Conditional symmetry preserving discretizations: the Boussinesq equation

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Plan of the talk

1. **Introduction: Lie point and conditional symmetries**

2. **Description of two dimentional lattices and PΔEs on it**

3. **Construction of nonlinear PDEs with conditional symmetries: the conditional symmetries of the Boussinesq equation**

4. **The conditionally discretized Boussinesq equation**

5. **Conclusions**

1 Introduction

Towards the end of the nineteenth century, Sophus Lie introduced the notion of **Lie group of symmetries to unify the study of the solutions of differential equations**.

- If an equation is invariant under a one-parameter Lie group of point transformations then we can simplify the construction of solutions by symmetry reducing the equation.
- Lie point symmetries map every solution of the system into another solution of the same system.

A partial differential equation (PDE) $\mathcal{E} = 0$ is invariant under a symmetry group if the corresponding infinitesimal symmetry generator $\hat{X}$ is such that

$$\mathrm{pr}\hat{X}\mathcal{E}\Big|_{\mathcal{E}=0} = 0, \tag{1.1}$$

$$\hat{X} = \xi(x, y, u)\partial_x + \eta(x, y, u)\partial_y + \phi(x, y, u)\partial_u. \tag{1.2}$$

A function $\mathcal{I}$ is an invariant of a symmetry if it is such that

$$\mathrm{pr}\hat{X}\mathcal{I} = 0. \tag{1.3}$$

Eq. (1.3) is a first order PDE which can be solved on the characteristic.

A PDE, invariant with respect to the infinitesimal generator $\hat{X}$, is written as:

$$\mathcal{E} = \mathcal{E}(\{\mathcal{I}_j\}) = 0, \qquad j = 0, 1, \cdots . \tag{1.4}$$

As often **the number of symmetries is limited**, one looks for more *symmetries*

- not always expressed in local form in terms of the dependent variable of the differential equations, (the potential symmetries introduced by Bluman et. al., the nonlocal symmetries by Vinogradov et. al.).

- not satisfying all the properties of a Lie group but just providing solutions (conditional symmetries).

What is a conditional symmetry?

Conditional symmetries were introduced by Bluman and Cole with the name *non-classical method* by adding an auxiliary first-order equation, build up in terms of the coefficients of the infinitesimal generator $\hat{X}$, namely

$$\mathcal{C} = \mathcal{C}(x, y, u, u_x, u_y) = \xi(x, y, u)u_x + \eta(x, y, u)u_y - \phi(x, y, u) = 0, \qquad (1.5)$$

Equation (1.5) will be determined together with the vector field $\hat{X}$. As

$$\mathrm{pr}\hat{X}\mathcal{C} = -(\xi_u u_x + \eta_u u_y - \phi_u)\,\mathcal{C}, \qquad (1.6)$$

we need just to apply the invariance condition

$$\mathrm{pr}\hat{X}\mathcal{E}\Big|_{\substack{\mathcal{E}=0\\ \mathcal{C}=0}} = 0. \qquad (1.7)$$

Eq. (1.7) gives nonlinear determining equations for ξ, η and ϕ which **provide at the same time the classical and non–classical symmetries**.

A PDE invariant under a conditional symmetry is given by

$$\mathcal{E}(\{\mathcal{I}_j\})\Big|_{\{\mathcal{C}=0\}} = 0, \qquad j = 0, 1, \cdots . \qquad (1.8)$$

Symmetry preserving discretization is the construction of **discrete equations which in the continuous limit go over to the given PDE and preserve its geometric structure**. As discrete equations defined on a fixed non transformable lattice have only dilation symmetries as its Lie point symmetries, a discrete equation with more general Lie symmetries will have a **transformable lattice**, will thus be described by what is called a *discrete scheme*.

In the following we will provide the **basis for the description of a difference scheme with a given symmetry** and then we will **discuss** a few **conditional symmetry preserving discretizations of the Boussinesq equation with different lattice properties**.

2 Description of two dimentional lattices and P∆Es on it

Geometry of 2-dimensional quadrilateral lattices

A sequence of points $P(x, y)$ in $\mathbb{R}^2$ will be characterized by two indices laying on a 2-dimensional quadrilateral lattice.

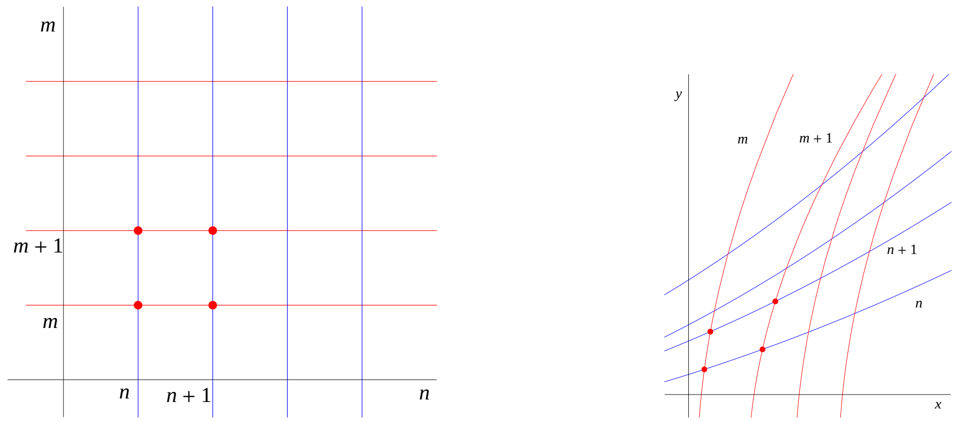

Figure 1: A 2-dimensional lattice of points,on the left the *computational* and on the right the *physical* coordinates.

