Multivariate Meixner, Charlier and Krawtchouk polynomials

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Abstaract

In **[5]**, we introduced some multivariate analogues of Meixner, Charlier and Krawtchouk polynomials and established their main properties (generating functions, orthogonality, difference equations (recurrence formulas)).

However, our proofs are based on harmonic analysis on the symmetric cones (special functions for matrix arguments). Hence, all our results need a restriction condition for the coupling constant $\beta = \frac{2}{d}$ (we proved only $d = 1, 2, 4$ cases).

Recently, we give new their proofs without using harmonic analysis on the symmetric cones, and succeed in extending all our previous results for any $d \in \mathbb{R}$.

Notations

Let $r \in \mathbb{Z}_{\geq 1}$, $\alpha, d \in \mathbb{R}$ and

$$\mathcal{P} := \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\},$$

$$\delta := (r-1, r-2, \ldots, 2, 1, 0) \in \mathcal{P},$$

$$(\alpha)_{\mathbf{m}} := \prod_{j=1}^{r} \left(\alpha - \frac{d}{2}(j-1)\right)_{m_j} \quad (\mathbf{m} \in \mathcal{P}),$$

$$e_k(\mathbf{z}) := \sum_{1 \leq i_1 < \cdots < i_k \leq r} z_{i_1} \cdots z_{i_k} \quad (k \in \mathbb{Z}_{\geq 0}),$$

$$E_k(\mathbf{z}) := \sum_{j=1}^{r} z_j^k \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0}),$$

$$D_k(\mathbf{z}) := \sum_{j=1}^{r} z_j^k \partial_{z_j}^2 + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^k}{z_j - z_l} \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0}).$$

The Jack and shifted (or interpolation) Jack polynomials

For any partition $\mathbf{m} \in \mathcal{P}$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$, the Jack polynomials $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ are defined by the following two conditions.

(1) $D_2(\mathbf{z}) P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) \sum_{j=1}^{r} m_j \left(m_j - 1 - d(r-j)\right)$.

(2) $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = m_{\mathbf{m}}(\mathbf{z}) + \sum_{\mathbf{k}<\mathbf{m}} c_{\mathbf{m}\mathbf{k}} m_{\mathbf{k}}(\mathbf{z})$.

Similarly, the shifted (or interpolation) Jack polynomials $P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ are defined by the following two conditions.

$(1)^{\mathrm{ip}}$ $P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right) = 0$, unless $\mathbf{k} \subset \mathbf{m} \in \mathcal{P}$

$(2)^{\mathrm{ip}}$ $P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ + (lower terms).

Further, we put

$$\Phi^{(d)}_{\mathbf{m}}(\mathbf{z}) := \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)}, \quad \Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) := \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)},$$

$$\binom{\mathbf{z}}{\mathbf{k}}^{(d)} := \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{z} + \frac{d}{2}\delta; \frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k} + \frac{d}{2}\delta; \frac{d}{2}\right)}$$ (generalized (or Jack) binomial coefficients).

Multivariate Meixner, Charlier and Krawtchouk polynomials

Meixner, Charlier and Krawtchouk polynomials

$$M_m(x;\alpha,c) := \sum_{k=0}^{m} \frac{k!}{(\alpha)_k}\binom{m}{k}\binom{x}{k}\left(1-\frac{1}{c}\right)^k,$$

$$C_m(x;a) := \sum_{k=0}^{m} k!\binom{m}{k}\binom{x}{k}\left(-\frac{1}{a}\right)^k,$$

$$K_m(x;p,N) := M_m(x;-N,p(p-1)^{-1}) \ \ (0 \le m \le N).$$

Multivariate Meixner, Charlier and Krawtchouk polynomials

$$M^{(d)}_{\mathbf{m}}(\mathbf{x};\alpha,c) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}\left(\mathbf{1};\frac{d}{2}\right)(\alpha)_{\mathbf{k}}}\binom{\mathbf{m}}{\mathbf{k}}^{(d)}\binom{\mathbf{x}}{\mathbf{k}}^{(d)}\left(1-\frac{1}{c}\right)^{e_1(\mathbf{k})},$$

