Hierarchies of discrete $q\mathrm{P}_{\mathrm{II}}, q\mathrm{P}_{\mathrm{III}}, q\mathrm{P}_{\mathrm{IV}}$ and their properties

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1. Objectives

To extend discrete q-Painlevé II, III, IV equations ($q\mathrm{P}_{\mathrm{II}}, q\mathrm{P}_{\mathrm{III}}, q\mathrm{P}_{\mathrm{IV}}$) [1] in higher dimensions and study some of their properties such as Lax pairs, Bäcklund transformations, special solutions, symmetry groups and solutions of the associated linear problems.

2. Method

We use periodic/geometric reductions of the discrete modified Koterweg – de Vries (mKdV) equation which is consistent around the cube (CAC).

3. Hierarchies of $q\mathrm{P}_{\mathrm{II}}, q\mathrm{P}_{\mathrm{III}}$

Let $\widetilde{T}$, $\widehat{T}$ and $\overline{T}$ be operators acting on parameters

$$\widetilde{T} : (b_0, b_1, \dots, b_{k-2}, b_{k-1}, \lambda, q, x) \to (b_1, b_2, \dots, b_{k-1}, qb_0, \lambda, q, x), \tag{1a}$$
$$\widehat{T} : (b_0, b_1, \dots, b_{k-2}, b_{k-1}, \lambda, q, x) \to (b_2, b_3, \dots b_{k-1}, b_0 q, b_1 q, \lambda, q, x), \tag{1b}$$
$$\overline{T} : (b_0, b_1, \dots b_{k-1}, \lambda, q, x) \to (b_0, b_1, \dots, b_{k-1}, q\lambda, q, x). \tag{1c}$$

and variables

$$\begin{cases} \widetilde{T} : (h_0, h_1, \dots, h_{k-2}, h_{k-1}) \to (h_1, h_2, \dots, h_{k-1}, h_k) \\ h_0 h_k = \dfrac{\lambda^2 (1 + b_0 h_1 h_2 \dots h_{k-1})}{h_1 \dots h_{k-1} (b_0 + h_1 h_2 \dots h_{k-1})}, \end{cases} \tag{2}$$

$$\begin{cases} \widehat{T} : \quad (h_0, h_1, \dots, h_{k-2}, h_{k-1}) \to (h_2, h_3, \dots, h_k, h_{k+1}) \\ h_0 h_k = \dfrac{\lambda^2 \left(1 + b_0 \prod_{j=1}^{k-1} h_j\right)}{\left(\prod_{j=1}^{k-1} h_j\right) \left(b_0 + \prod_{j=1}^{k-1} h_j\right)}, \\ h_1 h_{k+1} = \dfrac{\lambda^2 \left(1 + b_1 \prod_{j=2}^{k} h_j\right)}{\left(\prod_{j=2}^{k} h_j\right) \left(b_1 + \prod_{j=2}^{k} h_j\right)}. \end{cases} \tag{3}$$

(2) and (3) are $q\mathrm{P}_{\mathrm{II}}, q\mathrm{P}_{\mathrm{III}}$ hierarchies respectively.

Example 1 *In* (**??**) $b_0 = t, b_1 = pt, q = p^2$, *and* $h_1 = h(t), h_0 = h(t/p), h_2 = h(pt)$, *we obtain* qP_{II} *equation*

$$h(pt)h(t/p) = \frac{\lambda^2}{h(t)} \frac{1 + th(t)}{t + h(t)}. \tag{4}$$

In (**??**), *we take* $q = p$, $b_0 = t$, $b_1 = bt$ *and* $h_0 = g(t), h_1 = f(t), h_2 = g(pt), h_3 = f(pt)$, *we obtain the discrete* qP_{III}.

4. Multi-parametric mKdV

Consider the multi-parametric mKdV equation:

$$Q(w, w_1, w_2, w_{12}; \alpha_i, \beta_i) = \alpha_1 w w_2 - \alpha_2 w_1 w_{12} - \beta_1 w w_1 + \beta_2 w_2 w_{12} = 0, \tag{5}$$

where $w := w_{l,m}$ and subscripts $1, 2$ denote the shifts in the l and m directions, respectively.
This equation has the following properties:

- **CAC**: assign parameters (α_1, α_2) and (β_1, β_2) in the l and m directions, respectively.

