Rational Solutions to Quadrilateral Equations

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- Sato/KP Theory: General Dispersion Relation (DR)

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$$x_j = \frac{1}{j} \sum_i p_i^j n_i. \qquad (1)$$

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- Common τ function: $\tau[\mathbf{t} - \mathbf{x}]$
- We would like to investigate $\tau(P(\mathbf{x}))$

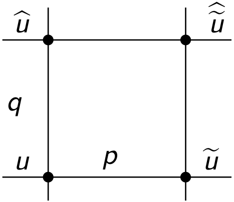
Outline of the talk

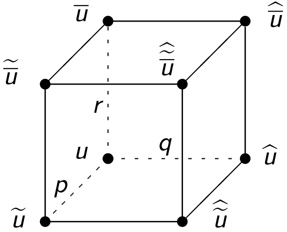
- Part I: Notations
- Part II: Main results
- Part III: Proof

Notations:

$$u \equiv u_{n,m}, \quad \widetilde{u} \equiv u_{n+1,m}, \quad \widehat{u} \equiv u_{n,m+1}, \quad \widehat{\widetilde{u}} \equiv u_{n+1,m+1}$$

Map:





Quadrilateral equations: $Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0$

lpKdV: $(u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) = p^2 - q^2,$

$$(u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) = p^2 - q^2.$$

Hirota-Miwa

Hirota-Miwa:

$$(a-b)\overline{\tau}\widehat{\widetilde{\tau}}+(b-c)\widetilde{\tau}\widehat{\overline{\tau}}+(c-a)\widehat{\tau}\widetilde{\overline{\tau}}=0$$

semi-discrete Hirota-Miwa:

$$(a-b)(\tau\widehat{\widetilde{\tau}}-\widetilde{\tau}\widehat{\tau})+\widetilde{\tau}_x\widehat{\tau}-\widetilde{\tau}\widehat{\tau}_x=0$$

NQC [Nijhoff, Quispel, Capel-PLA-1983]

NQC:

$$\begin{aligned}&\frac{1+(p-a)S(a,b)-(p+b)\widetilde{S}(a,b)}{1+(q-a)S(a,b)-(q+b)\widehat{S}(a,b)}\\=&\frac{1-(q+a)\widehat{\widetilde{S}}(a,b)+(q-b)\widetilde{S}(a,b)}{1-(p+a)\widehat{\widetilde{S}}(a,b)+(p-b)\widehat{S}(a,b)},\end{aligned} \tag{2}$$

where $S(a,b)=S(b,a)$ are functions of (n,m) with (a,b) being branch point parameters, p and q are spacing parameters of n and m, respectively.

ABS [Adler, Bobenko, Suris-CMP-2003]

$$
\begin{aligned}
\mathrm{Q3}_\delta:&\ \ P(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-Q(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})=(p^2-q^2)\Big((\widetilde{u}\widehat{u}+u\widehat{\widetilde{u}})+\frac{\delta^2}{4PQ}\Big),\\
\mathrm{Q2}:&\ \ (q^2-a^2)(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-(p^2-a^2)(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})\\
&\quad +(p^2-a^2)(q^2-a^2)(q^2-p^2)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})\\
&\ =(p^2-a^2)(q^2-a^2)(q^2-p^2)\big((p^2-a^2)^2+(q^2-a^2)^2-(p^2-a^2)(q^2-a^2)\big),\\
\mathrm{Q1}_\delta:&\ \ (q^2-a^2)(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-(p^2-a^2)(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})=\frac{\delta^2a^4(p^2-q^2)}{(p^2-a^2)(q^2-a^2)},\\
\mathrm{H3}_\delta:&\ \ P(a^2-q^2)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-Q(a^2-p^2)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})=\delta(p^2-q^2),\\
\mathrm{H2}:&\ \ (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+(p^2-q^2)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})=p^4-q^4,\\
\mathrm{H1}:&\ \ (u-\widehat{\widetilde{u}})(\widehat{u}-\widetilde{u})=p^2-q^2,
\end{aligned}
$$

where in $\mathrm{Q3}_\delta$ $(p,P)=\mathfrak{p}$ and $(q,Q)=\mathfrak{q}$ are the points on the elliptic curve

$$\{(x,X)|X^2=(x^2-a^2)(x^2-b^2)\}, \qquad (3)$$

and in $\mathrm{H3}_\delta$: $P^2=a^2-p^2,\ Q^2=a^2-q^2$.

ABS* [Atkinson, Nieszporski-IMRN-2014]

$$
\begin{aligned}
\mathrm{H2}^*:\ & (a^{-2}-b^{-2})\left[a^{-2}(v-\tilde{v}+\hat{v}-\widehat{\tilde{v}})^2-b^{-2}(v-\hat{v}+\tilde{v}-\widehat{\tilde{v}})^2\right]\\
& +(v-\widehat{\tilde{v}})(\tilde{v}-\hat{v})\left[(v-\widehat{\tilde{v}})(\tilde{v}-\hat{v})-2(a^{-2}-b^{-2})(v+\tilde{v}+\hat{v}+\widehat{\tilde{v}})\right]=0,\\
\mathrm{H3}^*(\delta):\ & (a^{-2}-b^{-2})\left[a^{-2}(U\hat{U}-\tilde{U}\widehat{\tilde{U}})^2-b^{-2}(U\tilde{U}-\hat{U}\widehat{\tilde{U}})^2\right]\\
& +(U-\widehat{\tilde{U}})(\tilde{U}-\hat{U})\left[(U-\widehat{\tilde{U}})(\tilde{U}-\hat{U})a^{-2}b^{-2}-4\delta^2(a^{-2}-b^{-2})\right]=0,\\
\mathrm{A1}^*:\ & (a^{-2}-b^{-2})\left[a^{-2}(w-\tilde{w}+\hat{w}-\widehat{\tilde{w}})^2-b^{-2}(w-\hat{w}+\tilde{w}-\widehat{\tilde{w}})^2\right]\\
& -(w-\widehat{\tilde{w}})(\tilde{w}-\hat{w})\left[a^{-2}(w+\hat{w})(\tilde{w}+\widehat{\tilde{w}})-b^{-2}(w+\tilde{w})(\hat{w}+\widehat{\tilde{w}})\right]=0,\\
\mathrm{Q2}^*:\ & (a^2-b^2)[a^2(w\tilde{w}-\hat{w}\widehat{\tilde{w}})(w+\tilde{w}-\hat{w}-\widehat{\tilde{w}})-b^2(w\hat{w}-\tilde{w}\widehat{\tilde{w}})(w-\tilde{w}+\hat{w}-\widehat{\tilde{w}})]\\
& -(w-\widehat{\tilde{w}})(\tilde{w}-\hat{w})\\
& \quad\times\left[a^2(w-\hat{w})(\tilde{w}-\widehat{\tilde{w}})-b^2(w-\tilde{w})(\hat{w}-\widehat{\tilde{w}})-a^2b^2(a^2-b^2)\right]=0.
\end{aligned}
$$

