Difference Moving Frames

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Based on joint work with: Gloria Marì Beffa (Madison, Wisconsin),

Applications with Linyu Peng (Waseda), Peter Hydon (Kent),

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This talk:

1. Moving frames - definitions, examples
   Why a discrete or difference moving frame?
   Joint v. discrete invariants.
   Difference moving frames arise naturally.
2. Difference Maurer Cartan invariants and their recurrence relations.
3. Pointers to the applications: difference variational methods, discrete integrable systems

Not this talk:

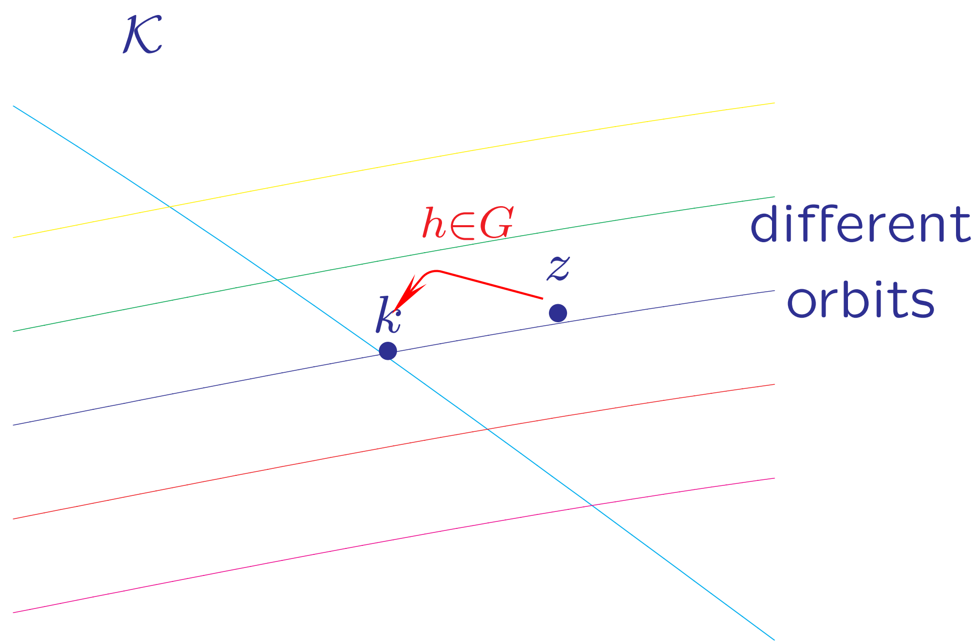
1. *From local to global*: Discrete moving frames on lattice based “manifolds” and global syzygies
2. *Multispace: a coherent framework for discrete and smooth frames*

Smooth moving frames can be used to describe complete, or generating, sets of differential invariants and their relations. There are excellent algorithms to manipulate quantities derived from moving frames in symbolic computation environments. Moving frames are flexible, to allow for ease of computation in specific applications, and they satisfy equations that allow them to be obtained numerically (if necessary).

Mansfield, A practical guide to the invariant calculus, Cambridge Univ. Press, 2010.

Fels and Olver, Kogan and Olver, Hubert, Hubert and Kogan, . . .

Moving Frame if $G \times M \to M$ is a regular, free action



$\rho : M \to G$ $\rho(z) = h$ is equivariant: $\rho(g \cdot z) = \rho(z) g^{-1}$

Calculation of a moving frame

Specify $\mathcal{K}$, the cross-section, as the locus of $\Phi(z) = 0$. Then solve $\Phi(g \cdot z) = 0$ for g. In practice, solve

$$\phi_j(g \cdot z) = 0, \qquad j = 1, \ldots, r = \dim(G)$$

for the r independent parameters describing g. Call the solution $\rho(z)$. Invoke Implicit Function Theorem. Unique solution yields

$$\rho(g \cdot z) = \rho(z) \cdot g^{-1}.$$

• local solutions only this way: but see Hubert and Kogan, FoCM **7** (2007) and J. Symb. Comp., **42** (2007).

