} = g^n N^{\chi}.$

Orthogonal Polynomial Approach to RMM

- Orthogonal polynomials : $\int_{-\infty}^{\infty} P_n(\lambda)P_m(\lambda)\ e^{-NV(\lambda)}\ d\lambda = h_n\ \delta_{mn}$.
- Partition function of RMM in terms of normalization constants

$$\begin{aligned} Z(g,N) &= \int_{\mathcal{H}_N} \mathcal{D}M\ e^{-N\ trV(M)} \\ &= 1N! \int \prod_{k=1}^{N} d\lambda_k \left[\det_{i,j} P_{j-1}(\lambda_i)\right]^2 e^{-N\sum_{l=1}^{N} V(\lambda_l)} \\ \underbrace{\Rightarrow}_{N=3} &\ 13! \int \prod_{k=1}^{3} d\lambda_k \left[\sum_{\sigma\in\pi_3} (-1)^{\sigma} P_0(\lambda_{\sigma(1)})P_1(\lambda_{\sigma(2)})P_2(\lambda_{\sigma(3)})\right]^2 e^{-3\sum_{l=1}^{3} V(\lambda_l)} \\ \Rightarrow &\ \prod_{n=0}^{N-1} h_n(g,N). \end{aligned} \tag{1}$$

- Three-term recurssive relation :

$$\text{quartic model} \Rightarrow \lambda P_n(\lambda) = P_{n+1}(\lambda) + r_n P_{n-1}(\lambda). \tag{2}$$

$$\text{cubic model} \Rightarrow \lambda P_n(\lambda) = P_{n+1}(\lambda) + \beta_n P_n(\lambda) + \alpha_n P_{n-1}(\lambda). \tag{3}$$

Discrete Painleve Equation from RMM

- $h_n = \int_{-\infty}^{\infty} P_{n-1}(\lambda)\,[\lambda P_n(\lambda)]\ \ e^{-NV(\lambda)}\ d\lambda = r_n h_{n-1} \Rightarrow r_n = h_n h_{n-1}$.
- Partition function for the RMM : $Z\,(g,N) = h_0^N\ \ \prod_{k=1}^{N-1}\ r_k^{N-k}$.
- Recursive equation(s) among recurrence coefficients

$$
\begin{aligned}
nh_{n-1} &= \int_{-\infty}^{\infty} P_{n-1}(\lambda)\,[dd\lambda P_n(\lambda)]\,e^{-NV(\lambda)} d\lambda \\
&= P_{n-1}(\lambda)P_n(\lambda)e^{-NV(\lambda)}{}^{\infty}_{-\infty} - \int_{-\infty}^{\infty} (dd\lambda P_{n-1})\,P_n e^{-NV} d\lambda \\
&+ N\int_{-\infty}^{\infty} V^{'}(\lambda)P_{n-1}(\lambda)P_n(\lambda)e^{-NV(\lambda)} d\lambda
\end{aligned}
\tag{4}
$$

$$\text{quartic model} \Rightarrow \frac{n}{N} = r_n - g r_n\,(r_{n+1} + r_n + r_{n-1})\,, \ \text{d-P.I Eq.} \tag{5}$$

$$\text{cubic model} \Rightarrow \frac{n}{N} = \alpha_n[1 - g(\beta_n + \beta_{n-1})], \tag{6}$$

$$\beta_n = g(\alpha_{n+1} + \beta_n^2 + \alpha_n). \tag{7}$$

Roadmap of the Computations (Part I)

- Partition Function of the Random Matrix Models (RMM)

$$Z = \prod_{l=0}^{N-1} h_l. \tag{8}$$

- Recursive coefficients of the associated monic orthogonal polynomial systems (for even potential $V(-x) = V(x)$)

$$x \cdot P_n(x) = P_{n+1}(x) + r_n P_{n-1}(x), \quad r_k = \frac{h_k}{h_{k-1}}. \tag{9}$$

$$\Rightarrow \ln Z = N(\ln h_0) + \sum_{k=1}^{N-1} (N-k) \ln r_k. \tag{10}$$

- Perturbative expansion of the recursive cofficients for the quartic model

$$r_n = \frac{1}{g} \sum_{m=0}^{\infty} r_{nm} \epsilon^{m+1}, \quad \epsilon := \frac{g}{N} \tag{11}$$

Roadmap of the Computations (Part II)

