Quasi-pfaffian identities and noncommutative discrete integrable systems

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Contents

Pfaffians

What is a Pfaffian?
Let A be a $2n \times 2n$ skewsymetric matrix then

$$\det(A) = pf(A)^2$$

$pf(A)$ is a polynomial in the matrix entries.
If A is $(2n+1) \times (2n+1)$ then the $\det(A) = 0$.
You can write a pfaffian as a triangular array, eg

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{vmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

Shorthand notation:

$$(1,2,3,4) = (12)(34) - (13)(24) + (14)(23)$$

Pfaffians

$$(1, 2, \cdots, 2N) = \sum_w \epsilon(w) \prod_{i=1}^{N} (w(2i-1)w(2i))$$

Where the sum is over permutations of w of $1, 2, 3..., 2N$ such that

$$w(1) < w(2),\ w(3) < w(4),\ ....,\ w(2N-1) < w(2N)$$

and $\quad w(1) < w(3), ... < w(2N-1).$
Examples:

$$\begin{aligned}
(1234) &= (12)(34) - (13)(24) + (14)(23) \\
(123456) &= (12)[(34)(56) - (35)(46) + (36)(45)] \\
&\quad - (13)[(24)(56) - (25)(46) + (26)(45)] \\
&\quad + (14)[(23)(56) - (25)(36) + (26)(35)] \\
&\quad - ......
\end{aligned}$$

Pfaffian Identities

Like determinants, pfaffians obey bilinear identities.
Let

$$(\bullet) = (1, 2, 3, 4, \cdots, 2n)$$

Simplest identity:

$$(\bullet\, a, b, c, d)\,(\bullet) = (\bullet\, a, b)(\bullet\, c, d) - (\bullet\, a, c)(\bullet\, b, d) + (\bullet\, a, d)(\bullet\, b, c)$$

$$= \left| \begin{array}{ccc} (\bullet\, a, b) & (\bullet\, a, c) & (\bullet\, a, d) \\ & (\bullet\, b, c) & (\bullet\, b, d) \\ & & (\bullet\, c, d) \end{array} \right|$$

Inverse of a Skew Symmetric Matrix

This has a rather interesting form: Take entries $(ij) = -(ji)$,

$$\begin{pmatrix} 0 & (12) & (13) & (14) & (15) & (16) \\ (21) & 0 & (23) & (24) & (25) & (26) \\ (31) & (32) & 0 & (34) & (35) & (36) \\ (41) & (42) & (43) & 0 & (45) & (46) \\ (51) & (52) & (53) & (54) & 0 & (56) \\ (61) & (62) & (63) & (64) & (65) & 0 \end{pmatrix}^{-1}$$

$$= \frac{1}{(123456)} \begin{pmatrix} 0 & (3456) & -(2456) & (2356) & -(2346) & (3456) \\ . & 0 & (1456) & -(1356) & (1346) & -(1345) \\ . & . & 0 & (1256) & -(1246) & (1245) \\ . & . & . & 0 & (1236) & -(1235) \\ . & . & . & . & 0 & (1236) \\ . & . & . & . & . & 0 \end{pmatrix}$$

Pfaffian Identities

General identity:
Take two sets of indices $\{\underline{\gamma}, \underline{\alpha}\}$ and $\{\underline{\gamma}, \underline{\beta}\}$, then you can form a bilinear identity

$$\sum_{r=1}^{m}(-1)^{m-r}(\underline{\gamma}, \underline{\alpha}/\alpha_r)(\underline{\gamma}, \underline{\beta}, \alpha_r)+\sum_{r=1}^{n}(-1)^{n-r}(\underline{\gamma}, \underline{\alpha}, \beta_r)(\underline{\gamma}, \underline{\beta}/\beta_r) = 0$$

(Hirota, Ohta)

For our simple identity take the sets $\{\bullet\, a\}$ and $\{\bullet\, b, c, d\}$. Giving

$$(\bullet\,)(\bullet\, b, c, d, a)+(\bullet\, a, b)(\bullet\, c, d)-(\bullet\, a, c)(\bullet\, b, d)+(\bullet\, a, d)(\bullet\, b, c) = 0$$

