Growth, invariants, Lagrangians and integrability for four-dimensional recurrence relations

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Outline of the Seminar

Bi-rational projective maps

- In this talk we will deal with recurrence relations.
- For us a recurrence relation of order n is a bi-rational map from the n-dimensional complex projective space into itself:

$$\varphi\colon [\mathbf{x}] \in \mathbb{CP}^n \to [\mathbf{x}'] \in \mathbb{CP}^n.$$

- Here $[\mathbf{x}] = [x_1 : x_2 : \cdots : x_{n+1}]$ and $[\mathbf{x}'] = \left[x_1' : x_2' : \cdots : x_{n+1}'\right]$ are the homogeneous coordinates in projective space and:

$$x_1' = P, \quad x_i' = x_{i-1} Q,\ i = 2, \dots, n, \quad x_{n+1}' = x_{n+1} Q,$$

  where $P, Q \in \mathbb{C}_h [x_1, \dots, x_{n+1}]$.
- We are interested in the study of the integrability of such maps.

Bi-rational maps

A rational map of algebraic varieties $\varphi\colon V \to W$ is bi-rational if there exists a rational map $\psi\colon W \to V$ such that it is the inverse of φ in the subset where both are defined.

Invariants of bi-rational maps

- Integrability for recurrence relations can be understood in different ways.
- Like for continuous differential equations, since we are in the autonomous case we can introduce the concept of invariants.

First integrals

An invariant for a bi-rational map is a homogeneous function $I\colon \mathbb{CP}^n \to \mathbb{C}$ such that $\varphi^*(I) = I$. With $\varphi^*(I)$ we mean the pullback of I through the map φ, i.e. $\varphi^*(I) = I(\varphi([\mathbf{x}]))$.

Degree pattern

Let $F \in \mathbb{C}_h[x_1, \dots, x_{n+1}]^N$, i.e. F is a vector of homogeneous polynomials. We define the degree pattern of F to be:

$$\operatorname{dp} F = \left(\deg_{x_1} F, \deg_{x_2} F, \dots, \deg_{x_n} F\right),$$

where the $\deg_{x_i} F$ is maximum of the degrees of the entries of F with respect to x_i.

Integrability for bi-rational maps

- Invariants are useful in solving recurrence relations because the existence of an invariant I means that the solutions of the recurrence relation must lie on the hypersurface $I = \kappa$.
- The value of the constant κ is determined by the initial conditions $\mathbf{x_0}$: $\kappa = I([\mathbf{x_0}])$.
- This means that, in principle, if we know $n-1$ functionally independent invariants of an n-dimensional map φ by solving the system

$$I_i = \kappa_i, \quad i = 1, \ldots, n-1$$

we can reduce the map to a 1-dimensional map $\widehat{\varphi}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$.
- The solution is then obtained by integrating the map $\widehat{\varphi}$ which will give a last invariant.

Integrability

An bi-rational map is a integrable if it possesses $n-1$ invariants.

Reduction and bi-rationality

In general the reduction to a lower-dimensional map can break the bi-rationality.

How many invariants are really needed for integrability?

- From the previous discussion we have that for a general n-dimensional bi-rational map to claim integrability it is necessary to produce $n-1$ invariants, and then a last invariant will come from integration.
- So to have integrability one needs n invariants, but if the map has some additional structure the number of integrals can be significantly lowered.
- In the continuous case, for even-order differential equations a natural additional structure to look at is the Lagrangian/Hamiltonian formulation.
- In particular in Hamiltonian mechanics a powerful result is the so-called Liouville theorem which states that for the integrability in the Hamiltonian case only half of the invariants are needed.
- In the map setting there is no commonly accepted equivalent of the Hamiltonian formulation, but it is still possible to define symplectic structures for even-dimensional maps.

Symplectic forms

Symplectic forms and symplectic maps

In affine coordinates $\mathbf{y} = (y_1, \ldots, y_{2r}) = [x_1/x_{2r+1} : \cdots : x_{2r}/x_{2r+1} : 1] \in \mathbb{C}^{2r}$ the 2-form:

$$\omega = \sum_{i<j} \omega_{i,j}(\mathbf{y})\, \mathrm{d}y_i \wedge \mathrm{d}y_j$$

is called symplectic form if is closed and non-degenerate a.e. in $\mathbb{C}^{2r}$.

Symplectic maps

A map $\varphi\colon \mathbb{C}^{2r} \to \mathbb{C}^{2r}$ is called a symplectic map if its pullback preserves the symplectic form ω: $\varphi^*(\omega) = \omega$.