τ function of our interest

Consider general plain wave factors (PWF)

$$\psi^{\pm}(\{n_i\};l) = \varrho^{\pm}(1 \pm k)^l \prod_i (1 \pm p_i k)^{n_i}, \tag{4}$$

where (with arbitrary constant γ_j)

$$\varrho^{\pm} = \pm\frac{1}{2}\exp\left[-\sum_{j=1}^{\infty}\frac{(\mp k)^j}{j}\gamma_j\right] \tag{5}$$

Expand $\psi^{\pm}(\{n_i\};l)$ as

$$\psi^{\pm}(\{n_i\};l) = \pm\frac{1}{2}\sum_{h=0}^{\infty}\alpha_h^{\pm}k^h, \quad \alpha_h^{\pm} = \pm\frac{2}{h!}\partial_k^h\psi^{\pm}|_{k=0}. \tag{6}$$

$$\psi^{\pm}(\{n_i\};l) = \pm\frac{1}{2}\exp\left[-\sum_{j=1}^{\infty}(\mp k)^j \mathring{x}_j\right], \quad \mathring{x}_j = x_j + l/j, \tag{7}$$

$$x_j = \frac{1}{j}(\gamma_j + \sum_i p_i^j n_i).$$

$\{\alpha_h^{\pm}\}$ in terms of $\{x_j\}$:

$$\alpha_h^{\pm} \doteq \alpha_h^{\pm}(\{n_i\}; l) = (\mp 1)^h \sum_{||\mu||=h} (-1)^{|\mu|} \frac{\mathring{\mathbf{x}}^{\mu}}{\mu!} \tag{8}$$

where

$$\mu = (\mu_1, \mu_2, \cdots), \quad \mu_j \in \{0, 1, 2, \ldots\}, \quad ||\mu|| = \sum_{j=1}^{\infty} j\mu_j,$$

$$|\mu| = \sum_{j=1}^{\infty} \mu_j, \quad \mu! = \prod_i \mu_i!, \quad \mathring{\mathbf{x}}^{\mu} = \prod_i (\mathring{x}_i)^{\mu_i}.$$

The first few α_h^{+} are

$$\alpha_0^{+} = 1, \quad \alpha_1^{+} = \mathring{x}_1, \quad \alpha_2^{+} = \frac{1}{2}\mathring{x}_1^2 - \mathring{x}_2, \quad \alpha_3^{+} = \frac{1}{6}\mathring{x}_1^3 - \mathring{x}_1\mathring{x}_2 + \mathring{x}_3,$$

$$\alpha_4^{+} = \frac{1}{24}(\mathring{x}_1^4 - 12\mathring{x}_1^2\mathring{x}_2 + 24\mathring{x}_1\mathring{x}_3 + 12\mathring{x}_2^2 - 24\mathring{x}_4).$$

τ functions defined by Casoratians

- $\tau(\alpha)$

$$\tau_N(\alpha) = |\alpha(\{n_i\};0), \alpha(\{n_i\};1), \cdots, \alpha(\{n_i\};N-1)| \qquad (9)$$

where

$$\alpha(\{n_i\};l) = (\alpha_1^+, \alpha_3^+, \cdots, \alpha_{2N-1}^+)^T, \qquad (10)$$

- $\tau(\beta)$

$$\tau_N(\beta) = |\beta(\{n_i\};0), \beta(\{n_i\};1), \cdots, \beta(\{n_i\};N-1)| \qquad (11)$$

where

$$\beta(\{n_i\};l) = (\alpha_0^+, \alpha_2^+, \cdots, \alpha_{2N-2}^+)^T. \qquad (12)$$

- Examples of $\tau_N(\alpha)$:

$$\tau_{N=1}(\alpha) = x_1,$$
$$\tau_{N=2}(\alpha) = \frac{1}{3}x_1^3 - x_3,$$
$$\tau_{N=3}(\alpha) = \frac{1}{45}x_1^6 - \frac{1}{3}x_1^3x_3 + x_1x_5 - x_3^2,$$

- Algebraic solutions of HM [Nimmo-JPA-1997]
- Dependency: x_{2j+1}
- Homogeneous
  Degree of monomial: $\mathcal{D}[\prod_{i\geq 1} x_i^{s_i}] = \sum_{i\geq 1} is_i$

Main Results

- Superposition/recursive formula for $\tau_N(\alpha)$:

$$\tau_{N+1;n_i+1}\tau_{N-1;n_i} - \tau_{N+1;n_i}\tau_{N-1;n_i+1} = p_i\tau_{N;n_i}\tau_{N;n_i+1} \quad (13)$$

- Properties of $\tau(\alpha)$
  - $\tau_{N+1}(\beta) = \tau_N(\alpha)$
  - $\tau_N(\alpha)$ is homogeneous with degree $\mathcal{D}[\tau_N] = \frac{N(N+1)}{2}$
  - $\tau_N(\alpha)$ depends only on $\{x_1, x_3, \cdots, x_{2N-1}\}$
  - $\tau_N(n, m)$ is positive in the first quadrant $\{n \geq 0, m \geq 0\}$ provided $p_i > 0$ and $\tau_N(0,0) > 0$; in construction, $\tau_N(0,0) > 0$ is guaranteed by successively choosing

$$(-1)^{N+1}\gamma_{2N-1} > -\frac{(2N-1)\tau_N(0,0)|_{\gamma_{2N-1}=0}}{\tau_{N-2}(0,0)}.$$