Equivariance is the key to success. In particular, we obtain:

Invariants: The components of $I(z) = \rho(z) \cdot z$ are invariant.

$$I(g \cdot z) = \rho(g \cdot z) \cdot (g \cdot z) = \rho(z) g^{-1} g \cdot z = \rho(z) \cdot z.$$

If $I(z_i)$ are the canonical invariants for $z = (z_1, z_2, \ldots, z_n)$, and $F(z_1, z_2, \ldots, z_n)$ is an invariant, then we have the

Replacement rule,

$$\begin{aligned} F(z_1, z_2, \ldots, z_n) &= F(g \cdot z_1, g \cdot z_2, \ldots, g \cdot z_n) \\ &= F(g \cdot z_1, g \cdot z_2, \ldots, g \cdot z_n)|_{\text{frame}} \\ &= F(I(z_1), I(z_2), \ldots, I(z_n)) \end{aligned}$$

Also symbolic invariant differentiation formulae and much more.

Main expository example: $G = (\mathbb{R} \setminus \{0\}) \ltimes \mathbb{R}$ where the action is scaling and translation of $x \in \mathbb{R}$ given by

$$(\lambda, \epsilon) \cdot x = \tilde{x} = \lambda x + \epsilon$$

has a standard matrix representation

$$\begin{pmatrix} \tilde{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda & \epsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

Joint invariants example: $G = (\mathbb{R} \setminus \{0\}) \ltimes \mathbb{R}$ or scaling and translation:

$$u_k \mapsto \tilde{u}_k = \lambda u_k + \epsilon$$

Olver: Joint invariant signatures, FoCM 2001

With the normalisation equations

$$\Phi : \qquad \tilde{u}_0 = 1, \quad \tilde{u}_1 = 0$$

the frame is

$$\rho : \qquad \lambda = -\frac{1}{u_1 - u_0}, \qquad \epsilon = \frac{u_1}{u_1 - u_0}$$

Easy to see equivariance in standard matrix representation:

$$\begin{pmatrix} -\frac{1}{\tilde{u}_1 - \tilde{u}_0} & \frac{\tilde{u}_1}{\tilde{u}_1 - \tilde{u}_0} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{u_1 - u_0} & \frac{u_1}{u_1 - u_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu} & -\frac{\delta}{\mu} \\ 0 & 1 \end{pmatrix}.$$

The canonical invariants given by $I(z) = \rho(z) \cdot z$ are then

$$I(u_k) = (\lambda u_k + \epsilon)\,|_{\text{frame}} = -\frac{u_k - u_1}{u_1 - u_0}$$

From the computational point of view, for discrete curvature flows or discrete calculus of variations, such invariants are a disaster as they perform poorly under the shift operator. The problem is the frame fixes the base point.

Preferably, one would want something more natural, such as

$$I_k = \frac{u_{k+2} - u_{k+1}}{u_{k+1} - u_k}, \qquad \text{so that} \qquad I_k = S^k I_0.$$

This can be achieved with a frame that allows for a shifting base point.

Consider the discrete variational problem:

$$\mathcal{L}[u] = \sum \frac{1}{2} I_{n+2}^2 = \sum_n \frac{1}{2} \left(\frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} \right)^2$$

