Integrability of quad equations: CAC vs. BT

Jarmo Hietarinta

Department of Physics and Astronomy
University of Turku, FIN-20014 Turku, Finland

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The concept of “partial integrability” is also interesting.

Integrability concepts for lattice equations:

For lattice equations the following properties have been considered:

1. Algebraic entropy: The analysis is based of growth of complexity as a function of the number of iterations n:
   - if growth is linear in $n \Rightarrow$ equation is linearizable.
   - if growth is polynomial in $n \Rightarrow$ equation is integrable.
   - if growth is exponential in $n \Rightarrow$ equation is chaotic.

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   This may be taken as a rigorous test for integrability.
2. Singularity confinement: Very efficient method, necessary but not sufficient in its basic form.
3. Symmetry analysis, somewhat complicated.

There is still another property and it is used here:

4. **Multidimensional consistency (MDC):**
   For an equation defined on a 2D lattice it should be possible to extend it into 3D **consistently**.

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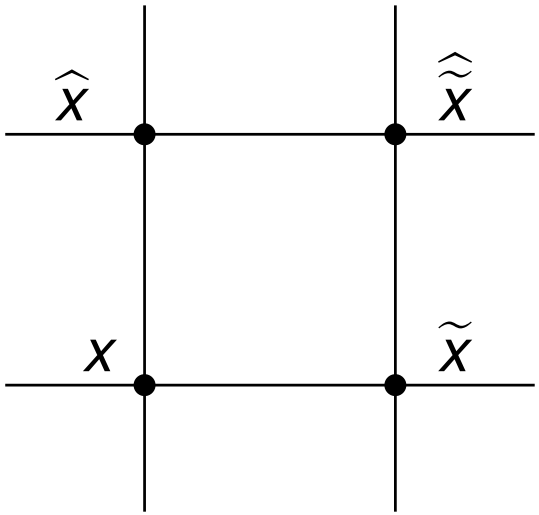
In this talk I will discuss the strength of this definition:

MDC is necessary in some form, but how strongly is it associated with integrability?

The lattice setting

The basic elementary square or quadrilateral:

$$x_{n,m} = x_{00} = x$$
$$x_{n+1,m} = x_{10} = x_{[1]} = \widetilde{x}$$
$$x_{n,m+1} = x_{01} = x_{[2]} = \widehat{x}$$
$$x_{n+1,m+1} = x_{11} = x_{[12]} = \widehat{\widetilde{x}}$$



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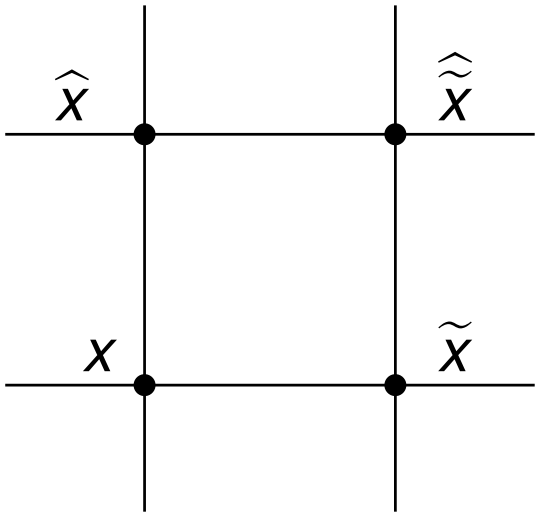
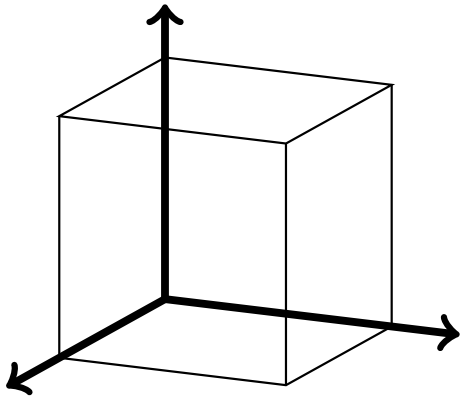


The four corner values are related by a multi-linear equation:

$$Q(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0$$

This allows propagation from initial data given on a staircase or on a corner.