Main Results

- Solutions in terms of $\tau_N(\alpha)$:

$$\begin{aligned}
\mathrm{Q1}(0):&\quad v_{N+2}=\overline{\overline{f}}/f,\\
\mathrm{lpmKdV}:&\quad V_{N+2}=\overline{f}/f,\\
\mathrm{Q1}(\delta):&\quad u_{N+2}=\frac{\overline{\overline{f}}+\delta^2\underline{\underline{f}}}{f},\\
\mathrm{H3}(\delta):&\quad Z_{N+2}=(-1)^{\frac{n+m}{2}+\frac{1}{4}}\frac{\overline{f}+(-1)^{n+m}\delta\underline{f}}{f},\\
\mathrm{H1}:&\quad u_{N+2}=x_{-1}-\frac{\partial_{x_1}f}{f},\\
\mathrm{H2}:&\quad v_{N+2}=u_{N+2}^2(\mathrm{H1})-\partial_{x_1}u_{N+2}(\mathrm{H1}),
\end{aligned}$$

where $f \doteq \tau_N,\ \overline{f}=\tau_{N+1},\ \underline{f}=\tau_{N-1},\ n_1=n,\ n_2=m.$

Main Results

- Q2: Rational solutions in terms of $\tau_N(\alpha)$:

$$w_{N+2} = \frac{u_{N+2}^2}{\delta^2} + \frac{\overline{f} + \delta \underline{f}}{\overline{f} - \delta \underline{f}} \left(\frac{-\overline{\overline{f}}^2}{\delta^2 f^2} + \frac{2\overline{\overline{f}}^2 \underline{f} + 2 f^2 \overline{\overline{\overline{f}}}}{\delta f^2 \overline{f}} + \theta_{N+2}^{(0)} - \frac{2\delta \underline{\underline{f}}^2 \overline{f} + 2\delta f^2 \underline{\underline{\underline{f}}}}{f^2 \underline{f}} + \frac{\delta^2 \underline{\underline{f}}^2}{f^2} \right),$$

where $u_{N+2} = u_{N+2}(Q1(\delta))$ and $\theta_{N+2}^{(0)}$ is determined by

$$\theta_{N+2}^{(0)} - \widetilde{\theta}_{N+2}^{(0)} = \frac{a^2}{f^2 \widetilde{f}^2} \left(\widetilde{\underline{f}}^2 \overline{f}^2 - \underline{f}^2 \widetilde{\overline{f}}^2 \right) + \frac{2(\widetilde{\overline{\overline{f}}} \underline{\underline{f}} - \widetilde{\underline{\underline{f}}} \overline{\overline{f}})}{f \widetilde{f}}.$$

- NQC:

$$S(a,b) = \frac{1}{a+b} \left(1 - \frac{E_{-n_3} E_{-n_4} f}{f} \right) \Big|_{n_3 = n_4 = 0}$$

Main Results

- $\mathrm{Q3}_\delta$:

$$u = AF(a,b)\left[1-(a+b)S(a,b)\right] + BF(a,-b)\left[1-(a-b)S(a,-b)\right] + CF(-a,b)\left[1+(a-b)S(-a,b)\right] + DF(-a,-b)\left[1+(a+b)S(-a,-b)\right], \quad (14)$$

in which

$$F(a,b) = \left(\frac{P}{(p-a)(p-b)}\right)^n \left(\frac{Q}{(q-a)(q-b)}\right)^m, \quad (15)$$

and P, Q are defined by (3); A, B, C and D are constants subject to

$$AD(a+b)^2 - BC(a-b)^2 = -\frac{\delta^2}{16ab}. \quad (16)$$

- Degeneration of $\mathrm{Q3}_\delta$ [1]

[1] Based on Nijhoff, Atkinson, Hietarinta-JPA-2009. Extended to RS.

Main Results

- Rational solutions of ABS*:

$$
\begin{aligned}
\mathrm{H2}^*: &\quad v = -(\ln f)_{x_1 x_1}, \\
\mathrm{H3}^*(\delta): &\quad U = \frac{\overline{f}^2 - \delta^2 \underline{f}^2}{f^2}, \\
\mathrm{A1}^*: &\quad w = \left(\ln \frac{\overline{f}}{f}\right)_{x_1}, \\
\mathrm{Q2}^*: &\quad w = \frac{\overline{\overline{f}}\,\underline{f}}{\overline{f}\,\overline{f}}.
\end{aligned}
$$

Part III: Proof of Main Results

Consistent Triplet

Discrete (difference) system:

$$\widetilde{v} - v = aV\widetilde{V},$$
$$\widehat{v} - v = bV\widehat{V},$$

- Solvable for v: $(E_n - 1)(E_m - 1)v = (E_m - 1)(E_n - 1)v$:

$$a(V\widetilde{V} - \widehat{V}\widehat{\widetilde{V}}) - b(V\widehat{V} - \widetilde{V}\widehat{\widetilde{V}}) = 0 \qquad \text{(lpmKdV)}$$

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- Solvable for V: $\widehat{\widetilde{V}} = \widetilde{\widehat{V}}$: $(p = a^2, q = b^2)$

$$p(v - \widehat{v})(\widetilde{v} - \widehat{\widetilde{v}}) - q(v - \widetilde{v})(\widehat{v} - \widehat{\widetilde{v}}) = 0 \qquad \text{(lSKdV/CR)}$$

Consistent Triplet

- **Consistent triplet:**

$$\mathrm{Q1}(0): \quad a^2(v-\widehat{v})(\widetilde{v}-\widehat{\widetilde{v}})-b^2(v-\widetilde{v})(\widehat{v}-\widehat{\widetilde{v}})=0 \quad \text{(ISKdV/CR)}$$

$$\widetilde{v}-v=aV\widetilde{V}, \quad \widehat{v}-v=bV\widehat{V}, \qquad \text{(BT)}$$

$$\mathrm{H3}(0): a(V\widetilde{V}-\widehat{V}\widehat{\widetilde{V}})-b(V\widehat{V}-\widetilde{V}\widehat{\widetilde{V}})=0 \qquad \text{(lpmKdV)}$$

- **Statement:**
  - Solution pair (v, V) of (BT) solves Q1(0)(v) and lpmKdV(V).

