Elliptic Stable Envelopes and Finite-dimensional Representation of Elliptic Quantum Group

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- H.K, "Elliptic Weight Functions and Elliptic q-KZ Equation", J.Int.Systems 2 (2017)
- H.K, "Elliptic Stable Envelopes and Finite-dim. Reps of Elliptic Quantum Group", J.Int.Systems 3 (2018).

SUSY Gauge Theories

- 4d, 5d, 6d: (Moduli sp. of Instantons, or VEV's (Higgs, Coulomb))

Nekrasov-Shatashvili Corresp. $\Longleftrightarrow$

Quantum Int. Systems

XXX,	rRS model,	Toda
XXZ,	RS model,	q-Toda
XYZ	Ruijsenaars	??
∼8VSOS,	model,	

AGT Corresp. $\Leftrightarrow$

R-matrix, q-KZ eq. $\Leftrightarrow$

Modules of Quantum Groups

$\mathcal{D}Y$

U_q

$U_{q,p}$

Nakajima $\Updownarrow$

Givental et.al. $\Updownarrow$

Geom.Rep. Theory $\Leftrightarrow$

?? $\leftrightarrow$ Quantum Diff. eq. (?)

Equiv. Cohom.

$\mathrm{H}^*_T(X)$

$\mathrm{K}_T(X)$

$\mathrm{E}_T(X)$

X: Quiver Var.

Maulik-Okounkov $\rightleftarrows$

Kähler parameters $\to 0$

Quantum Equiv. Cohom.

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$Q\mathrm{E}_T(X)$??

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$U_{q,p}(\widehat{\mathfrak{sl}}_N)$

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$X = T^*\mathcal{F}_\lambda$

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Definition of $U_{q,p}(\mathfrak{g})$, $\mathfrak{g}$: untwisted affine Lie algebra

(H.K'98, Jimbo-H.K-Odake-Shiraishi'99, Farghly-H.K-Oshima'13)

- $H = \bar{\mathfrak{h}} \oplus P_{\bar{\mathfrak{h}}} \oplus \mathbb{C}c$, $H^* = \bar{\mathfrak{h}}^* \oplus Q_{\bar{\mathfrak{h}}} \oplus \mathbb{C}\Lambda_0$,
- $\mathbb{F} = \mathcal{M}_{H^*}$: the field of merom. functions on H^*

$U_{q,p}(\mathfrak{g})$ is a topological algebra over $\mathbb{F}[[p]]$ gen. by $e_{j,m}, f_{j,m}, \alpha_{j,n}, K_j^{\pm}$ $(j \in \{1, 2, \cdots, l = \mathrm{rank}\mathfrak{g}\}, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0})$, d and the central element c.

- the elliptic currents :

$$e_j(z) = \sum_{m\in\mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m\in\mathbb{Z}} f_{j,m} z^{-m},$$

$$\psi_j^{\pm}(q^{\mp\frac{c}{2}} z) = K_j^{\pm} :\exp\left\{ \pm(q-q^{-1}) \sum_{n\neq 0} \frac{\alpha_{j,n} p^{\pm n}}{1-p^{\pm n}} z^{-n} \right\}: .$$

Remark 2.1

$U_{q,p}(\mathfrak{g})$ is an elliptic dynamical analogue of the quantum affine alg. $U_q(\mathfrak{g})$ in Drinfeld's new realization.

Theorem 2.2 (H.K '16)

$$U_{q,p}(\widehat{\mathfrak{gl}}_N) \cong E_{q,p}(\widehat{\mathfrak{gl}}_N)$$ *(: central extension of Felder's EQG)*

Defining Relations:

$\forall f(P), f(P+h) \in \mathcal{M}_{H^*}$

$$f(P+h)e_j(z) = e_j(z)f(P+h), \quad f(P)e_j(z) = e_j(z)f(P - < Q_{\alpha_j}, P >),$$

$$f(P+h)f_j(z) = f_j(z)f(P+h - < Q_{\alpha_j}, P+h >), \quad f(P)f_j(z) = f_j(z)f(P),$$

$$f(P+h)K_j^{\pm} = K_j^{\pm} f(P+h - < Q_{\alpha_j}, P+h >), \quad f(P)K_j^{\pm} = K_j^{\pm} f(P - < Q_{\alpha_j}, P >),$$

$$[\alpha_{i,m}, \alpha_{j,n}] = \delta_{m+n,0} \frac{[b_{ij}m]_q [cm]_q}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm}$$

: the elliptic bosons,

$$[\alpha_{i,m}, e_j(z)] = \frac{[b_{ij}m]_q}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \qquad [\alpha_{i,m}, f_j(z)] = -\frac{[b_{ij}m]_q}{m} z^m f_j(z),$$

$$z_1 \frac{(q^{b_{ij}} z_2/z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2/z_1; p^*)_\infty} e_i(z_1)e_j(z_2) = -z_2 \frac{(q^{b_{ij}} z_1/z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1/z_2; p^*)_\infty} e_j(z_2)e_i(z_1),$$

$$z_1 \frac{(q^{-b_{ij}} z_2/z_1; p)_\infty}{(pq^{b_{ij}} z_2/z_1; p)_\infty} f_i(z_1)f_j(z_2) = -z_2 \frac{(q^{-b_{ij}} z_1/z_2; p)_\infty}{(pq^{b_{ij}} z_1/z_2; p)_\infty} f_j(z_2)f_i(z_1), \qquad p^* = pq^{-2c}$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c} z_1/z_2) \psi_j^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1/z_2) \psi_j^+(q^{-\frac{c}{2}} z_2) \right),$$

$+$ Serre relations

The coefficients in z_1, z_2 are well defined in the p-adic topology.