$$C^{(d)}_{\mathbf{m}}(\mathbf{x};a) := \sum_{\mathbf{k}\subset\mathbf{m}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}\left(\mathbf{1};\frac{d}{2}\right)}\binom{\mathbf{m}}{\mathbf{k}}^{(d)}\binom{\mathbf{x}}{\mathbf{k}}^{(d)}\left(-\frac{1}{a}\right)^{e_1(\mathbf{k})},$$

$$K^{(d)}_{\mathbf{m}}(\mathbf{x};p,N) := M^{(d)}_{\mathbf{m}}(\mathbf{x};-N,p(p-1)^{-1}) \ \ (\mathbf{m}\subset N=(N,\dots,N)).$$

An interpretation of MM, MC, MK

Okunkov **[3]** prove the following binomial formula for the A-type Macdonald polynomials.

$$\frac{P_{\mathbf{m}}(\mathbf{z};q,t)}{P_{\mathbf{m}}(t^{\delta};q,t)} = \sum_{\mathbf{k}\subset\mathbf{m}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}(q^{\mathbf{k}}t^{\delta};q,t)}{P_{\mathbf{k}}(t^{\delta};q,t)} \frac{P^{\mathrm{ip}}_{\mathbf{k}}(q^{\mathbf{m}}t^{\delta};q,t)}{P^{\mathrm{ip}}_{\mathbf{k}}(q^{\mathbf{k}}t^{\delta};q,t)} \frac{P^{\mathrm{ip}}_{\mathbf{k}}(\mathbf{z};q,t)}{P^{\mathrm{ip}}_{\mathbf{k}}(q^{\mathbf{k}}t^{\delta};q,t)}.$$

Here, $P^{\mathrm{ip}}_{\mathbf{k}}(\mathbf{z};q,t)$: the interpolation Macdonald polynomials.

On the other hand,

$$C^{(d)}_{\mathbf{m}}(\mathbf{x};-1) = \sum_{\mathbf{k}\subset\mathbf{m}} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}\left(\mathbf{1};\frac{d}{2}\right)} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)} \frac{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{x}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P^{\mathrm{ip}}_{\mathbf{k}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)}.$$

Therefore, our multivariate Meixner polynomials are regarded as 2 parameter deformations of rational analogue for the Macdonald polynomials.

Duality and degeneration

By the definition, we immediately obtain a duality property and degenerations for these polynomials.

Duality

$$M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c) = M_{\mathbf{x}}^{(d)}(\mathbf{m};\alpha,c),$$
$$C_{\mathbf{m}}^{(d)}(\mathbf{x};a) = C_{\mathbf{x}}^{(d)}(\mathbf{m};a),$$
$$K_{\mathbf{m}}^{(d)}(\mathbf{x};p,N) = K_{\mathbf{x}}^{(d)}(\mathbf{m};p,N).$$

Degeneration

$$\lim_{\alpha\to\infty} M_{\mathbf{m}}^{(d)}\left(\mathbf{x};\alpha,\frac{a}{a+\alpha}\right) = C_{\mathbf{m}}^{(d)}(\mathbf{x};a),$$
$$\lim_{N\to\infty} K_{\mathbf{m}}^{(d)}\left(\mathbf{x};\frac{a}{N},N\right) = C_{\mathbf{m}}^{(d)}(\mathbf{x};a).$$

Generating functions

$$\prod_{j=1}^{r}(1-z_j)^{-\alpha}\Phi_{\mathbf{x}}^{(d)}\left(\frac{1-\frac{1}{c}\mathbf{z}}{1-\mathbf{z}}\right)=\sum_{\mathbf{n}\in\mathcal{P}}(\alpha)_{\mathbf{n}}M_{\mathbf{n}}^{(d)}(\mathbf{x};\alpha,c)\Psi_{\mathbf{n}}^{(d)}(\mathbf{z}), \tag{1}$$