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  - If (v, V) solves (BT), then $V_1 = v/V$ solves lpmKdV(V_1)
  - Recursive:

$$(v, V) \longrightarrow V_1 = v/V \longrightarrow v_1 \longrightarrow V_2 = v_1/V_1 \longrightarrow v_2 \longrightarrow \cdots$$

Theorem

For any solution pair (v_N, V_N) *of the (BT), define*

$$V_{N+1} = \frac{v_N}{V_N}, \qquad (17a)$$

which is a solution of (lpmKdV). Through the (BT), i.e.

$$\widetilde{v}_{N+1} - v_{N+1} = aV_{N+1}\widetilde{V}_{N+1}, \quad \widehat{v}_{N+1} - v_{N+1} = bV_{N+1}\widehat{V}_{N+1}, \qquad (17b)$$

v_{N+1} *is well defined and it solves (Q1(0)).*

Lemma

Define

$$v_{-N} = -\frac{1}{v_{N+1}}, \quad V_{-N} = (-1)^{N+1}\frac{1}{V_{N+2}}, \quad \text{for } N \geq 0. \qquad (18)$$

Then the iteration relation (17) *is valid for all* $N \in \mathbb{Z}$.

Rational solutions to Q1(0) and lpmKdV(H3(0))

$$v_1 = x_1, \quad V_1 = 1, \tag{19a}$$

$$v_2 = \frac{1}{3}x_1^3 - x_3, \quad V_2 = x_1, \tag{19b}$$

$$v_3 = \frac{1}{x_1}\left(\frac{1}{45}x_1^6 - \frac{1}{3}x_1^3 x_3 + x_1 x_5 - x_3^2\right), \quad V_3 = \frac{\frac{1}{3}x_1^3 - x_3}{x_1}, \tag{19c}$$

where

$$x_i = a^i n + b^i m + \gamma_i, \quad \gamma_i \in \mathbb{C}, \ (i = 1, 2, \cdots). \tag{20}$$

Theorem

The iteration relation (17) is meaningful in generating rational solutions from (19a) for Q1(0) and lpmKdV. These solutions are non-singular at least on quadrant $\{n \geq 0, m \geq 0\}$ if we take $a > 0, b > 0, V_N(0,0) > 0$.

To ABS: Rational Solutions via Bäcklund Transformations

$$
\begin{array}{ccccccc}
\mathrm{H2} & & \mathrm{Q1}(0) & \longrightarrow & \mathrm{Q1}(\delta) & \longrightarrow & \mathrm{Q2} \\
\updownarrow & & \updownarrow & & \downarrow & & \\
\mathrm{H1} & \longleftarrow & \mathrm{lpmKdV} & & \mathrm{A1}(\delta) & \longleftrightarrow & \mathrm{H3}(\delta)
\end{array}
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Fig.1 A map for generating rational solutions.

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$$
\begin{array}{ccccccc}
\mathrm{H2} & & \mathrm{Q1}(0) & \longrightarrow & \mathrm{Q1}(\delta) & \longrightarrow & \mathrm{Q2} \\
\updownarrow & & \updownarrow & & \downarrow & & \\
\mathrm{H1} & \longleftarrow & \mathrm{lpmKdV} & & \mathrm{A1}(\delta) & \longleftrightarrow & \mathrm{H3}(\delta)
\end{array}
$$

Fig.1 A map for generating rational solutions.

To ABS: Rational Solutions via Bäcklund Transformations

$$
\begin{array}{ccccccc}
\mathrm{H2} & & \mathrm{Q1}(0) & \longrightarrow & \mathrm{Q1}(\delta) & \longrightarrow & \mathrm{Q2} \\
\updownarrow & & \updownarrow & & \downarrow & & \\
\mathrm{H1} & \longleftarrow & \mathrm{lpmKdV} & & \mathrm{A1}(\delta) & \longleftrightarrow & \mathrm{H3}(\delta)
\end{array}
$$

Fig.1 A map for generating rational solutions.

Bilinear relation of V_N and v_N

Recursive BT:

$$V_{N+1} = \frac{v_N}{V_N}, \tag{21}$$

$$\widetilde{v}_{N+1} - v_{N+1} = aV_{N+1}\widetilde{V}_{N+1}, \quad \widehat{v}_{N+1} - v_{N+1} = bV_{N+1}\widehat{V}_{N+1}. \tag{22}$$

We express

$$V_N = \frac{P_{N-1}}{P_{N-2}} \tag{23a}$$

and it then follows from (21) that

$$v_N = \frac{P_N}{P_{N-2}}. \tag{23b}$$

From $V_2 = x_1$ we introduce "initial value"

$$P_0 = 1, \quad P_1 = x_1, \tag{24a}$$

and from (19) we find successively

$$P_2 = \frac{1}{3}x_1^3 - x_3, \tag{24b}$$

$$P_3 = \frac{1}{45}x_1^6 - \frac{1}{3}x_1^3 x_3 + x_1 x_5 - x_3^2, \tag{24c}$$

$$P_4 = \frac{1}{4725}x_1^{10} - \frac{1}{105}x_1^7 x_3 + \frac{1}{15}x_1^5 x_5 - x_1 x_3^3 - x_5^2 + x_1^2 x_3 x_5 - \frac{1}{3}x_1^3 x_7 + x_3 x_7. \tag{24d}$$

Viewing (23) as transformations, the BT (22) yields a bilinear form

$$\overline{\widetilde{P}}\,\underline{P} - \overline{P}\,\underline{\widetilde{P}} = aP\widetilde{P}, \tag{25a}$$

$$\overline{\widehat{P}}\,\underline{P} - \overline{P}\,\underline{\widehat{P}} = bP\widehat{P}, \tag{25b}$$

where $P \doteq P_N, \quad \overline{P} \doteq P_{N+1}, \quad \underline{P} \doteq P_{N-1}.$

Prove: Casoratian form of $P = \tau_N(\alpha)$

BT of lpKdV(H1):

$$(\tilde{u}-\bar{u})(\tilde{\bar{u}}-u)=a^{-2}-k^{-2}, \tag{26a}$$
$$(\hat{u}-\bar{u})(\hat{\bar{u}}-u)=b^{-2}-k^{-2}, \tag{26b}$$

BT for rational sol

$$(\tilde{u}-\bar{u})(\tilde{\bar{u}}-u)=a^{-2}, \tag{27a}$$
$$(\hat{u}-\bar{u})(\hat{\bar{u}}-u)=b^{-2}. \tag{27b}$$