Poisson bracket

A symplectic form defines a Poisson bracket through the equation

$$\{f, g\} := \sum_{i,j=1}^{2r} \Omega_{i,j}(\mathbf{y}) \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}, \quad \Omega = (\omega_{i,j})^{-1}.$$

Symplectic maps and Liouville theorem

For symplectic maps the following theorem holds:

Discrete Liouville Theorem (Bruschi and Ragnisco 1991, Veselov 1991)

If symplectic map $\varphi\colon \mathbb{C}^{2r} \to \mathbb{C}^{2r}$ possess r invariants $I_1, \dots, I_r$, such that $\{I_k, I_l\} \equiv 0$ for every $k, l = 1, \dots, r$ then it is possible to find the $r-1$ missing invariants and reduce the maps to a discrete integration. Moreover, if the invariants define compact and connected hypersurfaces it is possible to reduce the solution of the recurrence relation defined by φ to r trivial first order recurrence relations, defining the so-called action variables.

Liouville integrability

A symplectic map satisfying the hypotheses of discrete Liouville theorem is called a Liouville integrable map.

Liouville integrability implies integrability

Like in the continuous case a Liouville integrable map is a special integrable map which requires half of the invariants to claim integrability.

Integrability and growth

- In the case of the discrete equations given an initial condition we can always compute step by step the corresponding sequence.
- Then we can intuitively say that integrability is the ability to tell behaviour of the equation without the need of computing an infinite sequence.
- The concept of integrability is different from the one of solvability.
- A very simple example is given by the logistic map:

$$y_{n+1} = 4y_n(1 - y_n) \implies y_n = \frac{1}{2}\left[1 - \cos\left(2^n c_0\right)\right] \implies \frac{\mathrm{d}y}{\mathrm{d}c_0} = 2^{n-1}\sin\left(2^n c_0\right).$$

  The discrepancy between to arbitrarily near initial values grows exponentially!
- This simple observation led to the conjecture that a good way to establish if a projective map is integrable or not was to measure how fast the degree with respect to the initial conditions grows when iterated.
- This approach to integrability is called algebraic entropy method [Veselov 1992, Falqui and Viallet 1993 and Bellon and Viallet 1999].

Definition of algebraic entropy

- Algebraic entropy measure the rate of growth of the degrees of the iterates of a bi-rational map.
- Iterating the map $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ we define the sequence of the degrees with respect to the initial conditions $[\mathbf{x_0}]$:

$$d_k := \deg_{[\mathbf{x_0}]} \varphi^k, \quad k \in \mathbb{N}.$$

- The algebraic entropy of this sequence is then defined by the limit:

$$\varepsilon_\varphi = \lim_{k\to\infty} \frac{1}{k} \log d_k.$$

  This limit always exists due to the properties of bi-rational maps.
- Algebraic entropy is a bi-rational invariant of maps.
- From the definition if follows that $\varepsilon_\varphi \geq 0$. Moreover, $\varepsilon_\varphi = 0$ if and only if $d_k \simeq k^\nu$ for some $\nu \in \mathbb{N}$ as $k \to \infty$.

Algebraic entropy integrability

We say that a map $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ is integrable according to the algebraic entropy test if $\varepsilon_\varphi \equiv 0$. Moreover, if d_k is periodic (not constant) we say that the map is periodic, while if d_k is linear we say that the map is linearizable.

Computation of the algebraic entropy

- From the definition of algebraic entropy it seems that one has to compute all the iterates of a map to get its value.
- Hopefully enough for most application this is not needed since after a finite number of iterations the form of the sequence $\{d_k\}_{k\in\mathbb{N}}$ stabilizes.
- Then to obtain the explicit form of the sequence $\{d_k\}_{k\in\mathbb{N}}$ we can then search for a generating function, i.e. a function such that:

$$g(z) = \sum_{k=0}^{\infty} d_k z^k.$$

- If we are able to found a generating function then the algebraic entropy is given by:

$$\varepsilon = \log \left| \min \{z \in \mathbb{C} | g(z) \text{ has a pole in } z\} \right|^{-1}.$$

- A generating function is a predictive tool which can be used to test the successive members of a finite sequence.