Four points $(x_{n,m}, y_{n,m})$, $(x_{n+1,m}, y_{n+1,m})$, $(x_{n,m+1}, y_{n,m+1})$ and $(x_{n+1,m+1}, y_{n+1,m+1})$, will define a quadrilateral in $\mathbb{R}^2$.

$$x_{n+1,m} - x_{n,m} = h^x_{n,m}, \qquad x_{n,m+1} - x_{n,m} = \sigma^x_{n,m}, \tag{2.1}$$
$$y_{n+1,m} - y_{n,m} = \sigma^y_{n,m}, \qquad y_{n,m+1} - y_{n,m} = h^y_{n,m},$$

where $h^x_{n,m}$, $h^y_{n,m}$, $\sigma^x_{n,m}$ and $\sigma^y_{n,m}$ are given explicitly in Fig. 2.

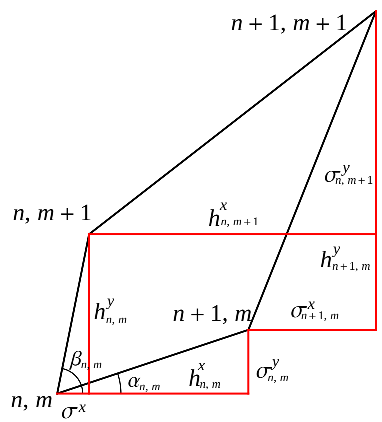

Figure 2: An elementary cell.

As the point $(x_{n+1,m+1}, y_{n+1,m+1})$ can be reached following two routes we have the consistency conditions,

$$h^{x}_{n,m+1} - h^{x}_{n,m} = \sigma^{x}_{n+1,m} - \sigma^{x}_{n,m}, \quad h^{y}_{n+1,m} - h^{y}_{n,m} = \sigma^{y}_{n,m+1} - \sigma^{y}_{n,m}.$$

The vectors defining two adjoint sides of the quadrilateral starting at the point of indices (n, m) must not be parallel,

$$h^{x}_{n,m} h^{y}_{n,m} - \sigma^{x}_{n,m} \sigma^{y}_{n,m} \neq 0. \tag{2.3}$$

Moreover the expression (2.3) must preserve the sign to avoid the folding of the lattice itself.

The quadrilateral will be a parallelogram if the sides are parallel:

$$\frac{h^{x}_{n,m+1}}{\sigma^{y}_{n,m+1}} = \frac{h^{x}_{n,m}}{\sigma^{y}_{n,m}} = a_n, \quad \frac{h^{y}_{n+1,m}}{\sigma^{x}_{n+1,m}} = \frac{h^{y}_{n,m}}{\sigma^{x}_{n,m}} = b_m. \tag{2.4}$$

Differential and difference operators

In the lowest order terms of the Taylor series expansion of $u_{n+1,m}$ and $u_{n,m+1}$:

$$\begin{aligned} u_{n+1,m} &= u_{n,m} + (x_{n+1,m} - x_{n,m})u_x + (y_{n+1,m} - y_{n,m})u_y + O(h^2) \\ u_{n,m+1} &= u_{n,m} + (x_{n,m+1} - x_{n,m})u_x + (y_{n,m+1} - y_{n,m})u_y + O(h^2), \end{aligned} \tag{2.5}$$

truncating it at the first order in h, we can substitute the first order partial derivatives u_x and u_y by the approximate discrete first order partial derivatives $\mathcal{D}_x u$ and $\mathcal{D}_y u$ and (2.5) reads

$$\begin{aligned} u_{n+1,m} &= u_{n,m} + h^x_{n,m}\mathcal{D}_x u + \sigma^y_{n,m}\mathcal{D}_y u \\ u_{n,m+1} &= u_{n,m} + \sigma^x_{n,m}\mathcal{D}_x u + h^y_{n,m}\mathcal{D}_y u. \end{aligned} \tag{2.6}$$

Inverting (2.6) we can approximate the partial differential equations on the lattice by introducing the following discrete partial differential derivatives in the x and y direction

$$\begin{aligned}
\mathcal{D}_x &= \frac{1}{h^x_{n,m}h^y_{n,m} - \sigma^x_{n,m}\sigma^y_{n,m}}(h^y_{n,m}\Delta_n - \sigma^y_{n,m}\Delta_m) \qquad (2.7)\\
\mathcal{D}_y &= \frac{1}{h^x_{n,m}h^y_{n,m} - \sigma^x_{n,m}\sigma^y_{n,m}}(-\sigma^x_{n,m}\Delta_n + h^x_{n,m}\Delta_m)
\end{aligned}$$

with $T_n f_{n,m} = f_{n+1,m}, \quad \Delta_n = T_n - 1$, and $T_m f_{n,m} = f_{n,m+1}, \quad \Delta_m = T_m - 1$. The functions $h^x_{n,m}$, $h^y_{n,m}$, $\sigma^x_{n,m}$ and $\sigma^y_{n,m}$ are such that

$$h^x_{n,m+1} - h^x_{n,m} = \sigma^x_{n+1,m} - \sigma^x_{n,m}, \quad h^y_{n+1,m} - h^y_{n,m} = \sigma^y_{n,m+1} - \sigma^y_{n,m} \qquad (2.8)$$

and

$$h^x_{n,m}h^y_{n,m} - \sigma^x_{n,m}\sigma^y_{n,m} \lesseqgtr 0. \qquad (2.9)$$

Any partial differential equation in two dependent variables can be constructed in term of the partial difference operators (2.7) in the directions x and y, provided the conditions (2.8, 2.9) are satisfied.