Hopf algebroid structure

(Etingof-Varchenko'98, Koelink-Rosengren'01, H.K'08)

- Modified tensor product $\widetilde{\otimes}$ defined by adding the extra condition:

$$f(P,p^*)a \ \widetilde{\otimes}\ b = a \ \widetilde{\otimes}\ f(P+h,p)b \qquad (p = p^* q^{2c})$$

- Two moment maps $\mu_l, \mu_r : \mathcal{M}_{H^*} \hookrightarrow (U_{q,p})_{0,0}$

$$\mu_l(f) = f(P+h,p), \quad \mu_r(f) = f(P,p^*)$$

Theorem 2.3 (H.K '08, '16)

The following (Δ, ε, S) *gives an* H*-Hopf algebroid str. of* $U_{q,p}(\widehat{\mathfrak{g}})$.

- $\Delta(L^+_{ij}(z)) = \sum_k L^+_{ik}(z)\widetilde{\otimes}L^+_{kj}(z)$,

  $\Delta(\mu_l(f)) = \mu_l(f)\widetilde{\otimes}1, \quad \Delta(\mu_r(f)) = 1\widetilde{\otimes}\mu_r(f)$,
- $\varepsilon(L^+_{ij}(z)) = \delta_{ij}e^{-Q_{\varepsilon_j}}, \quad \varepsilon(\mu_l(f)) = f(P+h,p), \quad \varepsilon(\mu_r(f)) = f(P,p^*)$
- $S(L^+(z)) = L^+(z)^{-1}, \quad S(\mu_l(f)) = \mu_r(f), \quad S(\mu_r(f)) = \mu_l(f)$

The L-operator : $\widehat{\mathfrak{sl}}_N$ case (Kojima-H.K '03)

Dynamical RLL-relation $\qquad L^+(z) \in \mathrm{End}(\mathbb{C}^N) \otimes U_{q,p}(\widehat{\mathfrak{sl}}_N)$

$$R^{+(12)}(z_1/z_2, P+h)L^{+(1)}(z_1)L^{+(2)}(z_2) = L^{+(2)}(z_2)L^{+(1)}(z_1)R^{+*(12)}(z_1/z_2, P)$$

$(R^{+*} = R^+|_{p\to p^*})$

Define the half currents $E^+_{l,j}(z),\ F^+_{j,l}(z),\ K^+_j(z)$ by

$$L^+(z) = \begin{pmatrix} 1 & F^+_{1,2}(z) & F^+_{1,3}(z) & \cdots & F^+_{1,N}(z) \\ 0 & 1 & F^+_{2,3}(z) & \cdots & F^+_{2,N}(z) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F^+_{N-1,N}(z) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K^+_1(z) & 0 & \cdots & 0 \\ 0 & K^+_2(z) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K^+_N(z) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E^+_{2,1}(z) & 1 & \ddots & & \vdots \\ E^+_{3,1}(z) & E^+_{3,2}(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E^+_{N,1}(z) & E^+_{N,2}(z) & \cdots & E^+_{N,N-1}(z) & 1 \end{pmatrix}.$$

Elliptic dynamical R-matrix : $\widehat{\mathfrak{sl}}_N$ case

$R^+(z,s) = \rho^+(z)\overline{R}(z,s),\ s = P$ or $P+h$

$$\overline{R}(z,s) = \sum_{j=1}^{N} E_{jj}\otimes E_{jj} + \sum_{1\le j<l\le N}\Big(b(z,s_{j,l})E_{jj}\otimes E_{ll} + \bar{b}(z)E_{ll}\otimes E_{jj}$$
$$+c(z,s_{j,l})E_{jl}\otimes E_{lj} + \bar{c}(z,s_{j,l})E_{lj}\otimes E_{jl}\Big),$$

$$b(z,s) = \frac{[s+1][s-1]}{[s]^2}\frac{[u]}{[u+1]}, \qquad \bar{b}(z) = \frac{[u]}{[u+1]},$$

$$c(z,s) = \frac{[s+u][1]}{[s][u+1]}, \qquad \bar{c}(z,s) = \frac{[s-u][1]}{[s][u+1]},$$

where $z = q^{2u},\ p = q^{2r} = e^{-\frac{2\pi i}{\tau}}$ and $[u] = \vartheta_1\left(\frac{u}{r}\middle|\,\tau\right)$.