Duality for bi-rational maps

- If the map φ possesses k invariants, namely I_l with $l = 1, \dots, k$ we can form the linear combination:

$$H = \alpha_1 I_1 + \cdots + \alpha_l I_l.$$

- If $\deg_{x_1} H, \deg_{x_n} H > 1$ for an unspecified recurrence

$$[x_1 : x_2 : \cdots : x_n : x_{n+1}] \mapsto [x_1' : x_1 : \dots, x_{n-1} : x_{n+1}]$$

  we have the following factorization:

$$H([\mathbf{x}']) - H([\mathbf{x}]) = A(x_1', [\mathbf{x}]) B(x_1', [\mathbf{x}]).$$

- Since H is a first integral of φ we will have that one of the possible solutions of $H([\mathbf{x}']) - H([\mathbf{x}]) = 0$ is just given by $[\mathbf{x}'] = \varphi([\mathbf{x}])$.
- We can assume that this corresponds to the annihilation of A.
- In some circumstances, i.e. when $\deg_{x_1} H, \deg_{x_n} H = 2$ we have that also the annihilation of B defines a bi-rational projective map.
- We call this map the dual map $\varphi^\vee$ [Quispel, Capel and Roberts 2005].

Duality and first integrals

- Assume that the invariants and the map φ depends on some arbitrary constants $I_l = I_l\left([\mathbf{x}]; a_1, \ldots, a_M\right)$, for $l = 1, \ldots, k$.
- Now assume that choosing some of the a_i in a way that there remains N arbitrary constants and such that for a subset $\{a_{i_j}\} \subset \{a_1, \ldots, a_N\}$ we can write H as:

$$H = a_{i_1} J_1 + a_{i_2} J_2 + \cdots + a_{i_K} J_K.$$

- Here $J_i = J_i\left([\mathbf{x}]; \alpha_1, \ldots, \alpha_k\right)$, $i = 1, 2, \ldots, N$ are new functions coming from the rearrangement of H.
- Then using the factorization of H we have that the J_i functions are invariants for the dual map.

Dual map and invariants

Depending on number of arbitrary constants $\{a_{i_j}\}$ the dual map is naturally equipped with a certain number of invariants.

Motivations

- In a recent paper [Joshi and Viallet, 2017] considered the autonomous limit of the second member of the $\mathrm{d}P_{\mathrm{I}}$ and $\mathrm{d}P_{\mathrm{II}}$ hierarchies, the $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations.
- The $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations are given by recurrence relations of order four, integrable according to the algebraic entropy approach.
- They showed that both maps possesses two first integrals I_{low} and I_{high}.
- Using these first integrals they showed that the dual maps of the $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations are integrable according to the algebraic entropy test and produced some integrals.
- Moreover they showed that the integrals of the $\mathrm{d}P_{\mathrm{I}}^{(2)}$ and $\mathrm{d}P_{\mathrm{II}}^{(2)}$ equations share some common properties: in both cases: $\operatorname{dp} I_{\mathrm{low}} = (1, 3, 3, 1)$ and $\operatorname{dp} I_{\mathrm{high}} = (2, 4, 4, 2)$.
- In the same paper they gave a scheme to construct recurrence relations with the assigned structure of integrals and they produced some new examples out of this construction.

A classification problem

The problem

We want to make a classification of the fourth-order maps $\varphi\colon [x:y:z:u:t] \mapsto [x':y':z':u':t']$ with the following hypotheses:

1. φ possesses a polynomial a symmetric first integral I_{low} such that $\operatorname{dp} I_{\text{low}} = (1,3,3,1)$ where the only non-zero coefficients are those appearing in the I_{low} of both $\mathrm{d}P_{\text{I}}^{(2)}$ and $\mathrm{d}P_{\text{II}}^{(2)}$ and such that the factorisation $I_{\text{low}}([\mathbf{x}']) - I_{\text{low}}([\mathbf{x}]) = (x-z)A(x',[\mathbf{x}])$ holds. This integral has 11 essential free parameters.
2. φ possesses a polynomial symmetric first integral I_{high} such that $\operatorname{dp} I_{\text{high}} = (2,4,4,2)$. This integral has 119 essential free parameters.
3. The first integrals I_{low} and I_{high} are functionally independent and non-degenerate.

Symmetric first integral

We say that a first integral φ is symmetric if it is invariant under the involution $\iota\colon [x:y:z:u:t] \mapsto [u:z:y:x:t]$.

Relation to other classifications

- The basic idea of this classification is similar to the one of [Quispel, Roberts and Thompson 1988], but of course in higher dimension.
- A possible extension to four dimension of the QRT approach was pursued in [Capel and Sahadevan, 2001], but it this case the author considered symplectic maps in four dimension with two first integrals of degree pattern $(2, 2, 2, 2)$. Let us notice that first integrals with this degree patterns where found in the autonomous limit of the fourth order qP equations [Hay, 2007].
- In the continuous case the method of imposing an ansatz on the form of the first integrals is widely used since the 60s in the theory of superintegrable systems [Friš, Mandrosov, Smorodinski, Uhlíř, and Winternitz, 1965] and yet produces new and interesting results [Gravel, 2008, Post and Winternitz, 2011].