Schemes for partial difference equations

In $\mathbb{R}^2$ we limit ourselves to a scheme of six points (n,m), $(n+1,m)$, $(n,m+1)$, $(n+2,m)$, $(n,m+2)$, $(n+1,m+1)$, the minimum number of points necessary to get all partial second derivatives as first order approximations. The variables x, y and $u(x,y)$ in all points correspond to 18 data, 12 related to the independent variables and 6 to the dependent one. Having 12 data for the independent variables we can construct from them 10 differences

$$
\begin{aligned}
x_{0,0},\ y_{0,0},\qquad & h^x_{0,0}=x_{1,0}-x_{0,0},\ h^y_{0,0}=y_{0,1}-y_{0,0},\ \sigma^x_{0,0}=x_{0,1}-x_{0,0},\ \sigma^y_{0,0}=y_{1,0}-y_{0,0},\\
& h^x_{1,0}=x_{2,0}-x_{1,0},\ h^y_{0,1}=y_{0,2}-y_{0,1},\ \sigma^x_{0,1}=x_{0,2}-x_{0,1},\ \sigma^y_{1,0}=y_{2,0}-y_{1,0},\\
& h^x_{0,1}=x_{1,1}-x_{0,1},\quad h^y_{1,0}=y_{1,1}-y_{1,0},
\end{aligned}
\tag{2.10}
$$

From the values of the dependent variables in the 6 points we can calculate the 6 quantities

$$
u_{0,0},\ D_x u_{0,0},\ D_y u_{0,0},\ [D_x]^2 u_{0,0},\ [D_y]^2 u_{0,0},\ D_x D_y u_{0,0}. \tag{2.11}
$$

$D_yD_xu_{0,0}$ is not independent from the 6 quantities (2.11). It can be written in term of (2.10) and (2.11). For a generic lattice we have:

$$\begin{aligned} D_yD_xu_{0,0} = &-D_xD_xu_{0,0}\frac{\left(h^x_{0,0}-h^x_{0,1}\right)\left(h^x_{0,0}-\sigma^x_{0,0}\right)}{h^x_{0,1}h^y_{0,0}+h^x_{0,0}\sigma^y_{0,0}-h^x_{0,1}\sigma^y_{0,0}-\sigma^x_{0,0}\sigma^y_{0,0}}+ \\ &+D_yD_yu_{0,0}\frac{\left(h^y_{0,0}-h^y_{1,0}\right)\left(h^y_{0,0}-\sigma^y_{0,0}\right)}{h^x_{0,1}h^y_{0,0}+h^x_{0,0}\sigma^y_{0,0}-h^x_{0,1}\sigma^y_{0,0}-\sigma^x_{0,0}\sigma^y_{0,0}}+ \\ &+D_xD_yu_{0,0}\frac{h^x_{0,0}h^y_{1,0}+h^y_{0,0}\sigma^x_{0,0}-h^y_{1,0}\sigma^x_{0,0}-\sigma^x_{0,0}\sigma^y_{0,0}}{h^x_{0,1}h^y_{0,0}+h^x_{0,0}\sigma^y_{0,0}-h^x_{0,1}\sigma^y_{0,0}-\sigma^x_{0,0}\sigma^y_{0,0}} \end{aligned} \tag{2.12}$$

Formulas (2.10–2.12) can be simplified if we require the validity of the Clairaut–Schwarz–Young theorem i.e. $D_xD_yu_{0,0}=D_yD_xu_{0,0}$, i.e.

$$\begin{aligned} \sigma^x_{n,m} &= \sigma^x_{n+1,m} \equiv \sigma^x_m, \qquad & h^x_{n,m} &= h^x_{n,m+1} \equiv h^x_n, \\ \sigma^y_{n,m} &= \sigma^y_{n,m+1} \equiv \sigma^y_n, \qquad & h^y_{n,m} &= h^y_{n+1,m} \equiv h^y_m. \end{aligned} \tag{2.13}$$

Using the operators D_x and D_y given in (2.7) we can transform the standard *discrete* prolongation

$$\mathrm{pr}\hat{X}_{n,m} = \hat{X}_{n,m} + \hat{X}_{n+1,m} + \hat{X}_{n+2,m} + \hat{X}_{n,m+1} + \hat{X}_{n,m+2} + \hat{X}_{n+1,m+1} \tag{2.14}$$

of the vector field

$$\hat{X}_{n,m} = \xi_{n,m}\partial_{x_{n,m}} + \tau_{n,m}\partial_{y_{n,m}} + \phi_{n,m}\partial_{u_{n,m}} \tag{2.15}$$