$$e^{e_1(\mathbf{z})}\Phi_{\mathbf{x}}^{(d)}\left(\mathbf{1}-\frac{1}{a}\mathbf{z}\right)=\sum_{\mathbf{n}\in\mathcal{P}}C_{\mathbf{n}}^{(d)}(\mathbf{x};a)\Psi_{\mathbf{n}}^{(d)}(z),$$

$$\prod_{j=1}^{r}(1+z_j)^{N}\Phi_{\mathbf{x}}^{(d)}\left(\frac{1-\frac{1-p}{p}\mathbf{z}}{1+\mathbf{z}}\right)=\sum_{\mathbf{n}\subset N}(-1)^{e_1(\mathbf{n})}(-N)_{\mathbf{n}}K_{\mathbf{n}}^{(d)}(\mathbf{x};p,N)\Psi_{\mathbf{n}}^{(d)}(z).$$

Orthogonality

For any partitions $\mathbf{m}$, $\mathbf{n} \in \mathcal{P}$,

$$\sum_{\mathbf{x}\in\mathcal{P}} (\alpha)_{\mathbf{x}} \Psi_{\mathbf{n}}^{(d)}(c) M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c) M_{\mathbf{n}}^{(d)}(\mathbf{x};\alpha,c) = \frac{1}{\Psi_{\mathbf{m}}^{(d)}(c)} \frac{(1-c)^{-r\alpha}}{(\alpha)_{\mathbf{m}}} \delta_{\mathbf{m},\mathbf{n}},$$

$$\sum_{\mathbf{x}\in\mathcal{P}} \Psi_{\mathbf{n}}^{(d)}(a) C_{\mathbf{m}}^{(d)}(\mathbf{x};a) C_{\mathbf{n}}^{(d)}(\mathbf{x};a) = \frac{e^{ra}}{\Psi_{\mathbf{m}}^{(d)}(a)} \delta_{\mathbf{m},\mathbf{n}},$$

$$\sum_{\mathbf{x}\subset N} \frac{(-N)_{\mathbf{x}}}{(1-p)^{rN}} \Psi_{\mathbf{n}}^{(d)}\left(\frac{p}{p-1}\right) K_{\mathbf{m}}^{(d)}(\mathbf{x};p,N) K_{\mathbf{n}}^{(d)}(\mathbf{x};p,N) = \left(\frac{1-p}{p}\right)^{|\mathbf{m}|} \binom{N}{\mathbf{m}}^{-1} \delta_{\mathbf{m},\mathbf{n}}.$$

Difference equations (recurrence relation)

$$
\begin{aligned}
&(c-1)e_1(\mathbf{m})M^{(d)}_{\mathbf{m}}(\mathbf{x};\alpha,c)\\
&\quad=\sum_{i=1}^{r}\prod_{1\le k\ne i\le r}\frac{x_i-x_k-\frac{d}{2}(i-k)+\frac{d}{2}}{x_i-x_k-\frac{d}{2}(i-k)}c\left(x_i+\alpha-\frac{d}{2}(i-1)\right)M^{(d)}_{\mathbf{m}}(\mathbf{x}+\epsilon_i;\alpha,c)\\
&\qquad-((1+c)e_1(\mathbf{x})+\alpha rc)M^{(d)}_{\mathbf{m}}(\mathbf{x};\alpha,c)\\
&\qquad+\sum_{i=1}^{r}\prod_{1\le k\ne i\le r}\frac{x_i-x_k-\frac{d}{2}(i-k)-\frac{d}{2}}{x_i-x_k-\frac{d}{2}(i-k)}\left(x_i+\frac{d}{2}(r-i)\right)M^{(d)}_{\mathbf{m}}(\mathbf{x}-\epsilon_i;\alpha,c),
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&-e_1(\mathbf{m})C^{(d)}_{\mathbf{m}}(\mathbf{x};a)\\
&\quad=\sum_{i=1}^{r}\prod_{1\le k\ne i\le r}\frac{x_i-x_k-\frac{d}{2}(i-k)+\frac{d}{2}}{x_i-x_k-\frac{d}{2}(i-k)}aC^{(d)}_{\mathbf{m}}(\mathbf{x}+\epsilon_i;a)\\
&\qquad-(e_1(\mathbf{x})+ar)C^{(d)}_{\mathbf{m}}(\mathbf{x};a)\\
&\qquad+\sum_{i=1}^{r}\prod_{1\le k\ne i\le r}\frac{x_i-x_k-\frac{d}{2}(i-k)-\frac{d}{2}}{x_i-x_k-\frac{d}{2}(i-k)}\left(x_i+\frac{d}{2}(r-i)\right)C^{(d)}_{\mathbf{m}}(\mathbf{x}-\epsilon_i;a).
\end{aligned}
$$