Factorization:

$$\tilde{u}-\bar{u}=\frac{f\tilde{\bar{f}}}{a\tilde{f}\bar{f}}, \quad \tilde{\bar{u}}-u=\frac{\tilde{f}\bar{f}}{af\tilde{\bar{f}}}, \tag{28a}$$
$$\hat{u}-\bar{u}=\frac{f\hat{\bar{f}}}{b\hat{f}\bar{f}}, \quad \hat{\bar{u}}-u=\frac{\hat{f}\bar{f}}{bf\hat{\bar{f}}}. \tag{28b}$$

Bilinearization of (28):

Introduce

$$u = x_{-1} - \frac{g}{f}, \quad \overline{u} = x_{-1} - \frac{\overline{g}}{\overline{f}}, \tag{29}$$

by which we bilinearize (28) as

$$\overline{g}\widetilde{f} - \overline{f}\widetilde{g} + \frac{1}{a}(\overline{f}\widetilde{f} - \widetilde{\overline{f}} f) = 0, \tag{30a}$$

$$g\widetilde{\overline{f}} - f\widetilde{\overline{g}} - \frac{1}{a}(\overline{f}\widetilde{f} - \widetilde{\overline{f}} f) = 0, \tag{30b}$$

$$\overline{g}\widehat{f} - \overline{f}\widehat{g} + \frac{1}{b}(\overline{f}\widehat{f} - \widehat{\overline{f}} f) = 0, \tag{30c}$$

$$g\widehat{\overline{f}} - f\widehat{\overline{g}} - \frac{1}{b}(\overline{f}\widehat{f} - \widehat{\overline{f}} f) = 0. \tag{30d}$$

- $$\widetilde{\overline{\overline{f}}} f - \overline{\overline{f}} \widetilde{f} = \lambda_1(m, N) \widetilde{\overline{f}}\, \overline{f}, \quad \widehat{\overline{\overline{f}}} f - \overline{\overline{f}} \widehat{f} = \lambda_2(n, N) \widehat{\overline{f}}\, \overline{f}.$$

- $$\widetilde{\overline{\overline{f}}}f - \overline{\overline{f}}\widetilde{f} = \lambda_1(m,N)\widetilde{\overline{f}}\,\overline{f}, \quad \widehat{\overline{\overline{f}}}f - \overline{\overline{f}}\widehat{f} = \lambda_2(n,N)\widehat{\overline{f}}\,\overline{f}.$$

- $$\gamma(N) = \frac{\widetilde{\overline{\overline{f}}}f - \overline{\overline{f}}\widetilde{f}}{a\widetilde{\overline{f}}\,\overline{f}} = \frac{\widehat{\overline{\overline{f}}}f - \overline{\overline{f}}\widehat{f}}{b\widehat{\overline{f}}\,\overline{f}}.$$

- $$\widetilde{\overline{\overline{f}}}\,f-\overline{\overline{f}}\,\widetilde{f}=\lambda_1(m,N)\widetilde{\overline{f}}\,\overline{f},\quad \widehat{\overline{\overline{f}}}\,f-\overline{\overline{f}}\,\widehat{f}=\lambda_2(n,N)\widehat{\overline{f}}\,\overline{f}.$$

- $$\gamma(N)=\frac{\widetilde{\overline{\overline{f}}}\,f-\overline{\overline{f}}\,\widetilde{f}}{a\widetilde{\overline{f}}\,\overline{f}}=\frac{\widehat{\overline{\overline{f}}}\,f-\overline{\overline{f}}\,\widehat{f}}{b\widehat{\overline{f}}\,\overline{f}}.$$

- $$f_N(\alpha(1,0;l))=a^N f_{N-1}(\alpha(0,0;l))+O(a^{N-1})$$

- $$\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}=\lambda_1(m,N)\widetilde{\overline{f}}\,\overline{f},\quad \widehat{\overline{\overline{f}}}f-\overline{\overline{f}}\widehat{f}=\lambda_2(n,N)\widehat{\overline{f}}\,\overline{f}.$$

- $$\gamma(N)=\frac{\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}}{a\widetilde{\overline{f}}\,\overline{f}}=\frac{\widehat{\overline{\overline{f}}}f-\overline{\overline{f}}\widehat{f}}{b\widehat{\overline{f}}\,\overline{f}}.$$

- $$f_N(\alpha(1,0;l))=a^N f_{N-1}(\alpha(0,0;l))+O(a^{N-1})$$

- $$\gamma(N)=\lim_{a\to\infty}\frac{\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}}{a\widetilde{\overline{f}}\,\overline{f}}\Bigg|_{n=m=0}=1.$$

- $$\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}=\lambda_1(m,N)\widetilde{\overline{f}}\,\overline{f},\quad \widehat{\overline{\overline{f}}}f-\overline{\overline{f}}\widehat{f}=\lambda_2(n,N)\widehat{\overline{f}}\,\overline{f}.$$

- $$\gamma(N)=\frac{\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}}{a\widetilde{\overline{f}}\,\overline{f}}=\frac{\widehat{\overline{\overline{f}}}f-\overline{\overline{f}}\widehat{f}}{b\widehat{\overline{f}}\,\overline{f}}.$$

- $$f_N(\alpha(1,0;l))=a^N f_{N-1}(\alpha(0,0;l))+O(a^{N-1})$$

- $$\gamma(N)=\lim_{a\to\infty}\frac{\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}}{a\widetilde{\overline{f}}\,\overline{f}}\Bigg|_{n=m=0}=1.$$

- $$\widetilde{\overline{\overline{f}}}f-\overline{\overline{f}}\widetilde{f}=a\widetilde{\overline{f}}\,\overline{f},\quad \widehat{\overline{\overline{f}}}f-\overline{\overline{f}}\widehat{f}=b\widehat{\overline{f}}\,\overline{f}, \tag{31}$$

where $f=\tau_N(\alpha)$.