to the new set of independent variables (2.10, 2.11). We get

$$\begin{aligned}
\mathrm{pr}\hat{X}_{n,m} \quad &= \quad \hat{X}_{n,m} + \sum_{(i,j)=0,1} \Big[\eta^{(x)}_{n+i,m+j}\partial_{h^x_{n+i,m+j}} + \chi^{(x)}_{n+i,m+j}\partial_{\sigma^x_{n+i,m+j}} + \eta^{(y)}_{n+i,m+j}\partial_{h^y_{n+i,m+j}} \\
&+ \quad \chi^{(y)}_{n+i,m+j}\partial_{\sigma^y_{n+i,m+j}}\Big] + \phi^{(1,x)}_{n,m}\partial_{D_x u_{n,m}} + \phi^{(1,y)}_{n,m}\partial_{D_y u_{n,m}} + \phi^{(2,xx)}_{n,m}\partial_{[D_x]^2 u_{n,m}} \\
&+ \quad \phi^{(2,xy)}_{n,m}\partial_{D_y D_x u_{n,m}} + \phi^{(2,yy)}_{n,m}\partial_{[D_y]^2 u_{n,m}},
\end{aligned} \tag{2.16}$$

where

$$
\begin{aligned}
\eta^{(x)}_{n+i,m+j} &= \xi_{n+1+i,m+j} - \xi_{n+i,m+j},\ \eta^{(y)}_{n+i,m+j} = \tau_{n+i,m+1+j} - \tau_{n+i,m+j},\\
\chi^{(x)}_{n+i,m+j} &= \xi_{n+i,m+1+j} - \xi_{n+i,m+j},\ \chi^{(y)}_{n+i,m+j} = \tau_{n+1+i,m+j} - \tau_{n+i,m+j},\\
\phi^{(1,x)}_{n,m} &= D_x\phi_{n,m} - D_x u_{n,m} D_x \xi_{n,m} - D_y u_{n,m} D_x \tau_{n,m}\\
\phi^{(1,y)}_{n,m} &= D_y\phi_{n,m} - D_x u_{n,m} D_y \xi_{n,m} - D_y u_{n,m} D_y \tau_{n,m},\\
\phi^{(2,xx)}_{n,m} &= D_x\phi^{(1,x)}_{n,m} - [D_x]^2 u_{n,m} D_x \xi_{n,m} - D_y D_x u_{n,m} D_x \tau_{n,m},\\
\phi^{(2,xy)}_{n,m} &= D_x\phi^{(1,y)}_{n,m} - D_x D_y u_{n,m} D_x \xi_{n,m} - [D_y]^2 u_{n,m} D_x \tau_{n,m},\\
\phi^{(2,yy)}_{n,m} &= D_y\phi^{(1,y)}_{n,m} - D_x D_y u_{n,m} D_y \xi_{n,m} - [D_y]^2 u_{n,m} D_y \tau_{n,m}.
\end{aligned}
$$

Applying the infinitesimal generator $\hat{X}$ onto the Schwarzian condition we get that both functions $\xi_{n,m}$ and $\tau_{n,m}$ must satisfy the discrete wave equations

$$
\begin{aligned}
\xi_{n,m+1} - \xi_{n,m} - \xi_{n+1,m+1} + \xi_{n+1,m} &= 0, \\
\tau_{n,m+1} - \tau_{n,m} - \tau_{n+1,m+1} + \tau_{n+1,m} &= 0,
\end{aligned} \tag{2.17}
$$

3 Construction of nonlinear PDEs with conditional symmetries: the conditional symmetries of the Boussinesq equation

The Boussinesq equation

$$u_{yy} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0, \tag{3.1}$$

was introduced in 1871 by Boussinesq to describe the propagation of long waves in shallow water and it is of considerable physical and mathematical interest.

Lie symmetries: $D = x\partial_x + 2t\partial_t - 2u\partial_u$, $P_1 = \partial_x$, $P_0 = \partial_t$.

Conditional symmetries: If $\eta \neq 0$ in $\hat{X}$ (1.2) we can always put it equal to one. If $\eta = 0$ and $\xi \neq 0$, we can put $\xi = 1$.

The conditional symmetries of the Boussinesq equation for $\eta \neq 0$ were obtained by L. and Winternitz. The case $\eta = 0$ can be found in a review by Clarkson.

$$
\begin{aligned}
\hat{X}_1 &= \partial_y + y\partial_x - 2y\partial_u && (3.2)\\
\hat{X}_2 &= \partial_y - \frac{x}{y}\partial_x + \Big(\frac{2}{y}u + \frac{6}{y^3}x^2\Big)\partial_u && (3.3)\\
\hat{X}_3 &= \partial_y + \Big(-\frac{x}{y} + y^4\Big)\partial_x + \Big(\frac{2}{y}u + \frac{6}{y^3}x^2 - 2y^2x - 4y^7\Big)\partial_u && (3.4)\\
\hat{X}_4 &= \partial_y + \Big(\frac{x}{2y} + y\Big)\partial_x - \frac{1}{y}(u + 2x + 4y^2)\partial_u && (3.5)\\
\hat{X}_5 &= \partial_y + \frac{1}{2}\frac{\dot{\wp}}{\wp}(x + \beta_2 W)\partial_x - \Big[\frac{\dot{\wp}}{\wp}u + 3\dot{\wp}x^2 + \frac{\beta_2}{2}\Big(\frac{1}{\wp} + 12\dot{\wp}W\Big) && (3.6)\\
&+ \frac{\beta_2^2}{2}W\Big(\frac{1}{\wp} + 6\dot{\wp}W\Big)\Big]\partial_u, \quad W(y) = \int_0^y \frac{\wp(s)}{[\dot{\wp}(s)]^2}ds, \\
\hat{X}_6 &= \partial_x + \Big[\frac{2}{x}u + \frac{48}{x^3}\Big]\partial_u, && (3.7)\\
\hat{X}_7 &= \partial_x + \Big[-2xQ + c_1Q + c_2Q\int_0^y \frac{ds}{[Q(s)]^2}\Big]\partial_u, && (3.8)
\end{aligned}
$$

where $Q = \wp(y + c_3, 0, g_3)$ and $\wp$ is a Weierstrass elliptic function.