which is overtly invariant under

$$u_k \mapsto \widetilde{u}_k = \lambda u_k + \epsilon.$$

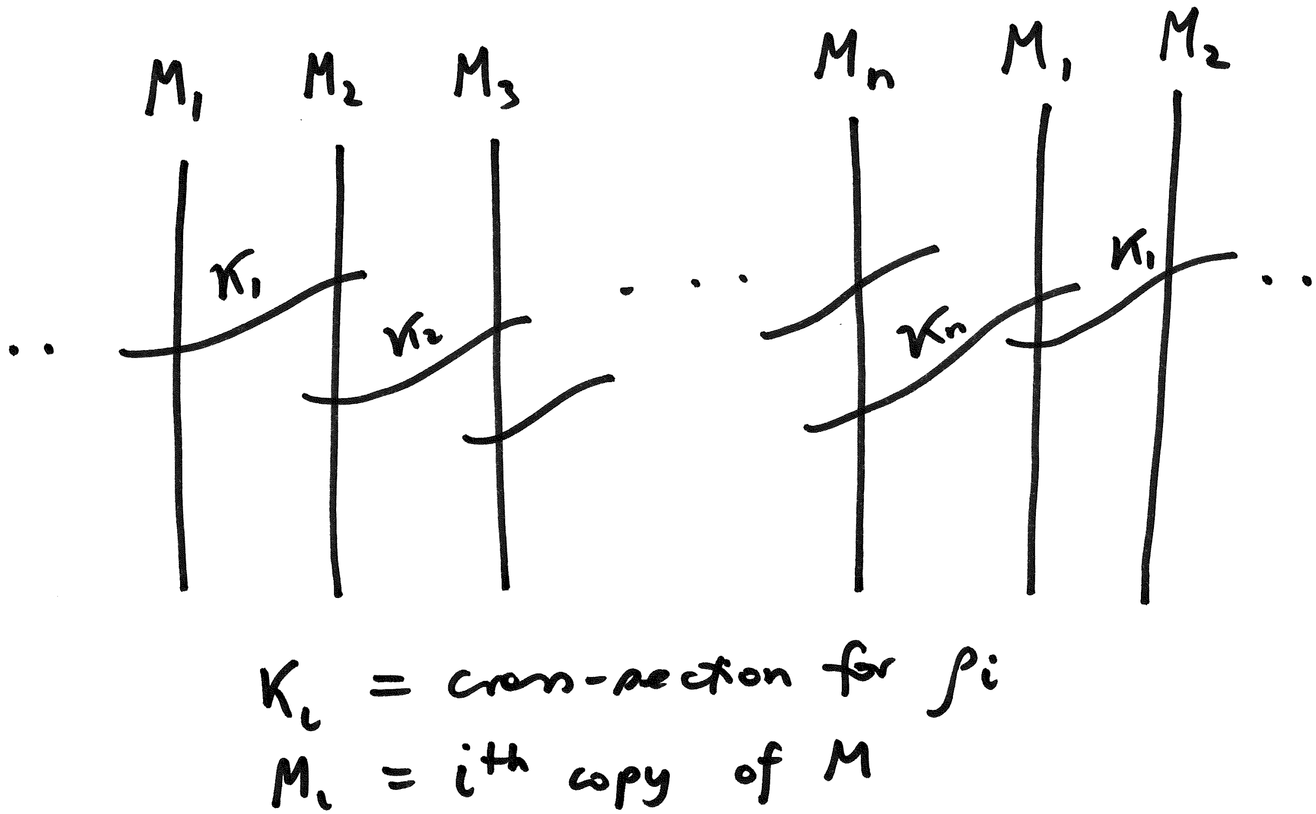
There are two conservation laws which can be written in the form

$$\mathbf{c} = A_n \mathbf{v}_n(I) = \begin{pmatrix} \dfrac{-1}{u_{n-1} - u_{n-2}} & 0 \\ \dfrac{-u_{n-1}}{u_{n-1} - u_{n-2}} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{I_{n+1}^2 - I_n^2}{I_n} \\ I_n^2 \end{pmatrix}$$

where each A_n is equivariant, that is, is a frame for each n.

Note: this is a moving frame free calcuation: the point is that indexed sets of frames arise naturally. In detail:

$$\begin{aligned}
\widetilde{A}_n &= \begin{pmatrix} \dfrac{-1}{\widetilde{u}_{n-1}-\widetilde{u}_{n-2}} & 0 \\ \dfrac{-\widetilde{u}_{n-1}}{\widetilde{u}_{n-1}-\widetilde{u}_{n-2}} & 1 \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{-1}{\lambda(u_{n-1}-u_{n-2})} & 0 \\ \dfrac{-\lambda u_{n-1}-\epsilon}{\lambda(u_{n-1}-u_{n-2})} & 1 \end{pmatrix} = \begin{pmatrix} 1/\lambda & 0 \\ \epsilon/\lambda & 1 \end{pmatrix} \begin{pmatrix} \dfrac{-1}{u_{n-1}-u_{n-2}} & 0 \\ \dfrac{-u_{n-1}}{u_{n-1}-u_{n-2}} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1/\lambda & 0 \\ \epsilon/\lambda & 1 \end{pmatrix} A_n
\end{aligned}$$



Picture of a discrete moving frame. If $\mathcal{K}_{i+1} = S(\mathcal{K}_i)$ call this a *difference* frame.