Superposition formula of $\tau_N(\alpha)$:

Theorem

The Casoratian $f = \tau_N(\alpha(n, m; l))$ *solves bilinear equation set*

$$\widetilde{\overline{\overline{f}}} f - \overline{\overline{f}}\widetilde{f} = a\widetilde{\overline{f}}\,\overline{f}, \tag{32a}$$

$$\widehat{\overline{\overline{f}}} f - \overline{\overline{f}}\widehat{f} = b\widehat{\overline{f}}\,\overline{f}. \tag{32b}$$

$P = \tau_N(\alpha(n, m; l))$ *provides a Casoratian form of solution to* (25). *By defining*

$$f_{-N} = (-1)^{[\frac{N}{2}]} f_{N-1}, \quad f_0 = 1, \tag{33}$$

one can consistently extend (32) *to* $N \in \mathbb{Z}$.

Generating BTs/Consistent triplet: Continued

- Many BTs can be found from:

  J. Atkinson, Bäcklund transformations for integrable lattice equations, JPA, 41 (2008) No.135202 (8 pp).

  J. Atkinson, M. Nieszporski, Multi-quadratic quad equations: Integrable cases from a factorized discriminant hypothesis, IMRN, (2014) No.15, 4215-40.

Generating BTs/Consistent triplet: Continued

We conduct a searching from the following system

$$h(u,\widetilde{u},p)=U\widetilde{U},\quad h(u,\widehat{u},q)=U\widehat{U}. \tag{34}$$

- Consistency w.r.t. U:

$$h(u,\widetilde{u},p)h(\widehat{u},\widehat{\widetilde{u}},p)-h(u,\widehat{u},q)h(\widetilde{u},\widehat{\widetilde{u}},q)=QP^2=0,$$

  where

$$Equ:\quad Q(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}};p,q)=0. \tag{35}$$

- where to meet the consistency w.r.t. u we require U satisfies

$$EqU:\quad F(U,\widetilde{U},\widehat{U},\widehat{\widetilde{U}};p,q)=0. \tag{36}$$

Consistent triplet: Equ, EQU, BT (34).

[2] Decompsition of quadrilateral equations [ABS].

Generating BTs/Consistent triplet: Continued

We start from a generic affine-linear polynomial

$$h(u, \widetilde{u}, p) = s_0(p) + s_1(p)u + s_2(p)\widetilde{u} + s_3(p)u\widetilde{u}, \qquad (37)$$

where $s_i(p)$ are functions of p.

Theorem

When $h(u, \widetilde{u}, p)$ in system (34) is defined by (37), then U-equation (36) is affine-linear if and only if either

$$h(u, \widetilde{u}, p) = s_0(p) + s_1(p)u + s_2(p)\widetilde{u} \qquad (38)$$

or (after a constant shift $u \to u - c$)

$$h(u, \tilde{u}, p) = s_0(p) + s_3(p)u\widetilde{u}. \qquad (39)$$

Theorem

For the system (34) *where h is affine-linear as given in* (38) *and* (39)*, consistent triplets: BT + two affine linear quadrilateral CAC equations: the exhausted results of case* (37) *are*

No.	BT(34)	u-equation	U-equation
1	$\frac{1}{p}(u-\widetilde{u}) = U\widetilde{U}$	Q1$(0; p^2, q^2)$	lpmKdV
2	$u+\widetilde{u}+p = U\widetilde{U}$	H2	H1$(2p, 2q)$
3	$\frac{1}{p}(u+\widetilde{u}) - \delta p = U\widetilde{U}$	A1$(\delta; p^2, q^2)$	H3$(\delta; 2p, 2q)$
4	$u\widetilde{u}+\delta p = U\widetilde{U}$	H3(δ)	H3$(-\delta)$
5	$\frac{1}{p}(u\widetilde{u}-1) = U\widetilde{U}$	(40)	H3(1) with $U \to U^{-1}$
6	$\frac{1}{\sqrt{1-p^2}}(1-pu\widetilde{u}) = U\widetilde{U}$	A2	A2$(\sqrt{1-p^2}, \sqrt{1-q^2})$

Table 1. *Affine linear consistent triplets*

$$q^2(u\widetilde{u}-1)(\widehat{\widetilde{u}}\widehat{\widetilde{\widetilde{u}}}-1) = p^2(u\widehat{u}-1)(\widetilde{u}\widehat{\widetilde{u}}-1), \text{ to Q1}(0; p^2, q^2) \text{ by } u \to u^{(-1)^{n+m}}. \quad (40)$$

Other cases:

No.	BT(34)	u-equation	U-equation
1	$\frac{1}{p}(\widetilde{u}-u)^2-\delta^2 p = U\widetilde{U}$	Q1(δ)	H3*(δ; $\frac{1}{p}$, $\frac{1}{q}$)
2	$\frac{1}{p}(\widetilde{u}+u)^2-\delta^2 p = U\widetilde{U}$	A1(δ)	H3*(δ; $\frac{1}{p}$, $\frac{1}{q}$)
3	$\frac{(pu\widetilde{u}-1)(u\widetilde{u}-p)}{(1-p^2)u\widetilde{u}} = U\widetilde{U}$	A2	A2*$\left(\frac{2(p^2+1)}{1-p^2}, \frac{2(q^2+1)}{1-q^2}\right)$
4	$(u\widetilde{u}+\delta p)u\widetilde{u} = U\widetilde{U}$	H3(δ)	H3*(δ; $\frac{4}{p^2}$, $\frac{4}{q^2}$)
5	$\frac{u\widetilde{u}-p}{pu\widetilde{u}-1} = U\widetilde{U}$	A2	A2
6	$\frac{u\widetilde{u}-p}{(pu\widetilde{u}-1)u\widetilde{u}} = U\widetilde{U}$	A2	A2*($2p^2-1, 2q^2-1$)

Table 2. BT(34) related to Q1(δ), A1(δ), H3(δ) and A2.