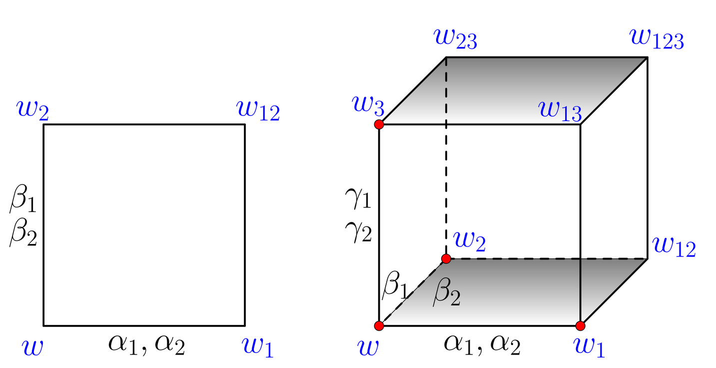

Figure 1: *CAC/ Bäcklund transformation*

- **Lax pair**: A Lax pair is given matrices L, M where

$$L(w, w_1, \alpha_1, \alpha_2, x) = \begin{pmatrix} -\frac{\alpha_1 w_0}{w_1} x & k_1 \\ k_2 & -\frac{\alpha_2 w_1}{w_0} x \end{pmatrix}, \tag{6}$$

  k_1, k_2 are constant and x is a spectral parameter, and M is obtained from L by replacing w_1 with w_2, α_i with β_i.

- **Bäcklund transformation**: replace w_3 with v, then all the side equations can be seen as a Bäcklund transformation between bottom equation and the top equation.

5. Properties of hierarchy of $q\mathrm{P}_{\mathrm{II}}$

5.1 Derive the hierarchy

We consider the $(k, -1)$ reduction of mKdV, i.e. we impose the periodicity condition $w_{l,m} = w_{l+k,m-1}$ (taking $u_n = w_{l,m}$, where $n = l + mk$).

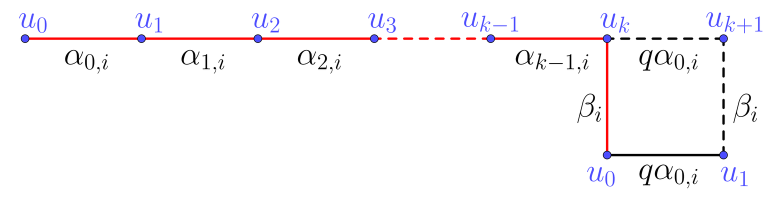

Figure 2: $(k, -1)$ *reduction (note that* $i = 1, 2$*)*

We obtain the map S:

$$(u_0, \dots, u_k, \alpha_{0,i}, \dots \alpha_{k-1,i}) \mapsto (u_1, \dots, u_{k+1}, \alpha_{1,i}, \dots \alpha_{k-1,i}, q\alpha_{0,i}),$$

where $i = 1, 2$ and u_{k+1} is solved from the equation $Q(u_0, u_1, u_k, u_{k+1}; q\alpha_{0,1}, q\alpha_{0,2}, \beta_1, \beta_2) = 0$. This implies

$$u_{k+1} = \frac{u_0 \left(q\alpha_{0,1} u_k - \beta_1 u_1\right)}{q\alpha_{0,2} u_1 - \beta_2 u_k}.$$

Introducing $h_i = u_{i+1}/u_i$, we obtain $q\mathrm{P}_{\mathrm{II}}$

$$h_0 h_1 \dots h_k = \frac{q\alpha_{0,1} h_1 h_2 \dots h_{k-1} - \beta_1}{q\alpha_{0,2} - \beta_2 h_1 h_2 \dots h_{k-1}}. \tag{7}$$

Taking $\alpha_{j,1} = ib_j\lambda, \alpha_{j,2} = ib_j/\lambda, \beta_1 = -iq\lambda$ and $\beta_2 = -iq/\lambda$, we obtain (2)

5.2 Lax pair

A Lax pair for $q\mathrm{P}_{\mathrm{II}}, q\mathrm{P}_{\mathrm{III}}, q\mathrm{P}_{\mathrm{IV}}$ takes the form

$$\Phi(qx, t) = A(x, t)\Phi(x, t), \tag{8}$$
$$T(\Phi(x, t)) = B(x, t)\Phi(x, t). \tag{9}$$