Pfaffian Identities

Second simple identity:
For this identity take the sets of labels $\{\bullet\, a, b\}$ and $\{\bullet\,, c, d\}$. The identity is:

$$(\bullet\, a)(\bullet\, b, c, d) - (\bullet\, b)(\bullet\, a, c, d) + (\bullet\, c)(\bullet\, a, b, d) - (\bullet\, d)(\bullet\, a, b, c) = 0$$

Quasi-determinants (Etingof, Gelfand and Retakh)

For a block matrix

$$\begin{pmatrix} A & B \\ C & d \end{pmatrix}$$

where d is a single entry, A is a square matrix (say $n \times n$) and B, C are column and row vectors, one quasi-determinant is

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B.$$

Indeed

$$\left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix}^{-1} \right\}_{n+1,n+1} = (d - CA^{-1}B)^{-1}.$$

If the entries in our matrix commute, then

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = \frac{\begin{vmatrix} A & B \\ C & d \end{vmatrix}}{|A|} \qquad \text{(Cramer's rule?)}$$

Quasi-determinants

Example 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are all single entries.
one quasi-determinant is

$$\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b.$$

There are other quasi determinants obtained by expanding about different entries:

$$\begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a \qquad \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c$$

and $$\begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d$$

Quasi-determinants

Example 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are all single entries.
one quasi-determinant is

$$\begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b \quad \text{(in commuting case } = \begin{vmatrix} a & b \\ c & d \end{vmatrix} / a)$$

There are other quasi determinants obtained by expanding about different entries:

$$\begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a \qquad \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c$$

and $$\begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d$$

Quasi Determinant identities

For Determinants we have Jacobi's identity:

$$\begin{vmatrix} A & B_1 & B_2 \\ C_1 & d_1 & d_2 \\ C_2 & d_3 & d_4 \end{vmatrix} |A| = \begin{vmatrix} \begin{vmatrix} A & B_1 \\ C_1 & d_1 \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & d_2 \end{vmatrix} \\ \\ \begin{vmatrix} A & B_1 \\ C_2 & d_3 \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_2 & d_4 \end{vmatrix} \end{vmatrix}$$

Quasi Determinant identities

Like normal determinants, quasideterminants obey identities. The analogue of the Jacobi identity for determinants is Sylvester's Identity

$$\begin{vmatrix} A & B_1 & B_2 \\ C_1 & d_1 & d_2 \\ C_2 & d_3 & \boxed{d_4} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} A & B_1 \\ C_1 & \boxed{d_1} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & \boxed{d_2} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_2 & \boxed{d_3} \end{vmatrix} & \boxed{\begin{vmatrix} A & B_2 \\ C_2 & \boxed{d_4} \end{vmatrix}} \end{vmatrix}$$

A second Quasi Determinant identities

There are other quasideterminant identities as well, for example:

$$\begin{vmatrix} A & B_1 & B_2 & B_3 \\ C_1 & d_1 & d_2 & d_3 \\ C_2 & d_4 & d_5 & d_6 \\ C_3 & d_7 & d_8 & \boxed{d_9} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} A & B_1 \\ C_1 & \boxed{d_1} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & \boxed{d_2} \end{vmatrix} & \begin{vmatrix} A & B_3 \\ C_1 & \boxed{d_3} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_2 & \boxed{d_4} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_2 & \boxed{d_5} \end{vmatrix} & \begin{vmatrix} A & B_3 \\ C_2 & \boxed{d_6} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_3 & \boxed{d_7} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_3 & \boxed{d_8} \end{vmatrix} & \boxed{\begin{vmatrix} A & B_3 \\ C_3 & \boxed{d_9} \end{vmatrix}} \end{vmatrix}$$

This turns out to be useful for skew-symmetric cases.

Quasi Pfaffians

Recall, a Pfaffian is the square root of a skew symmetric determinant. Eg

$$\begin{aligned}\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} &= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 \\ &= (1,2,3,4)^2.\end{aligned}$$

They typically turn out to give solutions to the BKP equation and related equations. Unfortunately in the non-commutative case we don't get this factorisation.