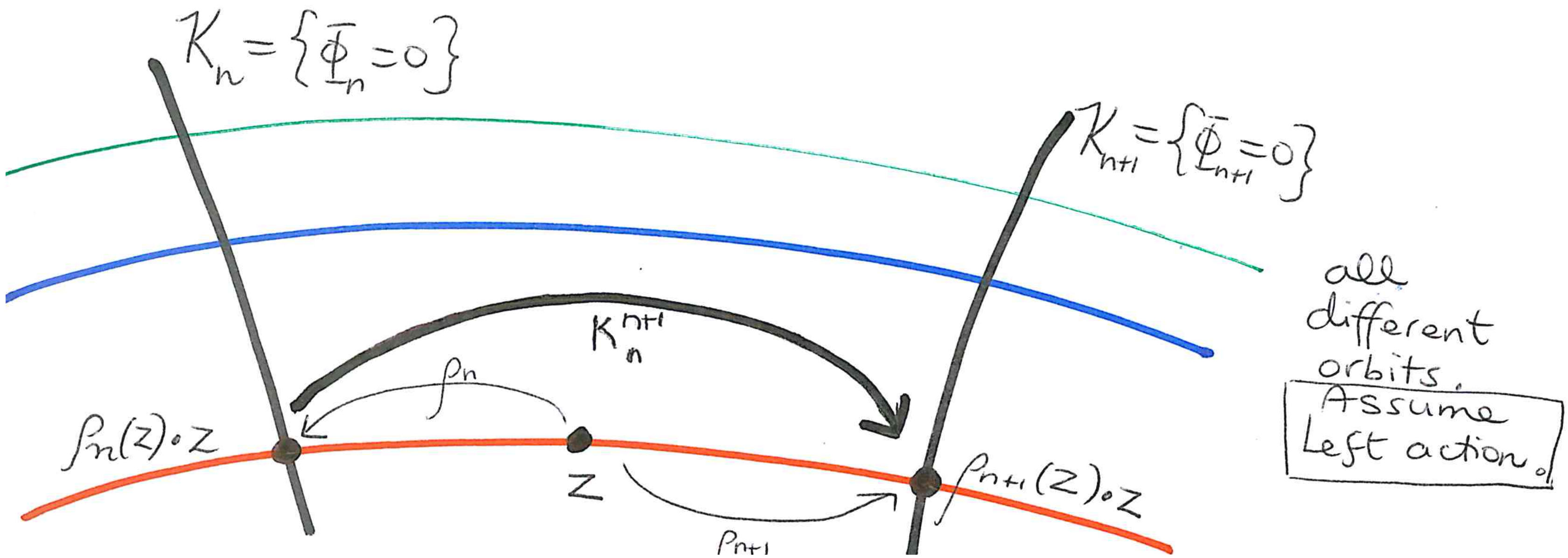
Since we have a sequence of moving frames, we have a plethora of invariants:

$$\rho_s \cdot z_r = I_r^s.$$

Need some canonical choices! For the differential case, the components of the Maurer Cartan matrices $\mathbf{K_i} = (\mathcal{D}_i \rho)\, \rho^{-1}$ are (almost) generating. Here, the components of

$$\mathbf{K}_r^{r+1} = \rho_{r+1} \rho_r^{-1}$$

are denoted the discrete Maurer Cartan invariants.
If $\mathcal{K}_{i+1} = S(\mathcal{K}_i)$ then $\rho_{r+1} = S(\rho_r)$ and we have *difference* Maurer Cartan invariants.



$$
\begin{aligned}
K_n^{n+1}(z) &= \rho_{n+1}(z)\rho_n(z)^{-1} = \rho_{n+1}(z)g^{-1}g\rho_n(z)^{-1} \\
&= \rho_{n+1}(g\cdot z)\rho_n(g\cdot z)^{-1} \\
&= K_n^{n+1}(g\cdot z) \quad \text{so } K_n^{n+1} \text{ is invariant} \\
&= K_n^{n+1}(\rho_n(z)\cdot z) \quad \text{setting } g = \rho_n(z) \\
&= \rho_{n+1}(\rho_n(z)\cdot z) \quad \text{as } \rho_n(\rho_n(z)\cdot z) = e
\end{aligned}
$$

In this way, action by the group element $K_n^{n+1}(z)$ provides a change of coordinates from one set of generating invariants to another.