How to prove these?

We prove these by using the multivariate Laguerre polynomials

$$\phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u}) := (\alpha)_{\mathbf{m}} \frac{P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)} \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{e_1(\mathbf{k})} \binom{\mathbf{m}}{\mathbf{k}}^{(d)} \frac{1}{(\alpha)_{\mathbf{k}}} \Phi_{\mathbf{k}}^{(d)}(\mathbf{u})$$

and their fundamental properties **[1]**.

<u>Differential equation</u>

$$\widetilde{D}^{(d)}(\alpha;\mathbf{u})\phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u}) = \phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u})e_1(\mathbf{m}).$$

Here,

$$E_0(\mathbf{u}) := \sum_{j=1}^{r} \partial_{u_j}, \quad E_1(\mathbf{u}) := \sum_{j=1}^{r} u_j \partial_{u_j},$$

$$D_1(\mathbf{u}) := \sum_{j=1}^{r} u_j \partial_{u_j}^2 + d \sum_{1\le j\ne l\le r} \frac{u_j}{u_j - u_l} \partial_{u_j},$$

$$\widetilde{D}^{(d)}(\alpha;\mathbf{u}) := -D_1^{(d)}(\mathbf{u}) + E_1(\mathbf{u}) - \left(\alpha + 1 - \frac{n}{r}\right) E_0(\mathbf{u}).$$

Generating functions

$$\prod_{j=1}^{r}(1-z_j)^{-\alpha}{}_0\mathcal{F}_0{}^{(d)}\left(;-\mathbf{u},\frac{\mathbf{z}}{\mathbf{1}-\mathbf{z}}\right)=\sum_{\mathbf{m}\in\mathcal{P}}\phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u})\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}), \quad (3)$$

$$e^{e_1(\mathbf{z})}{}_0\mathcal{F}_1{}^{(d)}\left(\genfrac{}{}{0pt}{}{}{\alpha};-\mathbf{u},\mathbf{z}\right)=\sum_{\mathbf{m}\in\mathcal{P}}\frac{1}{(\alpha)_{\mathbf{m}}}\phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u})\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}). \quad (4)$$

Here,

$${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{w}\right):=\sum_{\mathbf{m}\in\mathcal{P}}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})\Phi_{\mathbf{k}}^{(d)}(\mathbf{w})=\sum_{\mathbf{m}\in\mathcal{P}}\Phi_{\mathbf{k}}^{(d)}(\mathbf{z})\Psi_{\mathbf{k}}^{(d)}(\mathbf{w}),$$

$${}_0\mathcal{F}_1{}^{(d)}\left(\genfrac{}{}{0pt}{}{}{a};\mathbf{z},\mathbf{w}\right):=\sum_{\mathbf{m}\in\mathcal{P}}\frac{1}{(a)_{\mathbf{m}}}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})\Phi_{\mathbf{k}}^{(d)}(\mathbf{w})=\sum_{\mathbf{m}\in\mathcal{P}}\frac{1}{(a)_{\mathbf{m}}}\Phi_{\mathbf{k}}^{(d)}(\mathbf{z})\Psi_{\mathbf{k}}^{(d)}(\mathbf{w}).$$

Key summations

From (3) and (4), we have the following key summations.