Conditional invariant Boussinesq associated to $\hat{X}_1 = \partial_y + y\partial_x - 2y\partial_u$

Symmetry variables:

$$z = x - \frac{1}{2}y^2, \quad v(x, y) = w(z) - y^2. \tag{3.9}$$

The Boussinesq equation in the symmetry variables (3.9) is:

$$w''' + ww' - w - 2z + A = 0, \tag{3.10}$$

where A is an integration constant. Eq. (3.10) can be integrated to P_{II}.

Invariants of $\hat{X}_1$:

$$\begin{aligned}
\mathcal{I}_0 &= -2x + y^2, \quad \mathcal{I}_1 = 2x + u, \quad \mathcal{I}_2 = u_x, \quad \mathcal{I}_3 = 2y + yu_x + u_y, \quad \mathcal{I}_4 = u_{xx}, \\
\mathcal{I}_5 &= yu_{xx} + u_{xy}, \quad \mathcal{I}_6 = u_{yy} + 2yu_{xy} + 2(y^2 - x)u_{xx}, \quad \mathcal{I}_7 = u_{xxx} \\
&\ldots \quad , \mathcal{I}_{11} = u_{xxxx}.
\end{aligned} \tag{3.11}$$

The condition is given by $\mathcal{I}_3 = 0$ i.e. $\mathcal{C} = 2y + yu_x + u_y = 0$. Boussinesq equation in terms of the invariants is:

$$\mathcal{I}_1\mathcal{I}_4 + \mathcal{I}_6 + \mathcal{I}_{11} + \mathcal{I}_2^2\Big|_{\mathcal{C}_x=0} = 0. \tag{3.12}$$

$$\begin{aligned} &u_{yy} + 2y(u_{xy} + yu_{xx}) + uu_{xx} + u_{xxxx} + (u_x)^2\Big|_{u_{xy}+yu_{xx}=0} = \\ &= u_{yy} + uu_{xx} + u_{xxxx} + (u_x)^2 = 0. \end{aligned} \tag{3.13}$$

Conditional invariant Boussinesq associated to $\hat{X}_2 = \partial_y - \frac{x}{y}\partial_x + \left(\frac{2}{y}u + \frac{6}{y^3}x^2\right)\partial_u$

Symmetry variables:

$$z = xy, \quad v(x,y) = w(z)y^2 - (\frac{x}{y})^2 \tag{3.14}$$

Reduced Boussinesq equation: $w'' + \frac{1}{2}w^2 = c_1 z + c_0 \quad c_1 \neq 0$ Painlevé I

Invariants $\hat{X}_2$:

$$\mathcal{I}_0 = xy, \quad \mathcal{I}_1 = x^2 u + \frac{x^4}{y^2}, \quad \mathcal{I}_2 = x^3 u_x + 2\frac{x^4}{y^2}, \tag{3.15}$$
$$\mathcal{I}_3 = \frac{x^2}{y}(y u_y - x u_x - 2u - 6\frac{x^2}{y^2}), \quad \mathcal{I}_4 = x^4 u_{xx} + 2\frac{x^4}{y^2},$$
$$\mathcal{I}_5 = \frac{x^3}{y}\left(y u_{xy} - x u_{xx} - 3u_x - 12\frac{x}{y^2}\right),$$
$$\mathcal{I}_6 = \frac{x^2}{y}\left(y u_{yy} - 2x u_{xy} + \frac{x^2}{y} u_{xx} - 4u_y + 6\frac{x}{y} u_x + 6\frac{u}{y} + 42\frac{x^2}{y^3}\right),$$
$$\mathcal{I}_7 = x^5 u_{xxx}, \ldots, \mathcal{I}_{11} = x^6 u_{xxxx}.$$

The condition is given by $\mathcal{I}_3 = 0$ i.e. $\mathcal{C} = y u_y - x u_x - 2u - 6\frac{x^2}{y^2} = 0$

$$\mathcal{I}_1\mathcal{I}_4 + \mathcal{I}_{11} + \mathcal{I}_2^2\Big|_{\{C=0\}} = x^6\left[u_{yy} + u u_{xx} + u_{xxxx} + (u_x)^2\right]. \tag{3.16}$$

Conditionally invariant Boussinesq associated to $\hat{X}_6 = \partial_x + \left[\frac{2}{x}u + \frac{48}{x^3}\right]\partial_u$

Symmetry variables:

$$z = y, \quad v(x,y) = w(z)x^2 - \frac{12}{x^2} \tag{3.17}$$

Reduced Boussinesq equation: $w'' + 6w^2 = 0 \qquad w = \mathcal{P}(z + c_1, 0, c_2)$.