A glimpse on how to perform the classification

- Find the value of x' the relation

$$I_{\mathrm{low}}([\mathbf{x}']) - I_{\mathrm{low}}([\mathbf{x}]) = (x - z) A(x', [\mathbf{x}]),$$

  which is implied by the form of I_{low}.
- Substitute the obtained form of x' into the first integral condition for I_{high}.
- We can take coefficients with respect to the independent variables $[x : y : z : u : t]$.
- This yield a large system of nonlinear homogeneous equations.
- We proceed iteratively to solve the system starting from monomial equations then reducing the number of equations, until we obtain a collection of systems with no new monomial equations.
- This leaves us with 117 simpler systems. These systems can be solved to give 25 solutions which, upon relabeling of the parameters and degenerations yield us 6 fourth order recurrence relations with the desired properties.
- We denote by (P.j) for the main maps and by (Q.j) the dual maps where j is a small roman number.

Properties of the obtained equations

The properties of the obtained equation can be summarized by the following table:

Equation	dp I_{low}	dp I_{high}	Degree of growth
(P.i)	(1,3,3,1)	(2,4,4,2)	cubic
(Q.i)	(1,2,2,1)	(2,4,4,2)	cubic
(P.ii)	(1,3,3,1)	(2,4,4,2)	exponential
(Q.ii)	–	(2,4,4,2)	exponential
(P.iii)	(1,3,3,1)	(2,4,4,2)	quadratic
(Q.iii)	(1,2,2,1)	(2,4,4,2)	exponential
(P.iv) ($\mathrm{d}P_{\mathrm{I}}^{(2)}$)	(1,3,3,1)	(2,4,4,2)	quadratic
(Q.iv)	(1,2,2,1), (1,2,2,1)	(2,4,4,2)	quadratic
(P.v)	(1,3,3,1)	(2,4,4,2)	quadratic
(Q.v)	(1,2,2,1), (1,2,2,1)	(2,4,4,2)	quadratic
(P.vi) ($\mathrm{d}P_{\mathrm{II}}^{(2)}$)	(1,3,3,1)	(2,4,4,2)	quadratic
(Q.vi)	(1,2,2,1), (1,2,2,1)	(2,4,4,2)	quadratic

Selected examples: the map (P.i)

- The map $[\mathbf{x}] \mapsto \varphi_i([\mathbf{x}]) = [\mathbf{x}']$ is given by the following components:

$$
\begin{aligned}
x' &= -\{[\nu t^2(x+z) + uz^2]y + t^2\mu uz + (x+z)^2y^2\}d - at^4,\\
y' &= x^2 d(t^2\mu + xy), \quad z' = yxd(t^2\mu + xy),\\
u' &= zxd(t^2\mu + xy), \quad t' = txd(t^2\mu + xy).
\end{aligned}
\tag{P.i}
$$

- The map (P.i) has the following degrees of iterates:

$$\{d_n\}_{\mathrm{P.i}} = 1, 4, 12, 28, 52, 86, 130, 188, 260, 348, 452, 576, 720, 886, 1074, 1288 \ldots$$

- Its generating function is:

$$g_{\mathrm{P.i}}(s) = \frac{s^7 - 3s^6 + s^5 - s^4 + 3s^3 + 3s^2 + s + 1}{(s+1)(s^2+1)(s-1)^4}.$$

- This means that the map (P.i) is integrable according to the algebraic entropy test with cubic growth. Same considerations apply to the map (Q.i).

Cubic growth

- Besides integrability the cubic growth of (P.i) and (Q.i) gives more information.
- Alongside with information on the structure of the orbit can suggest the existence of non-algebraic first integrals.
- Indeed, if the additional first integral was rational the orbits would be confined to elliptic curves and the growth of the degree of the iterates would be quadratic, not cubic [Bellon, 1999].
- This means that I_{low} and I_{high} can define non-elliptic fibrations as already shown in [Joshi and Viallet, 2017].
- Moreover, the maps (P.i) and (Q.i) can be "deflated" to 3-dimensional maps with two integrals and quadratic growth.
- These 3-dimensional maps are *Liouville integrable* as they possess rank 2 Poisson structures and are reducible to 2-dimensional maps.
- The reduction of (Q.i) is a QRT-like map, while the reduction of (P.i) possess a bi-quadratic "anti-invariant" which lies in the framework of [Roberts and Jogia, 2015].