For the scaling and translation example, we calculate a difference moving frame using

$$\Phi_n : \qquad \widetilde{u}_n = 1, \quad \widetilde{u}_{n+1} = 0$$

giving in the standard matrix representation,

$$\text{matrix rep}(\rho_n) = \begin{pmatrix} -\frac{1}{u_{n+1}-u_n} & \frac{u_{n+1}}{u_{n+1}-u_n} \\ 0 & 1 \end{pmatrix}$$

We have by definition $\rho_n \cdot u_{n+k} = I^n_{n+k}$ and then

$$K^n_{n+1} = \begin{pmatrix} \frac{u_{n+2}-u_{n+1}}{u_{n+1}-u_n} & -\frac{u_{n+2}-u_{n+1}}{u_{n+1}-u_n} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -I^n_{n+2} & I^n_{n+2} \\ 0 & 1 \end{pmatrix}$$

Without solving for the frame: the normalisation equations and definition of I^n_{n+k} in matrix form are:

$$\rho_n \begin{pmatrix} \dots & u_{n-1} & u_n & u_{n+1} & u_{n+2} & u_{n+3} & \dots \\ & 1 & 1 & 1 & 1 & 1 & \end{pmatrix} = \begin{pmatrix} \dots & I^n_{n-1} & 1 & 0 & I^n_{n+2} & I^n_{n+3} & \dots \\ & 1 & 1 & 1 & 1 & 1 & \end{pmatrix}$$

for all n, so that

$$\rho_{n+1} \begin{pmatrix} u_{n+1} & u_{n+2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

yielding

$$\begin{pmatrix} 0 & I^n_{n+2} \\ 1 & 1 \end{pmatrix} = \rho_n \rho_{n+1}^{-1} \rho_{n+1} \begin{pmatrix} u_{n+1} & u_{n+2} \\ 1 & 1 \end{pmatrix} = \underbrace{\rho_n \rho_{n+1}^{-1}}_{K^n_{n+1}} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

This shows we have, for all n,

$$\rho_n \rho_{n+1}^{-1} = K_{n+1}^n = \begin{pmatrix} -I_{n+2}^n & I_{n+2}^n \\ 0 & 1 \end{pmatrix}$$

as before, only this time we didn't solve for the frame and have used I_{n+2}^n symbolically: we don't need to know what it is in order to obtain K_n in terms of the symbolic invariants I_{n+k}^n.

Finally,

$$S(I_k^n) = S(\rho_n \cdot z_k) = \rho_{n+1} \cdot z_{k+1} = \rho_{n+1}\rho_n^{-1}\rho_n \cdot z_{k+1} = K_n^{n+1} I_{k+1}^n$$

gives the recurrence relations between the Maurer Cartan invariants and their shifts.

The theory of discrete moving frames and the symbolic calculus of difference invariants is developed in *ELM, Marí Beffa and Wang, FoCM, 2013; Marí Beffa and ELM, FoCM, 2018.*

Application 1: Calculus of Variations

Given a difference Lagrangian which is invariant under an r-dimensional Lie symmetry group, use the symbolic calculus of difference invariants to find the difference Euler Lagrange equations directly in terms of the invariants.

Further, it is proved Noether's difference laws can be written as

$$0 = \sum_i (S_i - \mathrm{id})\,(v_i(I)\mathcal{A}d(\rho_0))$$

1. where $\mathcal{A}d(\rho)$ is the (left) Adjoint representation of an equivariant map from $M^n \to G$
2. and $v_i(I)$ is a (row) vector of invariants,

proving the equivariance of the laws.