Invariants $\hat{X}_6$:

$$\begin{aligned} &\mathcal{I}_0 = y, \qquad \mathcal{I}_1 = \frac{1}{x^4}\Big(12 + x^2 u\Big), \qquad \mathcal{I}_2 = \frac{u_y}{x^2}, \\ &\mathcal{I}_3 = \frac{1}{x^5}\Big(x^3 u_x - 2x^2 u - 48\Big), \qquad \mathcal{I}_4 = \frac{u_{yy}}{x^2}, \cdots, \\ &\mathcal{I}_{10} = \frac{1}{x^7}\Big(- 1440 - 24x^2 u + 18x^3 u_x + x^5 u_{xxx} - 6x^4 u_{xx}\Big). \end{aligned} \tag{3.18}$$

The condition is given by $\mathcal{I}_3 = 0$ i.e. $\mathcal{C} = u_x - \frac{2u}{x} - \frac{48}{x^3} = 0$

$$6\mathcal{I}_1^2 + \mathcal{I}_4\Big|_{\{C=0\}} = \frac{1}{x^2}\left[u_{yy} + uu_{xx} + u_{xxxx} + (u_x)^2\right], \tag{3.19}$$

4 The conditionally discretized Boussinesq equations

To **discretize the Boussinesq equation preserving the conditional symmetry** we, follow the continuous construction presented up above, **construct the invariant lattice** and the **same discrete invariants which appear in the continuous calculations**.

$\hat{X}_1 = \partial_y + y\partial_x - 2y\partial_u$ preserving discretization of the Boussinesq

Invariant lattice:

$$h^y_{i,j} = k, \; \sigma^y_{i,j} = 0, \; h^x_{i,j} = h, \; \sigma^x_{i,j} = ky \tag{4.1}$$

where h and k are two constant invariant spacing in the two independent directions and $x_{0,0} = x$, $y_{0,0} = y$ and $u_{0,0} = u$. The **lattice is non orthogonal** since σ^x depends on y but **with constant spacing and Schwarzian**.

On this lattice the two discrete derivatives are:

$$\mathcal{D}_x = \frac{1}{h}\Delta_n, \quad D_y = \frac{1}{k}\Delta_m - y\mathcal{D}_x. \tag{4.2}$$

The discrete invariants are:

$$\begin{aligned} \mathcal{I}_0 &= y^2 - 2x, \quad \mathcal{I}_1 = u + 2x, \quad \mathcal{I}_2 = \mathcal{D}_x u, \quad I_3 = 2y + \mathcal{D}_y u + y\mathcal{D}_x u, \\ \mathcal{I}_4 &= \mathcal{D}_x^2 u, \quad \mathcal{I}_6 = \mathcal{D}_y^2 u + y^2\mathcal{D}_x^2 u + 2y\mathcal{D}_x\mathcal{D}_y u, \quad \mathcal{I}_{11} = \mathcal{D}_x^4 u. \end{aligned} \tag{4.3}$$

The condition, like in the continuos, is just $C = \mathcal{I}_3 = 0 \quad \mathcal{D}_x C = \mathcal{D}_x\mathcal{D}_y u + y\,\mathcal{D}_x^2 u = 0$. The conditional symmetry $\hat{X}_1$ preserving discretization of the Boussinesq is then given by:

$$\begin{aligned} &(\mathcal{I}_1 + \mathcal{I}_0)\mathcal{I}_4 + \mathcal{I}_6 + \mathcal{I}_{11} + \mathcal{I}_2^2\Big|_{\mathcal{D}_x C=0} = \\ &= (y^2 + u)\mathcal{D}_x^2 u + \mathcal{D}_y^2 u + y^2\mathcal{D}_x^2 u + 2y\mathcal{D}_x\mathcal{D}_y u + \mathcal{D}_x^4 u + (\mathcal{D}_x u)^2\Big|_{\mathcal{D}_x\mathcal{D}_y u + y\,\mathcal{D}_x^2 u = 0} = \\ &= \mathcal{D}_y^2 u + u\mathcal{D}_x^2 u + \mathcal{D}_x^4 u + (\mathcal{D}_x u)^2 = 0, \end{aligned} \tag{4.4}$$

which, as a difference equation, reads:

$$\frac{1}{k^2}\Big[u_{0,2} - 2u_{0,1} + u\Big] - \frac{2y}{hk}\Big[u_{1,1} - u_{1,0} - u_{0,1} + u\Big] + \frac{y}{h^2}\Big[y\Big(u_{2,0} - 2u_{1,0} + u\Big) \quad (4.5)$$
$$+k\Big(u_{2,0} - u_{1,0}\Big)\Big] + \frac{u}{h^2}\Big[u_{2,0} - 2u_{1,0} + u\Big] + \frac{1}{h^4}\Big[u_{4,0} - 4u_{3,0} + 6u_{2,0} - 4u_{1,0} + u\Big]$$
$$+\frac{1}{h^2}\Big[u_{1,0} - u\Big]^2 = 0,$$

i.e an 8 points relation connecting $u_{0,0}$ to $u_{1,0}$, $u_{2,0}$, $u_{3,0}$, $u_{4,0}$, $u_{0,1}$, $u_{1,1}$, $\mathrm{u}_{0,2}$.

This relation, together with the lattice on which it is defined, is invariant under $\hat{X}_1$. So we can **do symmetry reduction with respect $\hat{X}_1$ and find an ordinary difference equation which goes in the continuous limit to P_{II}**.