The compatibility condition between (8) and (9) is given by

$$T(A(x, t))B(x, t) = B(qx, t)A(x, t), \tag{10}$$

where x and t are spectral and Painlevé variables, respectively and T is given in (1).
Equation (6) gives

$$L(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x) = L(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, qx).$$

Thus, for $q\mathrm{P}_{\mathrm{II}}$, A is given by the monodromy matrix along the staircase, i.e.

$$A = M^{-1}(u_0, u_k, \beta_i, x) L(u_{k-1}, u_k, \alpha_{k-1,i}, x) \dots L(u_0, u_1, \alpha_{0,i}, x),$$

and

$$B = L(u_0, u_1, \alpha_{0,i}, x).$$

5.3 Bäcklund transformation

To find a Bäcklund transformation for $q\mathrm{P}_{\mathrm{II}}$, we consider the third direction which associated with variables v_i and parameters (γ_1, γ_2). We impose the condition $v_{l+k,m-1} = v_{l,m}/d$.

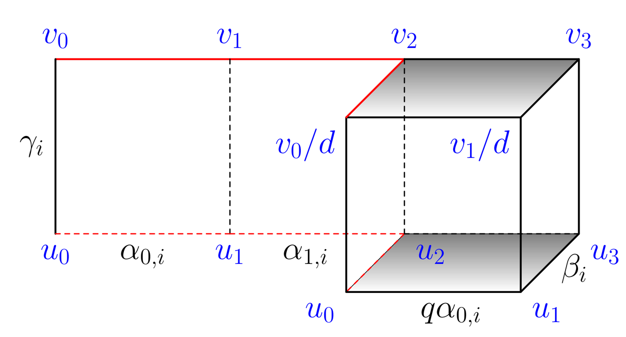

Figure 3: *Bäcklund transformation for* qP_{II}

A Bäcklund transformation between the bottom face and the top face (shaded faces) is given by the following system

$$\begin{aligned} Q(u_0, u_1, v_0, v_1, \alpha_{0,1}, \alpha_{0,2}, \gamma_1, \gamma_2) &= 0 \\ &\vdots \\ Q(u_{k-1}, u_k, v_{k-1}, v_k, \alpha_{k-1,1}, \alpha_{k-1,2}, \gamma_1, \gamma_2) &= 0 \\ Q(u_0, u_k, v_0/d, v_k, \beta_1, \beta_2, \gamma_1, \gamma_2) &= 0. \end{aligned}$$

Taking $\gamma_2 = 0$, we obtain a linear system of $v_0, v_1, \dots, v_k$.
We also want that this Bäcklund transformation is compatible with the map S, i.e $v_{i+1} = S(v_i)$; thus $d = q$.
Let $f_i = v_{i+1}/v_i$, and parameters are given in section 5.1 we obtain the Bäcklund transformation of equation (2) and the follwing equation

$$f_0 f_1 \dots f_k = \frac{\lambda^2 (qa_0 f_1 f_2 \dots f_{k-1} + 1)}{q (qf_1 f_2 \dots f_{k-1} + a_0)}. \tag{11}$$

Taking $g_i = q^{\frac{1}{k-1}} f_i$, we obtain a Bäcklund transformation $h_i \mapsto g_i$ of (2) where $\lambda^2 \mapsto q^{\frac{2}{k-1}} \lambda^2$.

Example 2 *For* $k = 2$, *we recreate the known Bäcklund transformation for* qP_{II}*:* $(h_i, \lambda^2) \mapsto (g_i, \lambda^2 q^2)$ *and*

$$g_1 = \frac{p (pth_2 + ph_1 h_2 + 1) \lambda^2}{h_1 \left(\lambda^2 p + th_2 + h_1 h_2\right)},$$

where $q = p^2, g_1 = g(t), h_1 = h(t), h_2 = h(pt)$.

5.4 Special solutions

We note that with $\lambda = \pm 1$, equation (2) has trivial solutions $h_i = 1$. Using a Bäcklund transformation in the previous section, we can give a rational solution for (2) with $\lambda^2 = q^{\frac{2n}{k-1}}$.

5.5 Symmetry group

- Start with an n-hypercube where $n = k + 1$ and vertex w is at the origin, and n-directions are denoted by subscripts $1, 2, \dots, n$.
- Geometric reduction: projecting this hypercube to the hyperplane $x_1 + x_2 + \dots + x_n = 0$ then w and $w_{123\dots n}$ project to the same point.
- Geometric reduction can be seen as a periodic reduction by taking $w_{12\dots i} = u_i$.