Dynamical Yang-Baxter eq.

$$R^+(z_1/z_2, P+h^{(3)})R^+(z_1,P)R^+(z_2,P+h^{(1)})$$
$$= R^+(z_2,P)R^+(z_1,P+h^{(2)})R^+(z_1/z_2,P)$$

The vertex operators of the level-1 $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-modules

Theorem 3.1 (Kojima-H.K '03 Cf. Asai-Jimbo-Miwa-Pugai '96)

The intertwiner $\Phi_V(z) : \mathcal{F}_{\Lambda_a,\nu}(\xi,\eta) \to V_z \widetilde{\otimes}\, \mathcal{F}_{\Lambda_{a'},\nu}(\xi,\eta)$ *is realized by*

$$\Phi_V(z) = \sum_{\mu=1}^{N} v_\mu \widetilde{\otimes}\, \Phi_\mu(z), \qquad V = \oplus_{\mu=1}^{N} \mathbb{C} v_\mu\ , \quad V_z = V \otimes \mathbb{C}[[z, z^{-1}]]$$

$$\Phi_N(z) =: \exp\left(\sum_{m\neq 0} (q^m - q^{-m}) \mathcal{E}_m (q^{N-1} z)^{-m} \right) : e^{-\bar{\varepsilon}_N} z^{h_{\bar{\varepsilon}_N}} z^{-\frac{1}{r}(P+h)_{\bar{\varepsilon}_N}},$$

$$\Phi_\mu(z) = F^{+}_{\mu,N}(q^{-1}z)\Phi_N(z) \qquad (\mu = 1, \cdots, N-1)$$

$$= \oint_{\mathbb{T}^{N-\mu}} \prod_{m=\mu}^{N-1} \frac{dt_m}{2\pi i t_m}\ \Phi_N(z) F_{N-1}(t_{N-1}) F_{N-2}(t_{N-2}) \cdots F_\mu(t_\mu)$$

$$\times \prod_{m=\mu}^{N-1} \frac{[v_{m+1} - v_m + (P+h)_{\mu,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m + \frac{1}{2}][(P+h)_{\mu,m+1}]}. \qquad z = q^{2u},\ t_a = q^{2v_a},\ v_N = u$$

Combinatorial notations

For $\quad \Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n) \qquad (\mu_j\in\{1,\cdots,N\})$

- For $l\in\{1,\cdots,N\}$, $I_l := \{\ i\in[1,n]\ |\ \mu_i=l\}$, $\quad \lambda_l := |I_l|\in\mathbb{Z}_{\geq 0}$, $\lambda := (\lambda_1,\cdots,\lambda_N)$. Then $I=(I_1,\cdots,I_N)$ is a partition of $[1,n]$

  i.e. $I_1\cup\cdots\cup I_N=[1,n], \quad I_k\cap I_l=\emptyset\ (k\neq l)$.
- We often denote resulting I as $I_{\mu_1\cdots\mu_n}$
- For $\lambda=(\lambda_1,\cdots,\lambda_N)$, $\quad |\lambda|=\lambda_1+\cdots+\lambda_N=n$,

  $\mathcal{I}_\lambda$: the set of all partitions $I=(I_1,\cdots,I_N)$ of $[1,n]$ with $|I_l|=\lambda_l$.
- Set also $\lambda^{(l)}:=\lambda_1+\cdots+\lambda_l$, $I^{(l)}:=I_1\cup\cdots\cup I_l =: \{i^{(l)}_1<\cdots<i^{(l)}_{\lambda^{(l)}}\}$.

Remark 3.2

Each partition $I\in\mathcal{I}_\lambda$ specifies the coordinate flag for the partial flag variety $\mathcal{F}_\lambda$ consisting of

$$0=V_0\subset V_1\subset\cdots\subset V_N=\mathbb{C}^n \quad \text{with} \quad \dim V_l/V_{l-1}=\lambda_l.$$

Theorem 3.3 (H.K '17)

$\mathbb{t} = (t_a^{(l)})\ (1 \le l \le N,\ 1 \le a \le \lambda^{(l)}),\ \mathbb{z} = (z_1, \cdots, z_n)$

$$\Phi_{\mu_1}(z_1) \cdots \Phi_{\mu_n}(z_n) \sim \oint_{\mathbb{T}^M} \underline{d\mathbb{t}}\ \widetilde{\Phi}(\mathbb{t}, \mathbb{z}) \widetilde{W}_I(\mathbb{t}, \mathbb{z}, \Pi), \qquad I = I_{\mu_1, \cdots, \mu_n}$$

- $\widetilde{\Phi}(\mathbb{t}, \mathbb{z}) = : \Phi_N(z_1) \cdots \Phi_N(z_n) :$

$$\times : F_{N-1}(t_1^{(N-1)}) \cdots F_{N-1}(t_{\lambda^{(N-1)}}^{(N-1)}) : \cdots : F_1(t_1^{(1)}) \cdots F_1(t_{\lambda^{(1)}}^{(1)}) :$$

$$\times \prod_{l=1}^{N-1} \prod_{1 \le a < b \le \lambda^{(l)}} < F_l(t_a^{(l)}) F_l(t_b^{(l)}) >^{Sym}$$

symmetric in $\mathbb{z}$ and $\{\mathbb{t}^{(l)}\}$ for each l, respectively

- $\widetilde{W}_I(\mathbb{t}, \mathbb{z}, \Pi)$

$$= \mathrm{Sym}_{t^{(1)}} \cdots \mathrm{Sym}_{t^{(N-1)}} \left[\prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \left(\left. \frac{[v_b^{(l+1)} - v_a^{(l)} + (P+h)_{\mu_s, l+1} - C_{\mu_s, l+1}][1]}{[v_b^{(l+1)} - v_a^{(l)} + 1][(P+h)_{\mu_s, l+1} - C_{\mu_s, l+1}]} \right|_{i_s^{(N)} = i_b^{(l+1)} = i_a^{(l)}} \right. \right.$$

$$\left. \left. \times \prod_{\substack{b=1 \\ i_b^{(l+1)} > i_a^{(l)}}}^{\lambda^{(l+1)}} \frac{[v_b^{(l+1)} - v_a^{(l)}]}{[v_b^{(l+1)} - v_a^{(l)} + 1]} \prod_{b=a+1}^{\lambda^{(l)}} \frac{[v_b^{(l)} - v_a^{(l)} + 1]}{[v_b^{(l)} - v_a^{(l)}]} \right) \right],$$

where $t_a^{(l)} = q^{2 v_a^{(l)}},\ z_s = q^{2u_s},\ v_s^{(N)} = u_s, \quad C_{\mu_s, l+1} = \sum_{j=s+1}^{n} < \bar{\epsilon}_{\mu_j}, h_{\mu_s, l+1} >.$

Elliptic hypergeom. integral solution to the ell.q-KZ eq.