$\hat{X}_2 = \partial_y - \frac{x}{y}\partial_x + \left(\frac{2}{y}u + \frac{6}{y^3}x^2\right)\partial_u$ preserving discretization of the Boussinesq

Invariant lattice:

$$h^y_{i,j} = k,\ \sigma^y_{i,j} = 0,\ h^x_{i,j} = \frac{h}{y_{i,j}},\ \sigma^x_{i,j} = -\frac{k x_{i,j}}{y_{i,j+1}} \tag{4.6}$$

where h and k are two constant related to the invariant spacing in the two independent directions and $x_{0,0} = x$, $y_{0,0} = y$ and $u_{0,0} = u$. The **lattice is non orthogonal** but with **non constant spacing in the x direction and non Schwarzian**.

On this lattice the two discrete derivatives are:

$$\mathcal{D}_x = \frac{y_{i,j}}{h}\Delta_i, \quad D_y = \frac{1}{k}\Delta_j + \frac{x_{i,j}}{y_{i,j+1}}\mathcal{D}_x. \tag{4.7}$$

Discrete invariants:

$$\begin{aligned}
\mathcal{I}_1 &= \frac{u}{y^2} + \frac{x^2}{y^4}, \quad \mathcal{I}_2 = x^3 \mathcal{D}_x u + 2\,\frac{x^4}{y^2} + \frac{x^3 h}{y^3}, \quad \mathcal{I}_4 = \frac{\mathcal{D}_x^2}{y^4} + 2\,y^{-6}, \\
I_3 &= -\frac{y^2 x^3 \mathcal{D}_x u}{(y+k)^3} + \frac{y^2 x^2\, duy}{(y+k)^2} - \frac{(k+2\,y)\,x^2 u}{(y+k)^2} \\
&- \frac{x^4\,(k+2\,y)\left(k^2+ky+y^2\right)\left(k^2+3\,ky+3\,y^2\right)}{y^2\,(y+k)^6}, \quad \mathcal{I}_{11} = \frac{\mathcal{D}_x^4 u}{y^6}.
\end{aligned} \tag{4.8}$$

The condition, like in the continuous, is just $C = \frac{(y+k)^2}{x^2\,y}\mathcal{I}_3$ $\quad \mathcal{Q} = y^2 \mathcal{D}_y C + \frac{x\,(2\,k^2+2\,k\,y+y^2)}{y+2\,k}\,\mathcal{D}_x C + y\,C$. The conditional symmetry $\hat{X}_2$ preserving discretization of the Boussinesq is then given by:

$$\mathcal{I}_1 \mathcal{I}_4 + \mathcal{I}_{11} + \mathcal{I}_2^2\Big|_{\mathcal{Q}=0} = 0 \tag{4.9}$$

i.e.

$$\frac{(\mathcal{D}_x u)^2}{y^6} + \frac{u\mathcal{D}_x^2 u}{y^6} + \frac{\mathcal{D}_x^4 u}{y^6} + \frac{(y+k)^3 x\mathcal{D}_y^2 u}{y^6 (xy+h)(k^2+ky+y^2)}$$
$$+ \frac{1}{y^9 (xy+h)(k^2+ky+y^2)(y+2k)}[-2x^2k^5 - 10k^4x^2y - 15k^3x^2y^2 - 13k^2x^2y^3$$
$$- 4kx^2y^4 + 12hk^3xy + 18hk^2xy^2 + 18hkxy^3 + 6hxy^4 + 4h^2k^3 + 6h^2k^2y + 6h^2ky^2$$
$$+ 2h^2y^3]\mathcal{D}_x u - \frac{k(y+k)x\mathcal{D}_y u}{y^6 (xy+h)(k^2+ky+y^2)} + \frac{x^2k^2 (y+k)\mathcal{D}_x\mathcal{D}_y u}{y^7 (xy+h)(k^2+ky+y^2)}$$
$$+ \frac{\left(-k^3x - k^2xy - 2kxy^2 + 2hk^2 + 2hky + 2hy^2\right) u}{y^8 (xy+h)(k^2+ky+y^2)} + \frac{1}{(y+2k)^4 (y+k)^3 y^{12} (k^2+ky+y^2)(xy+h)}$$
$$[-32k^{11}x^3y - 320k^{10}x^3y^2 - 1384k^9x^3y^3$$
$$- 3588k^8x^3y^4 - 6300k^7x^3y^5 - 7855k^6x^3y^6 - 6996k^5x^3y^7 - 4383k^4x^3y^8 - 1846k^3x^3y^9$$
$$- 469k^2x^3y^{10} - 54kx^3y^{11} - 16hk^{11}x^2 - 152hk^{10}x^2y - 516hk^9x^2y^2 - 902hk^8x^2y^3$$
$$- 890hk^7x^2y^4 - 431hk^6x^2y^5 + 40hk^5x^2y^6 + 163hk^4x^2y^7 + 58hk^3x^2y^8 - 13hk^2x^2y^9$$
$$- 12hkx^2y^{10} - 2hx^2y^{11} + 80h^2k^9xy + 480h^2k^8xy^2 + 1320h^2k^7xy^3 + 2200h^2k^6xy^4$$
$$+ 2445h^2k^5xy^5 + 1860h^2k^4xy^6 + 955h^2k^3xy^7 + 315h^2k^2xy^8 + 60h^2kxy^9 + 5h^2xy^{10}$$
$$+ 16h^3k^9 + 96h^3k^8y + 264h^3k^7y^2 + 440h^3k^6y^3 + 489h^3k^5y^4 + 372h^3k^4y^5 + 191h^3k^3y^6$$
$$+ 63h^3k^2y^7 + 12h^3ky^8 + h^3y^9]$$