Quasi Pfaffians

We could attempt to look at the quasideterminant of a skew symmetric matrix, this doesn't work:

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{0} \end{vmatrix} \qquad \text{with} \quad a_{ji} = -a_{ij}$$

or

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & 0 & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & 0 & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & \boxed{0} \end{vmatrix}$$

Quasi Pfaffians

For the quasi pfaffian we will take entries $a_{ij} = -a_{ji}^T$,
(possibly also need $a_{ii} = -a_{ii}^T$ rather than $a_{ii} = 0$)
For example

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{1\beta} \\ a_{21} & 0 & a_{23} & a_{24} & a_{2\beta} \\ a_{31} & a_{32} & 0 & a_{34} & a_{3\beta} \\ a_{41} & a_{42} & a_{43} & 0 & a_{4\beta} \\ a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 3} & \boxed{a_{\alpha\beta}} \end{vmatrix}$$

$$= (1, 2, 3, 4, \boxed{\alpha, \beta}).$$

Quasi Pfaffians

For the quasi pfaffian we will take entries $a_{ij} = -a_{ji}^T$,
(possibly also need $a_{ii} = -a_{ii}^T$ rather than $a_{ii} = 0$)
For example

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{1\alpha} & a_{1\beta} \\ a_{21} & 0 & a_{23} & a_{24} & a_{2\alpha} & a_{2\beta} \\ a_{31} & a_{32} & 0 & a_{34} & a_{3\alpha} & a_{3\beta} \\ a_{41} & a_{42} & a_{43} & 0 & a_{4\alpha} & a_{4\beta} \\ a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 4} & \boxed{\begin{matrix} 0 & a_{\alpha\beta} \\ a_{\beta\alpha} & 0 \end{matrix}} \\ a_{\beta 1} & a_{\beta 2} & a_{\beta 3} & a_{\beta 4} & & \end{vmatrix}$$

$$= \left(1, 2, 3, 4, \boxed{\boxed{\alpha, \beta}}\right).$$

Quasi Pfaffians (commutative case)

$$
\begin{vmatrix}
0 & a_{12} & a_{13} & a_{14} & a_{1\beta} \\
a_{21} & 0 & a_{23} & a_{24} & a_{2\beta} \\
a_{31} & a_{32} & 0 & a_{34} & a_{3\beta} \\
a_{41} & a_{42} & a_{43} & 0 & a_{4\beta} \\
a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 4} & \boxed{a_{\alpha\beta}}
\end{vmatrix}
$$

$$
= a_{\alpha\beta} - \begin{pmatrix} a_{\alpha 1} & a_{\alpha 2} & a_{\alpha 3} & a_{\alpha 4} \end{pmatrix}
\begin{pmatrix}
0 & a_{12} & a_{13} & a_{14} \\
a_{21} & 0 & a_{23} & a_{24} \\
a_{31} & a_{32} & 0 & a_{34} \\
a_{41} & a_{42} & a_{43} & 0
\end{pmatrix}^{-1}
\begin{pmatrix} a_{1\beta} \\ a_{2\beta} \\ a_{3\beta} \\ a_{4\beta} \end{pmatrix}
$$

$$
= \frac{(1\,2\,3\,4\,\alpha\,\beta)}{(1\,2\,3\,4)}.
$$

A Quasi-Pfaffian Identity

We can deduce a simple quasi-pfaffian identity based on Silvester's identity for quasi determinants:

$$(\bullet 12 \boxed{ij}) = \begin{vmatrix} (\bullet \boxed{12}) & (\bullet \boxed{1j}) \\ (\bullet \boxed{i2}) & \boxed{(\bullet \boxed{ij})} \end{vmatrix}$$

$$- \begin{vmatrix} (\bullet \boxed{12}) & (\bullet \boxed{11}) \\ (\bullet \boxed{i2}) & \boxed{(\bullet \boxed{i1})} \end{vmatrix} \begin{vmatrix} (\bullet \boxed{12}) & (\bullet \boxed{11}) \\ (\bullet \boxed{22}) & \boxed{(\bullet \boxed{21})} \end{vmatrix}^{-1} \begin{vmatrix} (\bullet \boxed{12}) & (\bullet \boxed{1j}) \\ (\bullet \boxed{22}) & \boxed{(\bullet \boxed{2j})} \end{vmatrix}$$

Here $\bullet$ is used to represent a set of indices (usually an even number of them). Here, terms such as $(\bullet \boxed{11})$ are not necessarily equal to zero.

What equations have Pfaffian solutions?