Generating BTs/Consistent triplet: Continued

Addition formulas:

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta), \tag{41a}$$

$$\sin(\alpha-\gamma)=\sin(\alpha)\cos(\gamma)-\cos(\alpha)\sin(\gamma). \tag{41b}$$

Introduce:

$$\alpha=an+bm+\alpha_0, \quad \beta=\widetilde{\alpha}, \quad \gamma=\widehat{\alpha}, \tag{42}$$

$$u=\cos(\alpha), \quad U=\sin(\alpha), \quad p=\sin(a), \quad q=\sin(b), \tag{43}$$

then (41) is written as

$$p=u\widetilde{U}-U\widetilde{u}, \quad q=u\widehat{U}-U\widehat{u}, \tag{44}$$

compatibility yields the lpmKdV:

$$p(U\widetilde{U}-\widehat{U}\widehat{\widetilde{U}})-q(U\widehat{U}-\widetilde{U}\widehat{\widetilde{U}})=0, \tag{45}$$

$$p(u\widetilde{u}-\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}-\widetilde{u}\widehat{\widetilde{u}})=0. \tag{46}$$

BTs from addition formulas:

No.	BT	u-equation	U-equation
1	$u\widetilde{U}-U\widetilde{u}=p$	lpmKdV	lpmKdV
2	$u\widetilde{u}(\widetilde{U}-U)=p$	lpmKdV	$\mathrm{Q1}(0;P_1,Q_1)$
3	$u^2\widetilde{U}-\widetilde{u}^2U=pu\widetilde{u}$	lpmKdV	$\mathrm{H3}^*(\delta=1;P_2,Q_2)$
4	$u\widetilde{u}+U\widetilde{U}=-\delta p$	$\mathrm{H3}(\delta)$	$\mathrm{H3}(\delta)$
5	$u\widetilde{u}(1+U\widetilde{U})=-\delta p$	$\mathrm{H3}(\delta)$	(47)
6	$(\delta p+u\widetilde{u})u\widetilde{u}=-U\widetilde{U}$	$\mathrm{H3}(\delta)$	$\mathrm{H3}^*(\delta;-P_2,-Q_2)$
7	$U\widetilde{U}(1+u\widetilde{u})^2=\delta^2p^2u\widetilde{u}$	(47)	$\mathrm{H3}^*(\delta;-P_2,-Q_2)$
8	$uU+\widetilde{u}\widetilde{U}=p(u\widetilde{U}+U\widetilde{u})$	$\mathrm{Q3}(0)$	$\mathrm{Q3}(0)$
9	$u\widetilde{u}(U+\widetilde{U})=p(u^2\widetilde{U}+\widetilde{u}^2U)$	$\mathrm{Q3}(0)$	$\mathrm{Q3}^*(0;P_3,Q_3)$
10	$pu\widetilde{u}(U+\widetilde{U})=u^2U+\widetilde{u}^2\widetilde{U}$	$\mathrm{Q3}(0)$	$\mathrm{Q3}^*(0;P_4,Q_4)$
11	$p^2u\widetilde{u}(U+\widetilde{U})^2=U\widetilde{U}(u+\widetilde{u})^2$	$\mathrm{Q3}^*(0;P_3,Q_3)$	$\mathrm{Q3}^*(0;P_4,Q_4)$
12	$u\widetilde{u}+U\widetilde{U}=p(1+u\widetilde{u}U\widetilde{U})$	A2	A2
13	$u\widetilde{u}+\frac{U\widetilde{U}}{u\widetilde{u}}=p(1+U\widetilde{U})$	A2	$\mathrm{A2}^*(P_4,Q_4)$
14	$u\widetilde{u}(1+U\widetilde{U})=p(1+u^2\widetilde{u}^2U\widetilde{U})$	A2	$\mathrm{A2}^*(P_3,Q_3)$
15	$u\widetilde{u}(1+U\widetilde{U})^2=p^2U\widetilde{U}(1+u\widetilde{u})^2$	$\mathrm{A2}^*(P_4,Q_4)$	$\mathrm{A2}^*(P_3,Q_3)$

Table 3-1. BTs with trigonometric functions.

No.	BT	u-equation	U-equation
16	$uU - \widetilde{u}\widetilde{U} = p(u\widetilde{u} - U\widetilde{U})$	lpmKdV	lpmKdV
17	$U - \widetilde{U} = p(u\widetilde{u} - \frac{U\widetilde{U}}{u\widetilde{u}})$	lpmKdV	Q3*(0;$-P_3$, $-Q_3$)
18	$u^2 U - \widetilde{u}^2\widetilde{U} = pu\widetilde{u}(1 - U\widetilde{U})$	lpmKdV	A2*($-P_4$, $-Q_4$)
19	$(u - \widetilde{u})^2 U\widetilde{U} = p^2 u\widetilde{u}(1 - U\widetilde{U})^2$	Q3*(0;$-P_3$, $-Q_3$)	A2*($-P_4$, $-Q_4$)
20	$\widetilde{u} - u = \frac{p(U\widetilde{U}-1)}{\widetilde{U}-U}$	Q1(1)	Q1(0)
21	$\widetilde{u} - u = \frac{-pU\widetilde{U}}{\widetilde{U}-U}$	Q1(0)	Q1(0)
22	$\widetilde{U}\widetilde{u} - Uu = \frac{-pU\widetilde{U}}{\widetilde{U}-U}$	Q2*	Q1(0)
23	$\widetilde{u} - u = pU\widetilde{U}$	Q1(0; P_1, Q_1)	lpmKdV
24	$\widetilde{u}\widetilde{U} - uU = pU\widetilde{U}$	lpmKdV	lpmKdV
25	$u + \widetilde{u} - \frac{\delta}{2}p^2 = pU\widetilde{U}$	A1(δ; $P_1/2$, $Q_1/2$)	H3(δ)
26	$u^2 - \widetilde{u}^2 = pu\widetilde{u}(U + \widetilde{U})$	lpmKdV	A1*(δ; $-P_2$, $-Q_2$)

Table 3-2. BTs with trigonometric functions.