Theorem 3.4 (H.K. '17)

$$\mathrm{tr}_{\mathcal{F}_{a,\nu}}\left(q^{-\kappa d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)\right) \sim \oint_{\mathbb{T}^M} \underline{d\mathbb{t}}\ \Phi(\mathbb{t},\mathbb{z})\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi),$$

$$\Phi(\mathbb{t},\mathbb{z}) = \mathrm{tr}_{\mathcal{F}_{a,\nu}}\left(q^{-\kappa d}\widetilde{\Phi}(\mathbb{t},\mathbb{z})\right) \qquad \textit{with } \sum_{j=1}^{n}\bar{\epsilon}_{\mu_j}=0$$

$$= \textit{const.} \prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}}(t_a^{(l)})^{\lambda_l - h_{\alpha_l}+\frac{1}{r}((P+h)_{\alpha_l}-\lambda^{(l)})}$$

$$\times\prod_{l=1}^{N-1}\left[\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}\frac{\Gamma(t_a^{(l)}/t_b^{(l+1)};p,q^{\kappa})}{\Gamma(p^*t_a^{(l)}/t_b^{(l+1)};p,q^{\kappa})}\prod_{1\le a<b\le\lambda^{(l)}}\frac{\Gamma(p^*t_a^{(l)}/t_b^{(l)},p^*t_b^{(l)}/t_a^{(l)};p,q^{\kappa})}{\Gamma(t_a^{(l)}/t_b^{(l)},t_b^{(l)}/t_a^{(l)};p,q^{\kappa})}\right],$$

$$\Gamma(z;p,q)=\frac{(pq/z;p,q)_\infty}{(z;p,q)_\infty}$$

Remark 3.5

This is an elliptic and dynamical analogue of Mimachi '96. Geometrically, this can be identified with Okounkov's vertex function with descendent .

Properties of the elliptic weight functions

It is convenient to consider

$$\mathcal{W}_I(\mathbb{t},\mathbb{z},\Pi) := \frac{\widetilde{W}_I(\mathbb{t},\mathbb{z},\Pi)H_\lambda(\mathbb{t},\mathbb{z})}{E_\lambda(\mathbb{t})}$$

$$= \mathrm{Sym}_{t^{(1)}}\cdots\mathrm{Sym}_{t^{(N-1)}}\left[\frac{\prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}} u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{\mu_{i_a^{(l)}},l+1}q^{-2C_{\mu_{i_a^{(l)}},l+1}})}{\prod_{l=1}^{N-1}\prod_{1\le a<b\le\lambda^{(l)}}[v_a^{(l)}-v_b^{(l)}][v_b^{(l)}-v_a^{(l)}-1]}\right],$$

where

$$H_\lambda(\mathbb{t},\mathbb{z}) = \prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}+1], \qquad E_\lambda(\mathbb{t}) = \prod_{l=1}^{N-1}\prod_{a,b=1}^{\lambda^{(l)}}[v_b^{(l)}-v_a^{(l)}+1],$$

and

$$u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{j,k})$$

$$= \prod_{\substack{b=1\\ i_b^{(l+1)}>i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}] \times \left.\frac{[v_b^{(l+1)}-v_a^{(l)}+(P+h)_{j,k}]}{[(P+h)_{j,k}]}\right|_{i_b^{(l+1)}=i_a^{(l)}} \times \prod_{\substack{b=1\\ i_b^{(l+1)}<i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}+1]$$

Quasi-periodicity

Proposition 4.1

For $I \in \mathcal{I}_\lambda$, the weight functions $\mathcal{W}_I(t, z, \Pi)$ have the following quasi-periodicity.

$$\mathcal{W}_I(\cdots, pt_a^{(l)}, \cdots, z, \Pi) = (-1)^{\lambda_{l+1}-\lambda_l+2}\mathcal{W}_I(\cdots, t_a^{(l)}, \cdots, z, \Pi),$$

$$\begin{aligned}
&\mathcal{W}_I(\cdots, e^{-2\pi i}t_a^{(l)}, \cdots, z, \Pi) \\
&= (-e^{-\pi i\tau})^{\lambda_{l+1}-\lambda_l+2} \\
&\times \exp\Bigg\{-\frac{2\pi i}{r}\Bigg((\lambda_{l+1}-\lambda_l)v_a^{(l)} - \sum_{b=1}^{\lambda^{(l+1)}} v_b^{(l+1)} + 2\sum_{b=1}^{\lambda^{(l)}} v_b^{(l)} - \sum_{b=1}^{\lambda^{(l-1)}} v_b^{(l-1)} \\
&\qquad -(P+h)_{l,l+1} - \lambda_{l+1}\Bigg)\Bigg\} \\
&\quad \times\mathcal{W}_I(\cdots, t_a^{(l)}, \cdots, z, \Pi) \qquad (1 \le l \le N-1, 1 \le a \le \lambda^{(l)})
\end{aligned}$$

- For each l, $\mathcal{W}_I$ has the same quasi-periodicity for all $t_a^{(l)}$.
- This and symm. property indicate that $\mathcal{W}_I$'s are merom. sections of certain line bundle over $E^{(\lambda^{(1)})} \times \cdots \times E^{(\lambda^{(N-1)})}$, where $E^{(k)} = E^k/\mathfrak{S}_k$.