Hirota and Ohta coupled soliton equations:

$$(4u_t - 6uu_x - u_{xxx})_x - 3u_{yy} + 24(v\bar{v})_{xx} = 0$$

$$2v_t + 3uv_x + 3\left(v_{xxx} + v\int^x u_y dx\right) = 0$$

$$2\bar{v}_t + 3u\bar{v}_x - 3\left(\bar{v}_{xxx} + \bar{v}\int^x u_y dx\right) = 0$$

With the transformation

$$u = 2(log\tau)_{xx}, \qquad v = \frac{\sigma}{\tau}, \qquad \bar{v} = \frac{\bar{\sigma}}{\tau}.$$

$$\tau = (1\,,2\,\cdots\,,2N),$$

$$\sigma = (1\,,2\,\cdots\,,2N, c_1, c_2), \qquad \bar{\sigma} = (1\,,2\,\cdots\,,2N, d_1, d_2),$$

Pfaffian entries have a skew symmetric form:

$$(i\,,j) = \int^x (f_i g_j - g_i f_j) dx, \quad (i\,,c_n) = \frac{\partial^n}{\partial x^n} g_i, \quad (i\,,d_n) = \frac{\partial^n}{\partial x^n} f_i$$

What equations have Pfaffian solutions?

Hirota and Ohta coupled soliton equations:

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$$\sigma = (1\,,2\,\cdots\,,2N, c_1, c_2), \qquad \bar{\sigma} = (1\,,2\,\cdots\,,2N, d_1, d_2),$$

$$(i\,,j) = \int^x (f_i g_j - g_i f_j) dx, \quad (i\,,c_n) = \frac{\partial^n}{\partial x^n} g_i, \quad (i\,,d_n) = \frac{\partial^n}{\partial x^n} f_i$$

The f's and g's obey simple linear relationships $(x_2 = y, x_3 = t)$

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \qquad \frac{\partial g_i}{\partial x_n} = (-1)^{n-1} \frac{\partial^n g_i}{\partial x^n}$$

The key to showing these are solutions is by differentiating the functions τ, σ and $\bar{\sigma}$.

$$\partial_x \tau = \partial_x (1\,,2\,\cdots\,,2N) = (1\,,2\,\cdots\,,2N, c_0, d_0).$$

What equations have Pfaffian solutions?

The (2+1) dimensional sine-Gordon System (Konopelchenko-Rogers):

$$u_{xyt} + u_x v_{yt} + u_y v_{xt} = 0$$
$$v_{xy} = u_x u_y$$

With the transformation

$$U = (v - iu)_{xy} = 2(log\tau)_{xy},$$
$$\text{i.e.} \qquad u = i\ log\tau/\bar{\tau}, \qquad v = log\tau\bar{\tau}.$$

Again the τ is just a pfaffian $\tau = (1\,, 2\,\cdots\,, 2N)$.
Pfaffian entries have a skew symmetric form:

$$(\theta_i,\,\theta_j)_x = \theta_i^T(\theta_j)_x - (\theta_i^T)_x\theta_j, \quad (\theta_i,\,\theta_j)_y = -\theta_i^T(\theta_j)_y + (\theta_i^T)_y\theta_j,$$
$$(\theta_i\,, c_n) = -\frac{\partial^n}{\partial x^n}\theta_i^T, \quad (d_n,\,\theta_j) = \frac{\partial^n}{\partial x^n}\theta_j$$

What equations have Pfaffian solutions?

The (2+1) dimensional sine-Gordon System (Konopelchenko-Rogers):

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With the transformation

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$$\text{i.e.} \qquad u = i\ log\tau/\bar{\tau}, \qquad v = log\tau\bar{\tau}.$$

Again the τ is just a pfaffian $\tau = (1, 2 \cdots, 2N)$.
The bilinear form of this system (Nimmo):

$$D_x D_y \tau \cdot \bar{\tau} = 0$$
$$D_x D_y D_t \tau \cdot \bar{\tau} = 0$$

Differentiation of a quasi-determinant

Consider the derivative

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B - C(A^{-1})'B - CA^{-1}B'$$
$$= d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'$$

Suppose now that A is a "gramian-like pfaffian", i.e.