- Polynomality of the Recursive Coefficients

$$r_{nm} = \sum_{l=0}^{\left[\frac{m+1}{2}\right]} n^{m+1-2l} 3^{m-l} C_m^{(l)}, \quad C^{(l)}(x) := \sum_{m=0}^{\infty} C_m^{(l)} x^m. \tag{12}$$

- Topological Expansion of the Recursive Coefficients

$$r_n(g,N) = \frac{n}{N} \sum_{l=0}^{\infty} (3n^2)^{-l} C^{(l)}(3n\epsilon) \tag{13}$$

$$= \frac{1}{3g} \sum_{l=0}^{\infty} (3n^2)^{-l} R^{(l)}(3n\epsilon), \quad R^l_{(x)} := xC^{(l)}_{(x)}. \tag{14}$$

- Solutions of the Topological Coefficients

$$C^{(0)} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} x^n, \tag{15}$$

$$C^{(1)} = \frac{2x^2}{(1-4x)^2} C^{(0)}(x) = \frac{x(1-\sqrt{1-4x})}{(1-4x)^2}, \tag{16}$$

$$C^{(2)} = \left[\frac{-28x^4}{(1-4x)^4} + \frac{98x^4}{(1-4)^{9/2}}\right] C^{(0)}(x) \tag{17}$$

Roadmap of the Computations (Part III)

- Free Energy of the Random Matrix Model

$$F(g,N) := \ln\left[\frac{Z(g,N)}{Z(0,N)}\right] \tag{18}$$

$$= N\ln\left[\frac{h_0(g,N)}{h_0(0,N)}\right] + \sum_{k=1}^{N-1}(N-k)\ln\left[\frac{r_k(g,N)}{r_k(0,N)}\right] \tag{19}$$

$$= N^2 e_0(g) + N^0 e_1(g) + N^{-2} e_2(g) + O(N^{-4}) \tag{20}$$

- Topological Expansion of the Finite N Matrix Model

$$e_0(g) = \sum_{k=0}^{\infty}\frac{(2k-1)!}{(k+2)!k!}(3g)^k \tag{21}$$

$$e_1(g) = \frac{1}{24}\sum_{k=1}^{\infty}\left[\frac{4^k}{k} - \frac{(2k)!}{(k+1)k!}\right](3g)^k \tag{22}$$

$$e_2(g) = \frac{1}{2^5\cdot 3^3\cdot 5}\sum_{k=3}^{\infty}[195\cdot 2^{2k-3}(k-1)$$

$$- \frac{(2k)!}{k!}(28k+9)(h-1)](3g)^k \tag{23}$$

Summary & Outlook (Part I)

- Take-home Message

$$
\begin{aligned}
Z(g,N) &:= \int_{\mathcal{H}_N} \mathcal{D}M \; e^{-N \; trV(M)} \\
F(g,N) &:= \ln\left[Z(g,N) Z(0,N)\right] = N^2 \sum_{n,h} g^n \; N^{-2h} \; C_{n,h}. \\
F(e^{-\beta}, e^{\gamma}) &= \sum_{n,h} e^{-\beta n + \gamma(2-2h)} \; C_{n,h}
\end{aligned}
$$

- We have performed systematic computations for counting DRS :

  – perturbative expansion in $g \Rightarrow F(g,N) = N^2 \; \sum_n g^n f_n(N^{-2})$

    $f_n(N^{-2}) := \sum_h N^{-2h} \; C_{n,h}$ finite (and even) polynomials in $1/N^2$.

  – topological expansion in $N^{-2} \Rightarrow F(g,N) = N^2 \; \sum_h N^{-2h} e_h(g)$

    $e_h(g) := \sum_n g^n \; C_{n,h}$ infinite series in g.

Summary & Outlook (Part II)

- Our calculations are based on the solutions of discrete difference equations as derived from the orthogonal polynomial approach to the random matrix integrals.

- Throughout our computations, we did not take any continuous approximation of summation. Furthermore, the relevant datum are obtained through the solutions of finite difference equations. Hence, our computations provide independent check of previous results, and lay down the foundation for a rigorous study of continuous/non-perturbative limit of 2-d quantum gravity.

- Hopeful, our analysis of perturbative solutions to the non-linear difference equations may shed some light for further explorations (e.g., asymptotic behaviors, exact solutions etc.)

- Double-Scaling Limit (DSL):

  generating function for DRS $\Rightarrow$ partition function for minimal string theory.