Triangular property : same as the trig. case (Rimanyi, Tarasov, Varchenko'15)

Define the partial ordering for $I, J \in \mathcal{I}_\lambda$ by

$$I \le J \ \Leftrightarrow \ i_a^{(l)} \le j_a^{(l)} \quad \text{for } l = 1, \cdots, N, \ a = 1, \cdots, \lambda^{(l)}.$$

Then the specialization $\mathbb{t} = \mathbb{z}_I$ i.e. $t_a^{(l)} = z_{i_a^{(l)}}$ yields

$$(\star) \qquad \mathcal{W}_J(\mathbb{z}_I, \mathbb{z}, \Pi) = 0 \text{ unless } I \le J.$$

Remark 4.2

In geometry, $\{I\}_{I \in \mathcal{I}_\lambda}$ labels the $(\mathbb{C}^\times)^n$-fixed points of $T^*\mathcal{F}_\lambda$.

$$\mathbb{t} = \mathbb{z}_I \quad \Leftrightarrow \quad \text{the restriction to the fixed point } I$$

$(\star) \quad \Leftrightarrow \quad$ the triangular property of $\mathrm{Stab}_{\mathfrak{C}}$ w.r.t. the restriction to the fixed points.

Transition property :

For $I = I_{\cdots\ \mu_i\mu_{i+1}\cdots}$,

$$\widetilde{W}_{s_i(I)}(\mathbb{t},\cdots,z_{i+1},z_i,\cdots,\Pi)$$
$$= \sum_{\mu_i',\mu_{i+1}'} R(z_i/z_{i+1},P+h+\sum_{j=1}^{i-1}h^{(j)})_{\mu_i\mu_{i+1}}^{\mu_i'\mu_{i+1}'}\ \widetilde{W}_I(\mathbb{t},\cdots,z_i,z_{i+1},\cdots,\Pi)$$

$$\because \quad \Phi_{\mu_n}(z_n)\cdots\Phi_{\mu_1}(z_1) \sim \oint \underline{d\mathbb{t}}\ \widetilde{\Phi}(\mathbb{t},\mathbb{z})\ \widetilde{W}_I(\mathbb{t},\mathbb{z};\Pi)$$

- LHS: $\Phi_\mu(z_{i+1})\Phi_\nu(z_i) = \sum_{\mu',\nu'} R(z_i/z_{i+1},P+h)_{\nu\mu}^{\nu'\mu'}\ \Phi_{\nu'}(z_i)\Phi_{\mu'}(z_{i+1})$
- RHS: $\widetilde{\Phi}(\mathbb{t},\mathbb{z})$ is symmetric in $z_i \leftrightarrow z_{i+1}$

Orthogonality :

For $J, K \in \mathcal{I}_\lambda$, $\sigma_0 \in \mathfrak{S}_n$: the longest element

$$\sum_{I \in \mathcal{I}_\lambda} \frac{\mathcal{W}_J(\mathbb{z}_I, \mathbb{z}, \Pi^{-1} q^{2\sum_{j=1}^n <\bar{\epsilon}_{\mu_j}, h>}) \mathcal{W}_{\sigma_0(K)}(\mathbb{z}_I, \sigma_0(\mathbb{z}), \Pi)}{Q(\mathbb{z}_I) R(\mathbb{z}_I)} = \delta_{J,K}$$

where

$$Q(\mathbb{z}_I) = \prod_{1 \le k < l \le N} \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a + 1],$$

$$R(\mathbb{z}_I) = \prod_{1 \le k < l \le N} \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a].$$

$\because$ triangular property, transition property and a property of the elliptic dynamical R-matrix w.r.t the transposition

Shuffle (Feigin-Odesskii) algebra structure :

Let $\lambda, \lambda' \in \mathbb{N}^N$, $|\lambda| = m, |\lambda'| = n$, $I \in \mathcal{I}_\lambda, I' \in \mathcal{I}_{\lambda'}$.

The following $\star$-product of $\widetilde{W}_I(t, z, \Pi_I)$ and $\widetilde{W}_{I'}(t', z', \Pi'_{I'})$ gives again an elliptic weight function $\widetilde{W}_{I+I'}(t \cup t', z \cup z', \Pi_{I+I'})$.