No.	BT	u-equation	U-equation
1	$uU+\widetilde{u}\widetilde{U}=p(u\widetilde{U}+U\widetilde{u})$	Q3(0)	Q3(0)
2	$u\widetilde{u}(U+\widetilde{U})=p(u^2\widetilde{U}+\widetilde{u}^2U)$	Q3(0)	Q3*(0; P_3, Q_3)
3	$pu\widetilde{u}(U+\widetilde{U})=u^2U+\widetilde{u}^2\widetilde{U}$	Q3(0)	Q3*(0; P_4, Q_4)
4	$p^2u\widetilde{u}(U+\widetilde{U})^2=U\widetilde{U}(u+\widetilde{u})^2$	Q3*(0; P_3, Q_3)	Q3*(0; P_4, Q_4)
5	$u\widetilde{u}+U\widetilde{U}=p(1+u\widetilde{u}U\widetilde{U})$	A2	A2
6	$u\widetilde{u}+\frac{U\widetilde{U}}{u\widetilde{u}}=p(1+U\widetilde{U})$	A2	A2*(P_4, Q_4)

Table 4. BTs with elliptic functions.

$$p^2(1+w\widehat{w})(1+\widetilde{w}\widehat{\widetilde{w}})=q^2(1+w\widetilde{w})(1+\widehat{w}\widehat{\widetilde{w}}), \qquad (47)$$

$$P_1=p^2,\quad Q_1=q^2,$$
$$P_2=4p^{-2},\quad Q_2=4q^{-2},$$
$$P_3=2p^2-1,\quad Q_3=2q^2-1,$$
$$P_4=2p^{-2}-1,\quad Q_4=2q^{-2}-1.$$

NQC:

$$\frac{1+(p-a)S(a,b)-(p+b)\widetilde{S}(a,b)}{1+(q-a)S(a,b)-(q+b)\widehat{S}(a,b)} = \frac{1-(q+a)\widehat{\widetilde{S}}(a,b)+(q-b)\widetilde{S}(a,b)}{1-(p+a)\widehat{\widetilde{S}}(a,b)+(p-b)\widehat{S}(a,b)}, \tag{48}$$

where $S(a,b)=S(b,a)$ are functions of (n,m) with (a,b) being branch point parameters, p and q are spacing parameters of n and m, respectively.

The way to NQC

$$1+(p-a)S(a,b)-(p+b)\widetilde{S}(a,b)=\widetilde{V}(a)V(b), \tag{49a}$$

$$1+(q-a)S(a,b)-(q+b)\widehat{S}(a,b)=\widehat{V}(a)V(b), \tag{49b}$$

$$p-q+\widehat{w}-\widetilde{w}=\frac{1}{\widehat{\widetilde{V}}(a)}((p-a)\widehat{V}(a)-(q-a)\widetilde{V}(a)) \tag{49c}$$

$$=\frac{1}{V(a)}((p+a)\widetilde{V}(a)-(q+a)\widehat{V}(a)), \tag{49d}$$

$$p+q+w-\widehat{\widetilde{w}}=\frac{1}{\widetilde{V}(a)}((p-a)V(a)+(q+a)\widehat{\widetilde{V}}(a)) \tag{49e}$$

$$=\frac{1}{\widehat{V}(a)}((p+a)\widehat{\widetilde{V}}(a)+(q-a)V(a)), \tag{49f}$$

$$(p-q+\widehat{w}-\widetilde{w})(p+q+w-\widehat{\widetilde{w}})=p^2-q^2, \tag{49g}$$

together with assuming symmetric property $S(a,b)=S(b,a)$, where $V(a)$ is a function of (n,m) with a as a parameter, $V(b)=V(a)|_{a\to b}$, w is a function of (n,m) but independent of (a,b).

Bilinearisation and τ for rational solutions

The above system can be bilinearised by taking

$$S(a,b)=\frac{1}{a+b}(1-\frac{\theta}{f}),\quad V(a)=\frac{h}{f},\quad V(b)=\frac{s}{f},\quad w=\frac{g}{f}. \qquad (50)$$

In particular, for rational solutions (i.e. $\tau(\alpha)=|\widehat{N-1}|$), we have

$$f=\tau(\alpha),\quad S(a,b)=\frac{1}{a+b}\left(1-\frac{E_{-n_3}E_{-n_4}f}{f}\right)\Big|_{n_3=n_4=0}, \qquad (51a)$$

$$h=E_{-n_3}f,\quad s=E_{-n_4}f,\quad \theta=T_{-n_3}T_{-n_4}f,\quad g=\partial_{x_1}f, \qquad (51b)$$

where (recall $\psi=\prod_i(1-p_ik)^{n_i}$)

$$n_1=n,\ n_2=m,\ p_1=1/p,\ p_2=1/q,\ p_3=1/a,\ p_4=1/b.$$

Eq.(51) satisfy HM and semi-discrete HM:[3]

$$(p_i-p_j)f(E_{n_i}E_{n_j}E_{n_k}f)-(p_i+p_k)(E_{n_i}f)(E_{n_j}E_{n_k}f)+(p_j+p_k)(E_{n_j}f)(E_{n_i}E_{n_k}f)=0,$$

$$(p_i-p_j)[f(E_{n_i}E_{n_j}f)-(E_{n_i}f)(E_{n_j}f)]+(E_{n_i}f_{x_1})(E_{n_j}f)-(E_{n_i}f)(E_{n_j}f_{x_1})=0.$$

[3] JJC Nimmo, JPA-1997; Y. Shi, JJC Nimmo, DJZ, JPA-2014.

From NQC to Q3

Theorem

The solution of $\mathrm{Q3}_\delta$ *is formulated by*[4]

$$u = A\mathcal{F}(a,b)\left[1-(a+b)S(a,b)\right] + B\mathcal{F}(a,-b)\left[1-(a-b)S(a,-b)\right] \\ + C\mathcal{F}(-a,b)\left[1+(a-b)S(-a,b)\right] \\ + D\mathcal{F}(-a,-b)\left[1+(a+b)S(-a,-b)\right], \tag{52a}$$

in which $S(a,b)$ *satisfies* (49) *and symmetry* $S(a,b)=S(b,a)$,

$$\mathcal{F}(a,b) = \left(\frac{P}{(p-a)(p-b)}\right)^n \left(\frac{Q}{(q-a)(q-b)}\right)^m, \tag{53}$$

and P, Q *are defined by* (3)*;* A*,* B*,* C *and* D *are constants subject to*

$$AD(a+b)^2 - BC(a-b)^2 = -\frac{\delta^2}{16ab}. \tag{54}$$

[4] Based on Nijhoff, Atkinson, Hietarinta-JPA-2009. Extended to RS.

Degeneration of $Q3_\delta$

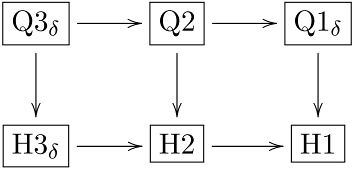

Fig.1 Degeneration relation

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Thank You!

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