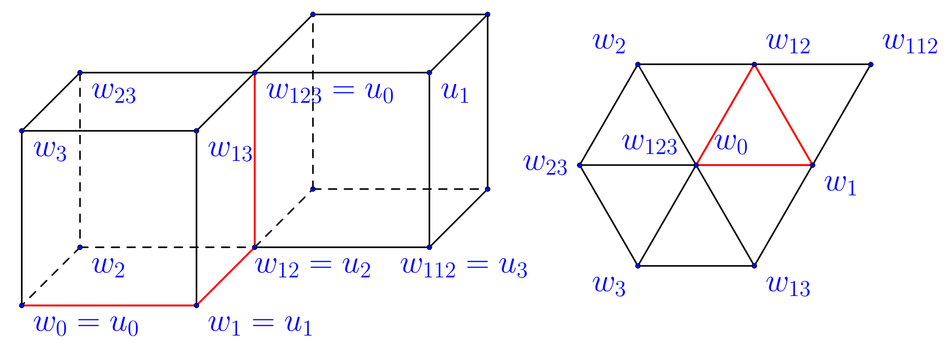

Figure 4: *Geometric reduction of a cube*

Let $b_i = a_0 a_1 \dots a_i$ for $i \le n - 2$ and $b_{n-1} = a_0 a_1 \dots a_{n-1} = q$.
The actions of generators of the $\widetilde{A}_{n-1}$ type on the parameters are given by

$$s_i(a_i) = 1/a_i, \; s_i(a_{i-1}) = a_{i-1} a_i, \; s_i(a_{i+1}) = a_{i+1} a_i,$$
$$s_i(a_j) = a_j, \text{ otherwise}$$

For $i \ge 1$, s_i is the transposition $(i, i + 1)$ of the permutation group S_n, i.e. we swap $w_{12\dots i}$ with $w_{12\dots(i-1)(i+1)}$. In addition, s_0 is the defined as the swap of between w and $w_{112\dots(n-1)}$.
The actions of these operators on h_i are given by the actions of them on w's.
We predict that the symmetry group is of the $(A_{n-1} + A_1)^{(1)}$ type, or a subgroup of it.

6. Hierarchy of $q\mathrm{P}_{\mathrm{III}}$

We note that $\widehat{T} = \widetilde{T}^2$. To find a hierarchy of $q\mathrm{P}_{\mathrm{III}}$, we just need to extend Figure 2 one more step to the right

7. Hierarchy of $q\mathrm{P}_{\mathrm{IV}}$

We find a hierarchy of $q\mathrm{P}_{\mathrm{IV}}$ using the so-called $(k, 0)$ reduction (after a scaling) as described below and taking $h_i = \frac{\lambda u_{i+1}}{u_i}$. The deformation operator $\overline{T}$ is defined in Section 3.

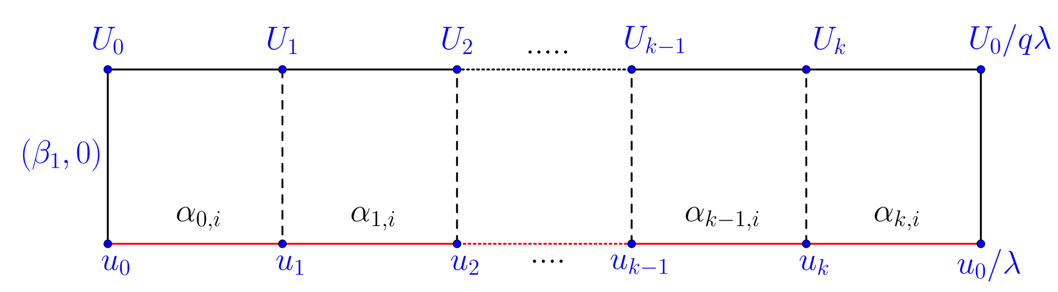

Figure 5: *The (k,0)-reduction*

8. Conclusions

- We have presented a method to derive hierarchies of $q\mathrm{P}_{\mathrm{II}}, q\mathrm{P}_{\mathrm{III}}, q\mathrm{P}_{\mathrm{IV}}$ explicitly.
- Lax pairs, Bäcklund transformations, special solutions (rational), symmetry groups also arise.
  We note that for the same order, these equations share the same Lax matrix A.
- Odd-order equations also arise, by one can reduce the order by one to get even-order equations.

References

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