$$
\begin{aligned}
&(\widetilde{W}_I \star \widetilde{W}_{I'})(t \cup t', z \cup z', \Pi_I \cup \Pi'_{I'}) \\
&= \frac{1}{\prod_{l=1}^{N-1} \lambda^{(l)}! \lambda'^{(l)}!} \\
&\quad \times \mathrm{Sym}^{(1)} \cdots \mathrm{Sym}^{(N-1)} \left[\widetilde{W}_I(t, z, \Pi_I q^{-2\sum_{j=1}^{n} <\bar{\epsilon}_{\mu'_j}, h>}) \, \widetilde{W}_{I'}(t', z', \Pi'_{I'}) \, \Xi(t, t', z, z') \right],
\end{aligned}
$$

where $I' = I'_{\mu'_1, \cdots, \mu'_n}$ and

$$
\Xi(t, t', z, z') = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \left(\prod_{b=1}^{\lambda'^{(l+1)}} \frac{[v_b'^{(l+1)} - v_a^{(l)}]}{[v_b'^{(l+1)} - v_a^{(l)} + 1]} \prod_{c=1}^{\lambda'^{(l)}} \frac{[v_c'^{(l)} - v_a^{(l)} + 1]}{[v_c'^{(l)} - v_a^{(l)}]} \right).
$$

Equiv. elliptic cohomology & elliptic stable envelopes

- $X = T^*\mathcal{F}_\lambda$ (Aganagic-Okounkov '16)
- $\mathrm{Ell}_T(X)$: T-equiv. elliptic cohomology, $T = A \times \mathbb{C}^\times_{q^2}$, $A = (\mathbb{C}^\times)^n$

$$\mathrm{Ell}_T(\mathrm{pt}) \cong T/p^{\mathrm{cochar(T)}} = E^n \times E, \qquad E = \mathbb{C}^\times/p^{\mathbb{Z}}$$

- $\mathcal{E}_{\mathrm{Pic}_T(X)} := \mathrm{Pic}_T(X) \otimes_{\mathbb{Z}} E$
- $\mathrm{E}_T(X) := \mathrm{Ell}_T(X) \times \mathcal{E}_{\mathrm{Pic}_T(X)}$,

$$\begin{array}{ccc} & \mathrm{Ell}_T(\mathrm{pt}) \times \mathcal{E}_{\mathrm{Pic}_T(X)} & \\ \nearrow & & \uparrow \\ z_1, \cdots, z_n, q^2 & & \Pi_{j,j+1} = q^{2(P+h)_{j,j+1}} \end{array}$$

the equiv. parameters   the Kähler parameters

- Elliptic stable envelope $\mathrm{Stab}_{\mathfrak{C}}$ ($\mathfrak{C}$: chamber of $\mathrm{Lie}A$)

$$\mathrm{Stab}_{\mathfrak{C}}: \text{ sheaves on } \mathrm{E}_T(X^A) \to \text{ sheaves on } \mathrm{E}_T(X)$$

  - Triangularity w.r.t. the restriction to the fixed point classes $\{[I]\}_{I \in \mathcal{I}_\lambda}$
  - $\cdots$

Identification of $\mathcal{W}_I$ with $\mathrm{Stab}_{\mathfrak{C}}$ for $X = T^*\mathcal{F}_\lambda$

$\mathrm{Stab}_{\mathfrak{C}}$ is obtained by the abelianization formula : (Shenfeld '13, Aganagic-Okounkov '16)

- $(T^*\mathrm{Gr}(k,n))_{S^{(k)}} = (T^*\mathbb{P}(\mathbb{C}^n))^k, \qquad S^{(k)} \subset GL(k)$,
- $(T^*\mathcal{F}_\lambda)_S = \text{“} \prod_{l=1}^{N-1} \left(T^*\mathrm{Gr}(\lambda^{(l)}, \lambda^{(l+1)})\right)_{S^{(\lambda^{(l)})}} \text{”}, \qquad S = \prod_{l=1}^{N-1} S^{(\lambda^{(l)})}$,

For a fixed component F_I in $X^A = \sqcup_{I\in\mathcal{I}_\lambda} F_I$,

$$\mathrm{Stab}_{\mathfrak{C}}(F_I)$$

$$= \mathrm{Sym}_{t^{(1)}} \cdots \mathrm{Sym}_{t^{(N-1)}} \left[\frac{\prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \text{“} \mathrm{Stab}_{\mathfrak{C}^{(\lambda^{(l+1)})}}^{T^*\mathbb{P}(\mathbb{C}^{\lambda^{(l+1)}})}(F_{i^{(l+1)}_{\mu(a)}}) \text{”}}{\prod_{l=1}^{N-1} \prod_{1\le a<b\le\lambda^{(l)}} [v_a^{(l)} - v_b^{(l)}][v_b^{(l)} - v_a^{(l)} - 1]} \right],$$

where $t_a^{(l)} = q^{2v_a^{(l)}}$ are the Chern roots of the tautological vec. b'dles on X.

$\mathrm{Stab}_{\mathfrak{C}}(F_I)$ is identical to

$$\mathcal{W}_I(\mathbb{t},\mathbb{z},\Pi)$$

$$=\mathrm{Sym}_{t^{(1)}}\cdots\mathrm{Sym}_{t^{(N-1)}}\left[\frac{\prod_{l=1}^{N-1}\prod_{a=1}^{\lambda^{(l)}} u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{\mu_{i_a^{(l)}},l+1}q^{-2C_{\mu_{i_a^{(l)}},l+1}})}{\prod_{l=1}^{N-1}\prod_{1\le a<b\le\lambda^{(l)}}[v_a^{(l)}-v_b^{(l)}][v_b^{(l)}-v_a^{(l)}-1]}\right].$$

with

$$u_I^{(l)}(t_a^{(l)},\mathbb{t}^{(l+1)},\Pi_{j,k})$$

$$=\prod_{\substack{b=1\\ i_b^{(l+1)}>i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}]\times\left.\frac{[v_b^{(l+1)}-v_a^{(l)}+(P+h)_{j,k}]}{[(P+h)_{j,k}]}\right|_{i_b^{(l+1)}=i_a^{(l)}}\times\prod_{\substack{b=1\\ i_b^{(l+1)}<i_a^{(l)}}}^{\lambda^{(l+1)}}[v_b^{(l+1)}-v_a^{(l)}+1]$$