This equation, together with the lattice on which it is defined, is invariant under $\hat{X}_2$. So we can **do symmetry reduction with respect $\hat{X}_2$ and find an ordinary difference equation which goes in the continuous limit to P_I.**

$\hat{X}_6 = \partial_x + \left[\frac{2}{x}u + \frac{48}{x^3}\right]\partial_u$ preserving discretization of the Boussinesq

Invariant lattice:

$$h_{i,j}^y = k, \ \sigma_{i,j}^y = 0, \ h_{i,j}^x = h, \ \sigma_{i,j}^x = 0 \tag{4.10}$$

where h and k are two constant invariant spacing in the two independent directions and $x_{0,0} = x$, $y_{0,0} = y$ and $u_{0,0} = u$. The **lattice is orthogonal** with **constant spacing and Schwarzian**.

On this lattice the two discrete derivatives are:

$$\mathcal{D}_x = \frac{1}{h}\Delta_n, \quad D_y = \frac{1}{k}\Delta_m. \tag{4.11}$$

Discrete invariants:

$$\mathcal{I}_0 = y, \quad \mathcal{I}_1 = \frac{1}{x^4}\Big[12 + x^2\, u\Big], \quad I_4 = \frac{1}{x^2} D_y^2 u, \tag{4.12}$$

$$\begin{aligned}\mathcal{I}_3 &= \frac{1}{x^4(x+h)^4}\Big[6D_x u - 2x^5 u - 48x^3 + h(2x^5 D_x u - 5x^4 u - 72x^2) + \\ &+ h^2(x^4 D_x u - 4x^3 u - 48x) - h^3(x^2 u + 12)\Big].\end{aligned}$$

The discretized Boussinesq equation is obtained considering the invariants:

$$6\mathcal{I}_1^2 + \mathcal{I}_4 = 6\,\frac{u^2}{x^4} + 144\,\frac{u}{x^6} + \frac{864}{x^8} + \frac{D_{yy}\,u}{x^2}, \tag{4.13}$$

when the condition is taken from $\mathcal{I}_3$ to be

$$\mathcal{C} = D_x\, u - \frac{(h+2\,x)\,u}{x^2} - 48\,\frac{1}{x\,(x+h)^2} - 12\,\frac{h\left(h^2 + 4\,hx + 6\,x^2\right)}{x^4\,(x+h)^2} = 0 \tag{4.14}$$

We get the discretized Boussinesq equation from (3.1) by taking $\frac{2u^2}{x^4}$ from

$uD_x\mathcal{C}$, $\frac{4u^2}{x^4}$ from $\frac{\mathcal{C}^2}{x^2}+4\frac{u\mathcal{C}}{x^3}$ and $D_x^4 u$ from $D_x^4\mathcal{C}$. We get

$$\frac{1}{x^2}\Big\{D_y^2 u + (D_x u)^2 + D_x^4 u + uD_x^2 u + \tag{4.15}$$

$$-\frac{h}{x^{10}\,(x+h)^4\,(x+2\,h)^2\,(x+3\,h)^2\,(x+4\,h)^2}\Big[\left(576\,u^2x^4+13824\,x^2u+82944\right)h^{11}+$$

$$+\left(5856\,u^2x^5+140544\,x^3u+843264\,x\right)h^{10}+\left(23764\,u^2x^6+570336\,u\,x^4+3422016\,x^2\right)h^9+$$

$$+\left(52964\,u^2x^7+1298784\,x^5u+7958592\,x^3\right)h^8+(73373\,u^2x^8+1910712\,u\,x^6+$$

$$+12528720\,x^4)h^7+\left(66978\,u^2x^9+1938288\,u\,x^7+14637600\,x^5\right)h^6+(41347\,u^2x^{10}+$$

$$+1381752\,u\,x^8+13061232\,x^6)h^5+\left(17332\,u^2x^{11}+683424\,u\,x^9+8632512\,x^7\right)h^4+$$

$$+\left(4851\,u^2x^{12}+226536\,u\,x^{10}+3983040\,x^8\right)h^3+\left(866\,u^2x^{13}+47472\,u\,x^{11}+1189440\,x^9\right)h^2+$$

$$+\left(89\,u^2x^{14}+5640\,u\,x^{12}+203616\,x^{10}\right)h+4\,u^2x^{15}+288\,ux^{13}+14976\,x^{11}\Big]\Big\}=0.$$

This is an seven point relation connecting $u_{0,0}$ to u_{01}, $u_{0,2}$, $u_{1,0}$,u$_{2,0}$ $u_{3,0}$ and $u_{4,0}$.

This relation, together with the lattice on which it is defined, is invariant under $\hat{X}_6$. So we can **do symmetry reduction with respect $\hat{X}_6$ and find an ordinary difference equation which goes in the continuous limit to an elliptic function**.

5 Concluding notes

- We have shown that **we can construct conditional symmetry discretized equations**.
- In this case the **condition** is constructed by analogy of the continuous case in term of the corresponding discrete invariant and thus it is **no more related to the characteristic form of the vector field** as is given by a **discrete equation**.
- The **discrete equation may be very complicate** especially in the case of a non Schwarzian lattice as it contains many terms which in the continuous limit go to zero.
- A long standing problem is to understand the relevance of the form of the lattice in the property of the solution. How important is the **Schwarzianity** of the lattice for the regularity of the solution?

Thank you for your attention