$$(\alpha)_{\mathbf{k}} \prod_{j=1}^{r} (1 - z_j)^{-\alpha} \Psi_{\mathbf{k}}^{(d)} \left(\frac{\mathbf{z}}{\mathbf{1} - \mathbf{z}} \right) = \sum_{\mathbf{m} \in \mathcal{P}} \binom{\mathbf{m}}{\mathbf{k}} (\alpha)_{\mathbf{m}} \Psi_{\mathbf{m}}^{(d)}(\mathbf{z}), \tag{5}$$

$$e^{e_1(\mathbf{z})} \Psi_{\mathbf{k}}^{(d)}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathcal{P}} \binom{\mathbf{m}}{\mathbf{k}} \Psi_{\mathbf{m}}^{(d)}(\mathbf{z}). \tag{6}$$

<u>One variable case</u>

$$(\alpha)_k (1 - z)^{-\alpha} \frac{1}{k!} \left(\frac{z}{1 - z} \right)^k = \sum_{m \geq 0} \binom{m}{k} (\alpha)_m \frac{z^m}{m!},$$

$$e^z \frac{z^k}{k!} = \sum_{m \geq 0} \binom{m}{k} \frac{z^m}{m!}.$$

Key lemma

From these key summations (5) and (6), we obtain the following key lemma (generating function of generating functions for the multivariate Meixner polynomials).

$$\sum_{\mathbf{x},\mathbf{m}\in\mathcal{P}} M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c)e^{-\frac{c}{1-c}e_1(\mathbf{u})}\left(\frac{c}{1-c}\right)^{e_1(\mathbf{x})}\Psi_{\mathbf{m}}^{(d)}(\mathbf{z})\Psi_{\mathbf{x}}^{(d)}(\mathbf{u})$$

$$= \sum_{\mathbf{m}\in\mathcal{P}} \frac{1}{(\alpha)_{\mathbf{m}}}\phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u})\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) = e^{e_1(\mathbf{z})}{}_0\mathcal{F}_1{}^{(d)}\left({}_{\alpha};-\mathbf{u},\mathbf{z}\right), \qquad (7)$$

$$\sum_{\mathbf{x},\mathbf{m}\in\mathcal{P}} M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c)(\alpha)_{\mathbf{m}}e^{-\frac{c}{1-c}e_1(\mathbf{u})}\left(\frac{c}{1-c}\right)^{e_1(\mathbf{x})}\Psi_{\mathbf{m}}^{(d)}(\mathbf{z})\Psi_{\mathbf{x}}^{(d)}(\mathbf{u})$$

$$= \sum_{\mathbf{m}\in\mathcal{P}} \phi_{\mathbf{m}}^{(d)}(\alpha;\mathbf{u})\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) = \prod_{j=1}^{r}(1-z_j)^{-\alpha}{}_0\mathcal{F}_0{}^{(d)}\left(;-\mathbf{u},\frac{\mathbf{z}}{1-\mathbf{z}}\right). \qquad (8)$$

Proofs of fundamental properties

Generating function

By comparing the coefficients of $\Psi_{\mathbf{x}}(u)$ on

$$\sum_{\mathbf{x},\mathbf{m}\in\mathcal{P}} M_{\mathbf{m}}^{(d)}(\mathbf{x};\alpha,c)(\alpha)_{\mathbf{m}}\left(\frac{c}{1-c}\right)^{e_1(\mathbf{x})}\Psi_{\mathbf{m}}^{(d)}(\mathbf{z})\Psi_{\mathbf{x}}^{(d)}(\mathbf{u})$$
$$=\prod_{j=1}^{r}(1-z_j)^{-\alpha}e^{\frac{c}{1-c}e_1(\mathbf{u})}{}_0\mathcal{F}_0{}^{(d)}\left(;-\mathbf{u},\frac{\mathbf{z}}{1-\mathbf{z}}\right),$$

we obtain the generating function of the multivariate Meixner polynomials.

Orthogonality

It follows from (1) and another type generating function of generating functions for the multivariate Meixner polynomials.

Difference equation

By applying $\widetilde{D}^{(d)}(\alpha;\mathbf{u})$ to (7) or (8), and comparing the coefficients of $\Psi_{\mathbf{x}}^{(d)}(\mathbf{u})$, we have (2).

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