Theorem 4.3 (H.K '18)

(1) $\quad \mathrm{Stab}_{\mathfrak{C}}(F_I)=\mathcal{W}_{\sigma_0(I)}(\tilde{\mathbb{t}},\sigma_0(\mathbb{z}^{-1}),\Pi^{-1}),$

(2) $\quad \mathrm{Stab}_{\mathfrak{C}}(F_I)|_{F_J}=\mathcal{W}_{\sigma_0(I)}(\tilde{\mathbb{t}},\sigma_0(\mathbb{z}^{-1}),\Pi^{-1})|_{\tilde{\mathbb{t}}=\mathbb{z}_J^{-1}}$

where $\sigma_0\in\mathfrak{S}_n$: the longest element, and $\tilde{t}^{(l)}_{\sigma_0^{(l)}(a)}=t_a^{(l)}$.

The Gelfand-Tsetlin basis on $V_{z_1}\widetilde{\otimes}\cdots\widetilde{\otimes}V_{z_n}$

- The Gelfand-Tsetlin basis

  $\overset{\text{def}}{\Leftrightarrow}$ the eigenbasis of the Gelfand-Tsetlin subalgebra $\mathfrak{G}$

- The Gelfand-Tsetlin subalgebra $\mathfrak{G}$:
  a commutative subalgebra of $U_{q,p}(\widehat{\mathfrak{gl}}_N)$ at level-0 generated by $K_j^+(z)$ $(j=1,\cdots,N)$

$$L^+(z)$$
$$=\begin{pmatrix}1 & F^+_{1,2} & \cdots & F^+_{1,N}\\ 0 & 1 & & \vdots\\ \vdots & & \ddots & F^+_{N-1,N}\\ 0 & \cdots & 0 & 1\end{pmatrix}\begin{pmatrix}K_1^+ & 0 & \cdots & 0\\ 0 & K_2^+ & & \\ \vdots & & \ddots & \vdots\\ 0 & \cdots & & K_N^+\end{pmatrix}\begin{pmatrix}1 & 0 & \cdots & 0\\ E^+_{2,1} & 1 & & \vdots\\ \vdots & & \ddots & 0\\ E^+_{N,1} & \cdots & E^+_{N,N-1} & 1\end{pmatrix}$$

Construction of the GT basis

(Gorbounov, Rimanyi, Tarasov, Varchenko'13,'15 : rational & trig. cases)

- Realization of $\mathfrak{S}_n$ in terms of the elliptic dynamical R:
  Define $\widetilde{S}_i(P)$ by $\widetilde{S}_i(P) := \mathcal{P}^{(ii+1)} R^{(ii+1)}(z_i/z_{i+1}, P + \sum_{j=1}^{i-1} h^{(j)}) s_i^z,$

  $$\mathcal{P} : v \widetilde{\otimes} w \mapsto w \widetilde{\otimes} v, \qquad s_i^z : z_i \leftrightarrow z_{i+1}$$

  Then DYBE and the unitarity of R yields

  $$\widetilde{S}_i(P)\widetilde{S}_{i+1}(P)\widetilde{S}_i(P) = \widetilde{S}_{i+1}(P)\widetilde{S}_i(P)\widetilde{S}_{i+1}(P),$$
  $$\widetilde{S}_i(P)\widetilde{S}_j(P) = \widetilde{S}_j(P)\widetilde{S}_i(P) \qquad (|i-j|>1)$$
  $$\widetilde{S}_i(P)^2 = 1$$

- For $\lambda = (\lambda_1, \cdots, \lambda_N)$, $I = I_{\mu_1 \cdots \mu_n} \in \mathcal{I}_\lambda$, set $v_I := v_{\mu_1} \widetilde{\otimes} \cdots \widetilde{\otimes} v_{\mu_n}$.
  Define GT bases $\{\xi_I\}_{I \in \mathcal{I}_\lambda}$ by

  $$\xi_{I^{max}} := v_{I^{max}}, \qquad \text{where} \quad I^{max} = I_{\underbrace{N \cdots N}_{\lambda_N} \cdots \underbrace{1 \cdots 1}_{\lambda_1}}$$

  $$\xi_{s_i(I)} := \widetilde{S}_i(P)\xi_I$$

Explicit realization of the Gelfand-Tsetlin basis

Theorem 5.1 (H.K '18)

$$\xi_I = \sum_{J\in\mathcal{I}_\lambda} \Xi_{IJ}(\mathbb{z}, P) v_J,$$

$$\Xi_{IJ}(\mathbb{z}, P) = \widetilde{W}_J(\mathbb{z}_I^{-1}, \mathbb{z}^{-1}, \Pi q^{2\sum_{j=1}^{n} <\bar{\epsilon}_{\mu_j}, h>}).$$

$\because$ the transition property of $\widetilde{W}_J(\mathbb{z}_I, \mathbb{z}, \Pi)$