In 1-d problems, the laws give additional, algebraic equations for the frame, easing the integration problem for the extremals.

1. Solve the Euler Lagrange equations for the generating invariants, then use the recurrence relations to obtain all the needed additional $I^s_r = \rho_s \cdot z_r$.
2. Solve the algebraic equation $\mathbf{c} = v(I)\mathcal{A}d(\rho)$ and as necessary, the recurrence equation $\rho_{n+1} = K_n^{n+1}\rho_n$, for ρ_n. Note $v(I)$, K_n^{n+1} are known, since the invariants have been solved for.
3. Obtain z_r from the algebraic equation, $z_r = \rho_s^{-1} \cdot (I^s_r)$.

ELM, Rojo-Echeburúa, Hydon and Peng, arxiv.org/abs/1804.00317; ELM, Rojo-Echeburúa, arxiv.org/abs/1808.03606

The key to getting the finite difference approximation to match all the smooth laws, bypassing the Ge and Marsden theorem, seems to be, to match the smooth with the discrete moving frames.

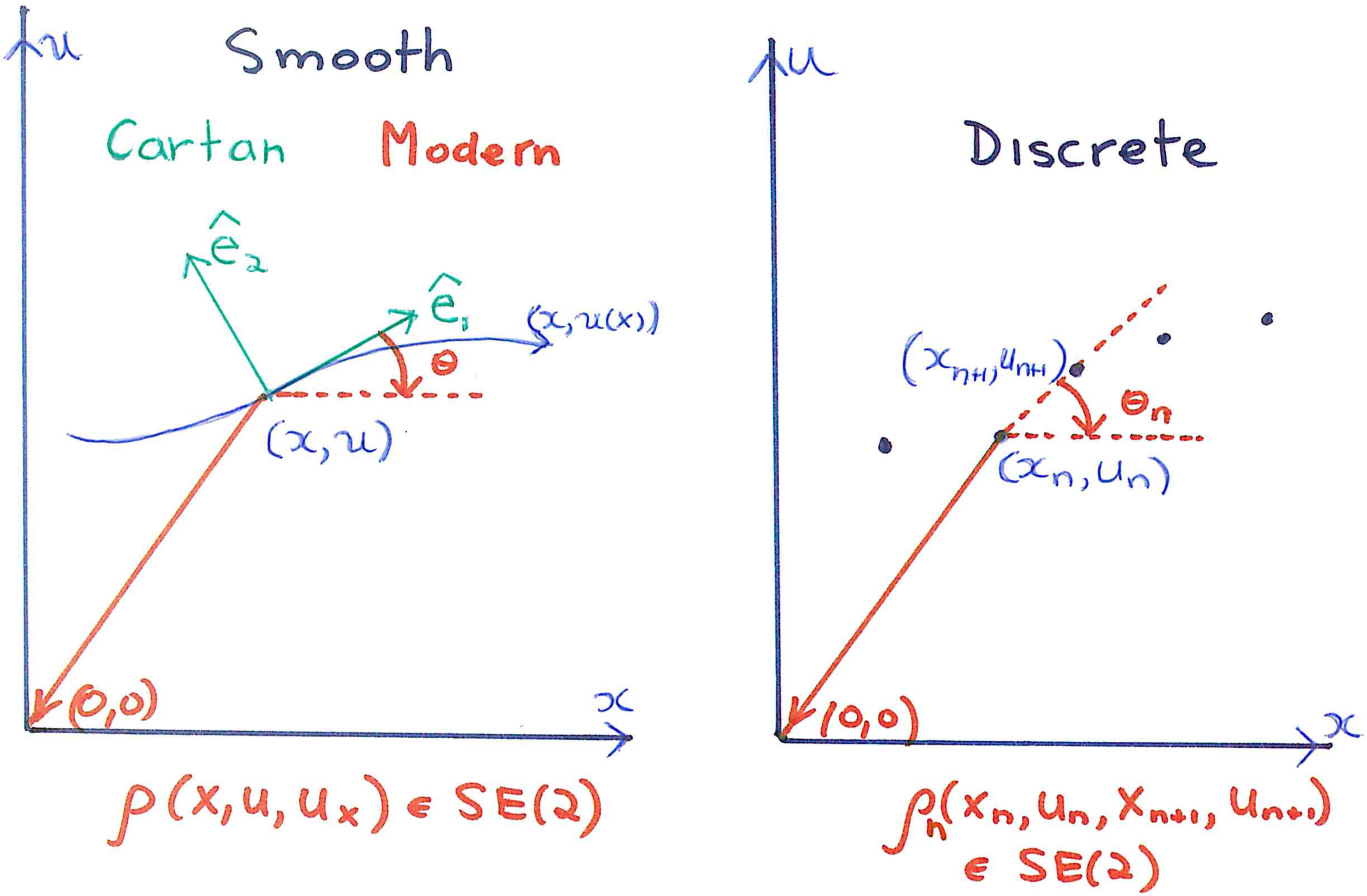
Example: Euler's Elastica:

$$\mathcal{L}[u] = \int \kappa^2 \, \mathrm{d}s, \qquad \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \quad \mathrm{d}s = \sqrt{1+u_x^2}\mathrm{d}x$$

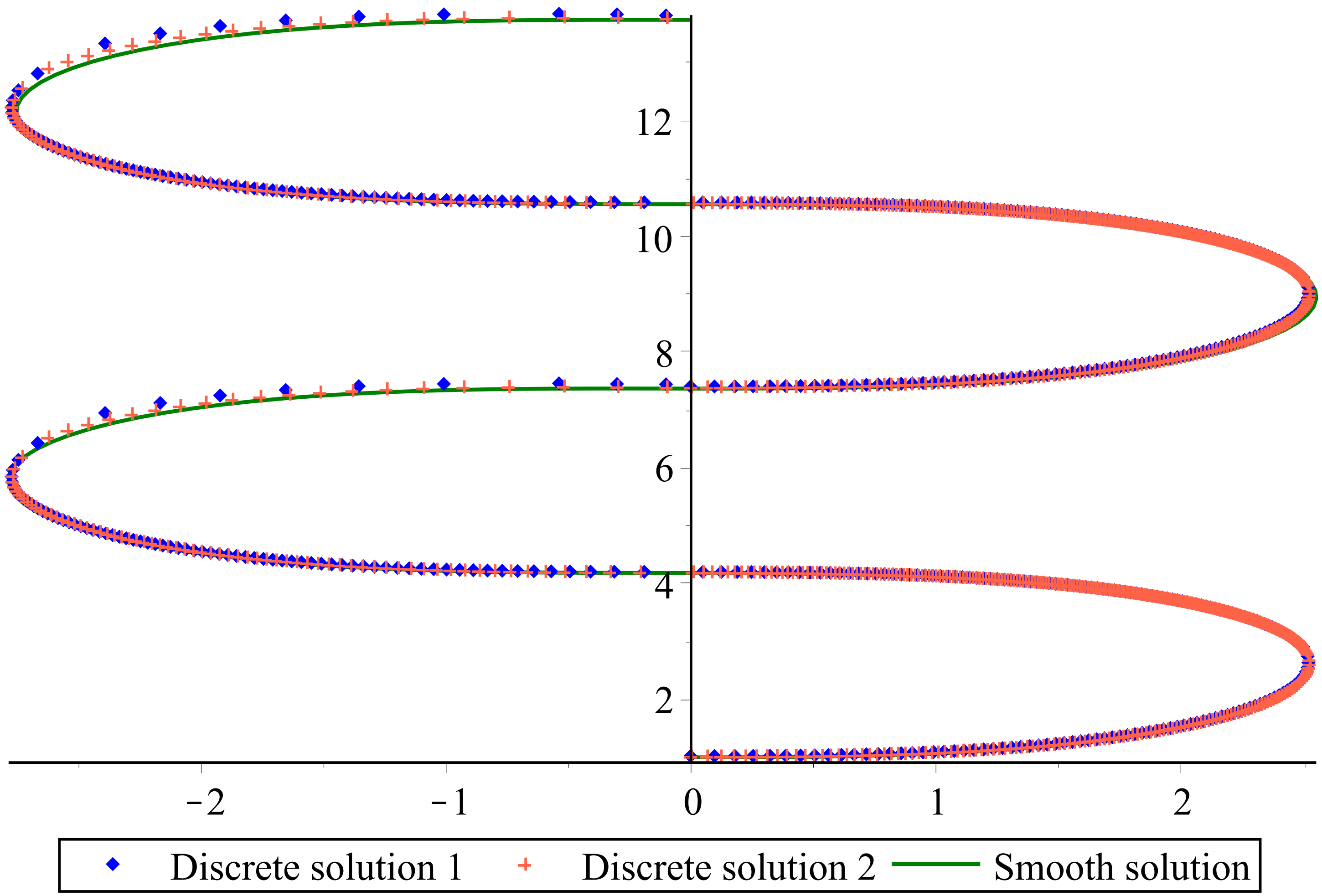
Euler Lagrange equation: $\kappa_{ss} + \frac{1}{2}\kappa^3 = 0.$