Selected examples: the map (P.ii)

- A rather peculiar example is given by the map $[\mathbf{x}] \mapsto \varphi_{ii}([\mathbf{x}]) = [\mathbf{x}']$:

$$
\begin{aligned}
x' &= \left[(x^2+z^2)y - uz^2\right]\mu - t^2(u-2y), \\
y' &= x(t^2+\mu x^2), \quad z' = y(t^2+\mu x^2), \\
u' &= z(t^2+\mu x^2), \quad t' = t(t^2+\mu x^2).
\end{aligned}
\tag{P.ii}
$$

- By construction the map (P.ii) has two integrals and it has the following degrees of iterates:

$$\{d_n\}_{\mathrm{P.ii}} = 1, 3, 9, 21, 45, 93, 189, 381, 765, 1533\dots$$

- This sequence has generating function:

$$g_{\mathrm{P.ii}}(s) = \frac{1+2s^2}{(2s-1)(s-1)}.$$

- The main map is non-integrable since it has positive entropy $\varepsilon = \log 2$!
- Similar considerations hold for the dual map (Q.iii).

How integrability arises in the list?

- We showed that maps from our list can have rather different properties, even though all of them possess two invariants.
- To understand why some of the maps with two integrals are non-integrable we need to understand if the discrete Liouville theorem is applicable or not.
- It is known from [Byrnes, Haggar, and Quispel, 1999] that from a four dimensional, measure preserving map with two integrals one can build a a pre-symplectic structure i.e. a "symplectic structure" where the 2-form ω is degenerate.
- This is not enough to apply the discrete Liouville's theorem.
- So, instead of searching directly for symplectic structures which is very difficult we searched for discrete Lagrangians for the equations in the list.
- Discrete Lagrangians naturally lead to non-degenerate symplectic structures, and vice-versa a symplectic structure can always be inverted to give Euler-Lagrange equations.
- So disproving the existence of a discrete Lagrangian is equivalent to disproving the existence of a symplectic structure.

Discrete Lagrangians and integrability

- A recurrence relation of order $2k$ in affine coordinates it is said to be Lagrangian if there exists a function $L = L(x_{n+k}, \dots, x_n)$ such that the Euler-Lagrange equation:

$$\sum_{l=0}^{k} \frac{\partial L}{\partial x_n}(x_{n+k-l}, x_{n+k-l-1}, \dots, x_{n-l}) = 0,$$

  is equivalent to the given recurrence relation.
- The function L is called the discrete Lagrangian.
- We developed a simple and algorithmic test to check if a given equation possesses or not a discrete Lagrangian.
- This study led to understand that the slow growth for equations with no Lagrangians arises from the existence of other integrals or by the mechanism of deflation.

Updated properties of the obtained equations

So we can write down an updated table with all the properties of the obtained solutions:

Equation	Degree pattern of integrals	Degree of growth	Lagrangian
(P.i)	(1,3,3,1), (2,4,4,2)	cubic (deflatable)	no
(Q.i)	(1,2,2,1), (2,4,4,2)	cubic (deflatable)	no
(P.ii)	(1,2,2,1), (1,3,3,1)	exponential	no
(Q.ii)	(2,4,4,2)	exponential	no
(P.iii)	(1,3,3,1), (2,4,4,2), (2,5,5,2)	quadratic	no
(Q.iii)	(1,2,2,1), (2,4,4,2)	exponential	no
(P.iv) ($\mathrm{d}P_{\mathrm{I}}^{(2)}$)	(1,3,3,1), (2,4,4,2)	quadratic	yes
(Q.iv)	(1,2,2,1), (1,2,2,1), (2,4,4,2)	quadratic	no
(P.v)	(1,3,3,1), (2,4,4,2)	quadratic	yes
(Q.v)	(1,2,2,1), (1,2,2,1), (2,4,4,2)	quadratic	no
(P.vi) ($\mathrm{d}P_{\mathrm{II}}^{(2)}$)	(1,3,3,1), (2,4,4,2)	quadratic	yes
(Q.vi)	(1,2,2,1), (1,2,2,1), (2,4,4,2)	quadratic	no

Summary

- In this talk we found a certain number of fourth order bi-rational maps possessing two first integrals of degree pattern (1,3,3,1) and (2,4,4,2).
- This family contains integrable, superintegrable and non-integrable members. These proprieties have been linked and explained through the existence of discrete Lagrangians.
- Search is going on for other fourth order $\mathrm{d}P$ equations and their autonomous limits.
- We are also addressing is the geometry of the surfaces defined by the intersections of the first integrals.
- Another problem arising regarding these equations is the one of de-autonomisation. In fact it can be shown that using the "strict" procedure of de-autonomisation several maps do not admit non-autonomous extensions.

Thank you for your attention!