$\Leftrightarrow$ recursion formula for Ξ obtained from $\xi_{s_i(I)} = \widetilde{S}_i(P)\xi_I$

Action of the elliptic currents on the GT basis

Theorem 5.2 (H.K '18)

$$\psi_j^{\pm}(w)\xi_I = \prod_{a\in I_j} \left.\frac{[u_a - v + 1]}{[u_a - v]}\right|_{\pm} \prod_{b\in I_{j+1}} \left.\frac{[u_b - v - 1]}{[u_b - v]}\right|_{\pm} e^{-Q_{\alpha_j}}\xi_I$$

$$E_j(w)\xi_I = a^* \sum_{i\in I_{j+1}} \delta(z_i/w) \prod_{\substack{k\in I_{j+1}\\ \neq i}} \frac{[u_i - u_k + 1]}{[u_i - u_k]} e^{-Q_{\alpha_j}} \xi_{I^{i'}}$$

$$F_j(w)\xi_I = a \sum_{i\in I_j} \delta(z_i/w) \prod_{\substack{k\in I_j\\ \neq i}} \frac{[u_k - u_i + 1]}{[u_k - u_i]} \xi_{I'^{i}}$$

where $z_j = q^{2u_j}$, $\quad I = (I_1, \cdots, I_N)$

$$(I^{i'})_j = I_j \cup \{i\}, \quad (I^{i'})_{j+1} = I_{j+1} - \{i\}, \quad (I^{i'})_k = I_k \quad (k \neq j, j+1),$$
$$(I'^{i})_j = I_j - \{i\}, \quad (I'^{i})_{j+1} = I_{j+1} \cup \{i\}, \quad (I'^{i})_k = I_k \quad (k \neq j, j+1)$$

Remark 5.3

In the trig. and non-dynamical limit, the combinatorial str. coincides with the geom. rep. of $U_q(\widehat{\mathfrak{sl}}_N)$ on the equiv. K-theory of the quiver variety of type A_{N-1} obtained by Ginzburg, Vasserot '98 and Nakajima '00.

$\mathrm{Stab}_{\mathfrak{C}}(F_I)$ and the fixed point class in $\mathrm{E}_T(X)$

By definition, "the stable classes" $\mathrm{Stab}_{\mathfrak{C}}(F_J)$ are triangular w.r.t the fixed point classes $\{[I]\}_{I\in\mathcal{I}_\lambda}$

$$\mathrm{Stab}_{\mathfrak{C}}(F_J) = \sum_{I\in\mathcal{I}_\lambda} \frac{\mathrm{Stab}_{\mathfrak{C}}(F_J)|_{F_I}}{R(z_I^{-1})}[I] \qquad \text{Cf. A.Smirnov '14}$$

Here we take $R(z_I^{-1}) = \prod_{1\le k<l\le N} \prod_{a\in I_k} \prod_{b\in I_l} [u_a - u_b]$.

Under the identification

$$\mathrm{Stab}_{\mathfrak{C}}(F_J)|_{F_I} = \mathcal{W}_{\sigma_0(J)}(\mathbb{z}_I^{-1}, \sigma_0(\mathbb{z}^{-1}), \Pi^{-1}),$$

the orthogonality of $\mathcal{W}_J$ yields

$$[I] = \sum_{J\in\mathcal{I}_\lambda} \widetilde{W}_J(\mathbb{z}_I^{-1}, \mathbb{z}^{-1}, \Pi q^{2\sum_j <\bar{\epsilon}_{\mu_j}, h>})\ \mathrm{Stab}_{\mathfrak{C}}(F_J).$$

This is identical to $\xi_I = \sum_{J\in\mathcal{I}_\lambda} \widetilde{W}_J(\mathbb{z}_I^{-1}, \mathbb{z}^{-1}, \Pi q^{2\sum_j <\bar{\epsilon}_{\mu_j}, h>}) v_J$ (Thm 5.5) !

Hence

the Gelfand-Tsetlin base $\xi_I \quad \Leftrightarrow \quad$ the fixed point class $\quad [I]$,

the standard base $\quad v_J \quad \Leftrightarrow \quad$ the stable class $\mathrm{Stab}_{\mathfrak{C}}(F_J)$

Remark 6.1

In different contexts, correspondence between GT bases and the fixed point classes has been studied by Nagao'09, Feigin, Finkelberg, Frenkel, Negut, Rybnikov'11, Tsymbaliuk'10.

Theorem 6.2 (H.K '18)

The following gives an action of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ on $\mathrm{E}_T(X)$.

$$\psi_j^{\pm}(w)[I] = \prod_{a\in I_j} \frac{[u_a - v + 1]}{[u_a - v]}\Bigg|_{\pm} \prod_{b\in I_{j+1}} \frac{[u_b - v - 1]}{[u_b - v]}\Bigg|_{\pm} \; e^{-Q_{\alpha_j}}[I]$$

$$E_j(w)[I] = a^* \sum_{i\in I_{j+1}} \delta(z_i/w) \prod_{\substack{k\in I_{j+1}\\ \neq i}} \frac{[u_i - u_k + 1]}{[u_i - u_k]} \; e^{-Q_{\alpha_j}}[I^{i'}]$$

$$F_j(w)[I] = a \sum_{i\in I_j} \delta(z_i/w) \prod_{\substack{k\in I_j\\ \neq i}} \frac{[u_k - u_i + 1]}{[u_k - u_i]} \; [{I'}^{i}].$$