$$A' = \sum_{i=1}^{k} \mathbf{E}_i \mathbf{F}_i, \qquad \text{(for instance } A_{ij} = \int^x (f_i g_j - g_i f_j) dx\text{)}$$

where $\mathbf{E}_i$ ($\mathbf{F}_i$) are column (row) vectors of appropriate length. Then we can factorise the BLUE term on the RHS to obtain

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{0} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{0} \end{vmatrix}.$$

Derivatives of a Quasipfaffian

Note slight change of notation here:

$$\left(\theta_1 \cdots \theta_{2n} \boxed{\theta_a \theta_b}\right) = \begin{vmatrix} (\theta_1\theta_1) & \cdots & (\theta_1\theta_{2n}) & (\theta_1\theta_b) \\ \vdots & \ddots & & \vdots \\ (\theta_{2n}\theta_1) & & (\theta_{2n}\theta_{2n}) & (\theta_{2n}\theta_b) \\ (\theta_a\theta_1) & \cdots & (\theta_a\theta_{2n}) & \boxed{(\theta_a\theta_b)} \end{vmatrix}$$

Let the entries in the quasi pfaffian be

$$(\theta_i, \theta_j) = \int [\theta_i^T (\theta_j)_x - (\theta_i^T)_x \theta_j] dx$$

$$(1, \theta_j) = \int [1\,(\theta_j)_x - 1_x\,\theta_j] dx = \theta_j, \qquad (d_n, \theta_j) = \frac{\partial^n}{\partial x^n} \theta_j$$

$$(\theta_i, 1) = -\theta_i^T, \qquad (\theta_i, c_n) = -\frac{\partial^n}{\partial x^n} \theta_i^T$$

Note that for $j > i$ $\qquad (\theta_j, \theta_i) = -(\theta_i, \theta_j)^T$

Derivatives of a Quasipfaffian

Notation:

$$\left(\theta_1 \cdots \theta_{2n} \boxed{\theta_a,\ c_j}\right) = \begin{vmatrix} (\theta_1\theta_1) & \cdots & (\theta_1\theta_{2n}) & -\theta_1^{(j)\,T} \\ \vdots & \ddots & & \vdots \\ (\theta_{2n}\theta_1) & & (\theta_{2n}\theta_{2n}) & -\theta_{2n}^{(j)\,T} \\ (\theta_a\theta_1) & \cdots & (\theta_a\theta_{2n}) & \boxed{-\theta_a^{(j)\,T}} \end{vmatrix}$$

$$\left(\theta_1 \cdots \theta_{2n} \boxed{d_i,\ \theta_b}\right) = \begin{vmatrix} (\theta_1\theta_1) & \cdots & (\theta_1\theta_{2n}) & (\theta_1\theta_b) \\ \vdots & \ddots & & \vdots \\ (\theta_{2n}\theta_1) & & (\theta_{2n}\theta_{2n}) & (\theta_{2n}\theta_b) \\ \theta_1^{(i)} & \cdots & \theta_{2n}^{(i)} & \boxed{\theta_b^{(i)}} \end{vmatrix}$$

Derivatives of a Quasipfaffian

$$\left(\theta_1\cdots\theta_{2n}\boxed{d_r,\ c_s}\right) = \begin{vmatrix} (\theta_1\theta_1) & \cdots & (\theta_1\theta_{2n}) & -{\theta_1^T}^{(s)} \\ \vdots & \ddots & & \vdots \\ (\theta_{2n}\theta_1) & & (\theta_{2n}\theta_{2n}) & -{\theta_{2n}^T}^{(s)} \\ {\theta_1}^{(r)} & \cdots & {\theta_{2n}}^{(r)} & \boxed{0} \end{vmatrix}$$

Derivatives

$$\begin{aligned} \frac{\partial}{\partial x}\left(\theta_1\cdots\theta_{2n}\boxed{\theta_a\,,\theta_b}\right) &= \left(\theta_1\cdots\theta_{2n}\boxed{\theta_a,\ c_1}\right)\left(\theta_1\cdots\theta_{2n}\boxed{d_0,\ \theta_b}\right) \\ &\quad - \left(\theta_1\cdots\theta_{2n}\boxed{\theta_a,\ c_0}\right)\left(\theta_1\cdots\theta_{2n}\boxed{d_1,\ \theta_b}\right) \end{aligned}$$

$$Q[\theta_a,\theta_b]_x = Q[\theta_a,c_1]Q[d_0,\theta_b] - Q[\theta_a,c_0]Q[d_1,\theta_b]$$