invariant under $SO(2) \ltimes \mathbb{R}^2$.

Method used: match famous frame for $SE(2)$ with an obviously matching discrete frame, in order to read off the discrete invariants which matches the Euclidean curvature, as well as the infinitesimal arclength, to build the discrete Lagrangian.



Smooth
Cartan
Modern
u
x
$\hat{e}_2$
$\hat{e}_1$
$(x, u(x))$
θ
(x, u)
$(0,0)$
$\rho(x, u, u_x) \in SE(2)$
Discrete
u
x
(x_{n+1}, u_{n+1})
θ_n
(x_n, u_n)
$(0,0)$
$\rho_n(x_n, u_n, x_{n+1}, u_{n+1})$
$\in SE(2)$



12
10
8
6
4
2
−2
−1
0
1
2
Discrete solution 1
Discrete solution 2
Smooth solution

Discrete integrable systems

Just as smooth integrable systems can occur in pairs, with one being the Lie symmetry reduction of another, e.g. vortex filament equations and the nonlinear Schrödinger equation, so can discrete integrable systems.

We have for an evolving difference frame, the identity

$$\frac{\partial}{\partial_t} K_0 = \frac{\partial}{\partial_t}\left(\rho_1 \rho_0^{-1}\right) = S(N_0^t) K_0 - K_0 N_0^t, \qquad N_0^t = \left(\frac{\partial}{\partial_t}\rho_0\right)\rho_0^{-1}$$

yielding a linear difference *syzygy* operator $\mathcal{H}$ satisfying

$$\frac{\partial}{\partial_t}\kappa_0 = \mathcal{H}\,\sigma_0^t$$

where κ_0 is a (vector) of Maurer Cartan invariants and σ_0^t are the generating t-differential invariants, $\sigma_0^t = \rho_0(z) \cdot z_{0,t}$.

Given two evolutions, and the (easy to compute) syzygies between the σ_0^t and the σ_0^s being written as

$$\mathcal{C}(\sigma_0^t, \sigma_0^s) = \frac{\partial}{\partial_t}\sigma_0^s - \frac{\partial}{\partial_s}\sigma_0^t + M_{st}$$

then we have both

$$\frac{\partial}{\partial_t}\kappa_0 = \mathcal{H}\,\sigma_0^t, \qquad \frac{\partial}{\partial_s}\kappa_0 = \mathcal{H}\,\sigma_0^s$$

and remarkably,

$$[\partial_t, \partial_s]\,\kappa_0 = \mathcal{H}\,\mathcal{C}(\sigma_0^t, \sigma_0^s),$$

showing commuting flows factor through the syzygy operator, mirroring the smooth result.

ELM and P. van der Kamp, J. Geom. Phys., (2006)
Rojo-Echeburúa, ELM and JP Wang, in preparation

Conjecture (JP Wang): the syzygy operator $\mathcal{H}$ is pre-Hamiltonian.

Much more about discrete moving frames and discrete integrability can be said, for example:

Gloria Marì Beffa and Jing Ping Wang, (2013) Nonlinearity.
B. Wang, X-K Chang, X-B Hu and S-H Li, (2018) J. Phys., A.
SIDE '13 Poster: Bao Wang.

Thank you!!