Derivatives of a Quasipfaffian (2+1 sine-Gordon case)

$$(\theta_i,\, \theta_j)_x = \theta_i^T(\theta_j)_x - (\theta_i^T)_x\theta_j, \quad (\theta_i,\, \theta_j)_y = -\theta_i^T(\theta_j)_y + (\theta_i^T)_y\theta_j,$$

Derivatives

$$\begin{aligned}
Q[\theta_a, \theta_b]_x &= Q[\theta_a, c_{(1,0)}]Q[d_0, \theta_b] - Q[\theta_a, c_0]Q[d_{(1,0)}, \theta_b] \\
Q[\theta_a, c_n]_x &= Q[\theta_a, c_{n+1}] + Q[\theta_a, c_1]Q[d_0, c_n] - Q[\theta_a, c_0]Q[d_1, c_n] \\
Q[d_n, \theta_b]_x &= Q[d_{n+1}, \theta_b] + Q[d_n, c_1]Q[d_0, \theta_b] - Q[d_n, c_0]Q[d_1, \theta_b] \\
Q[d_n, c_m]_x &= Q[d_{n+1}, c_m] + Q[d_n, c_{m+1}] \\
&\qquad + Q[d_n, c_1]Q[d_0, c_m] - Q[d_n, c_0]Q[d_1, c_m] \\
Q[\theta_a, \theta_b]_y &= Q[\theta_a, c_{(0,1)}]Q[d_0, \theta_b] - Q[\theta_a, c_0]Q[d_{(0,1)}, \theta_b] \\
Q[\theta_a, c_n]_y &= \ldots\ldots \\
Q[d_n, \theta_b]_y &= \ldots\ldots \\
Q[d_n, c_m]_y &= \ldots\ldots
\end{aligned}$$

Moutard Transformations (1878)

Consider the linear equation

$$\psi_{xy} + U\psi = 0$$

This equation is invariant when we apply a Moutard transformation to an eigenfunction ψ using the eigenfunction θ_1

$$\psi \rightarrow \psi_1 = (\theta_1^T)^{-1}(\theta_1, \psi)$$
$$U \rightarrow U_1 = U - 2{(\theta_1^T)^{-1}}_x(\theta_1)_y$$

A second Moutard transformation can be carried out, we will define it in the following way:

$$\psi_1 \rightarrow \psi_{12} = \begin{vmatrix} (\theta_1\theta_1) & (\theta_1\theta_2) & (\theta_1\psi) \\ (\theta_2\theta_1) & (\theta_2\theta_2) & (\theta_2\psi) \\ \theta_1 & \theta_2 & \boxed{\psi} \end{vmatrix} = s\left(\theta_1\theta_2\boxed{1\ \psi}\right).$$

Moutard Transformations

We can carry out further Moutard transformations in a consistent manor. For an even number of applications:

$$\psi_{1,\cdots,2n} = \begin{vmatrix} (\theta_1\theta_1) & \cdot\cdot & (\theta_1\theta_{2n}) & (\theta_1\psi) \\ \vdots & & \vdots & \vdots \\ (\theta_{2n}\theta_1) & \cdot\cdot & (\theta_{2n}\theta_{2n}) & (\theta_{2n}\psi) \\ (1\ \theta_1) & \cdot\cdot & (1\ \theta_{2n}) & \boxed{(1\ \psi)} \end{vmatrix} = \left(\theta_1 \cdot \cdot \theta_{2n} \boxed{1\ \psi}\right)$$

and for an odd amount of applications:

$$\psi_{1,\cdot\cdot,2n-1} = \left(\theta_1 \cdots \theta_{2n-1}\ 1\ \boxed{c_0\ \psi}\right)$$

Could we use these objects to build solutions to a noncommutative pfaffian type equation?

Conclusions

I've introduced the idea of a Quasi-Pfaffian and considered some ideas

- There exist Quasipfaffian identities, can we use them to build (integrable) equations involving quasipfaffians?
- Differentiation of Quasipfaffians give very simple things
- Non-commutative Moutard transformations