Consistency-Around-a-Cube



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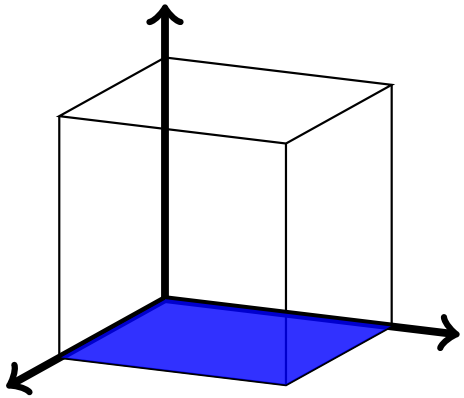

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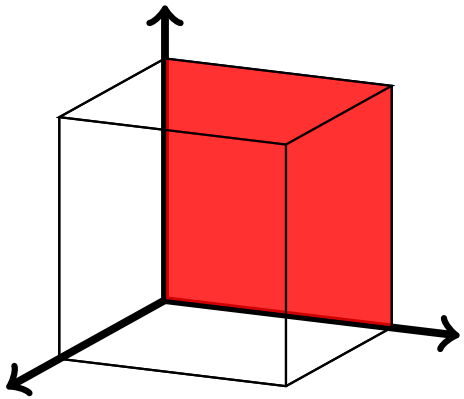

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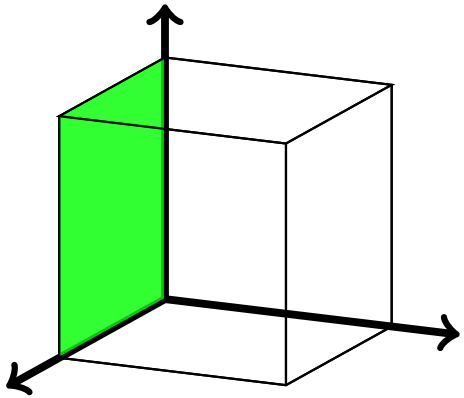

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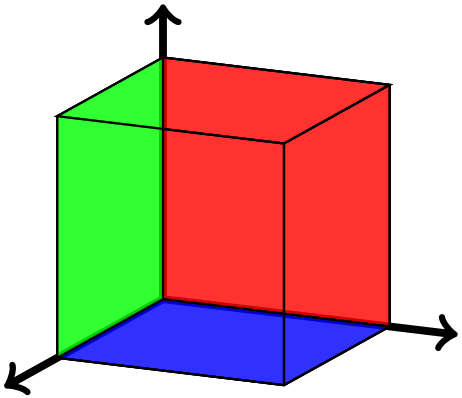

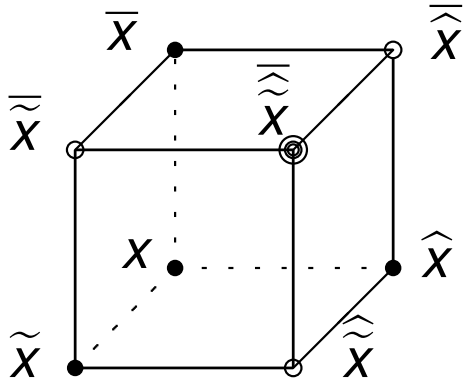

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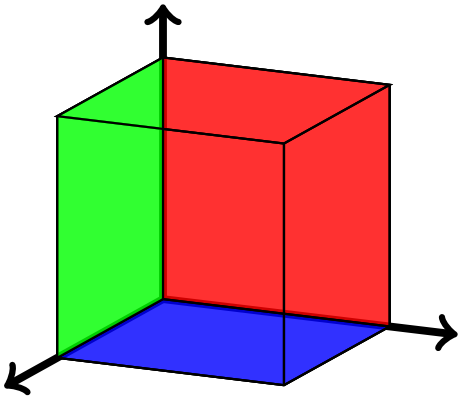
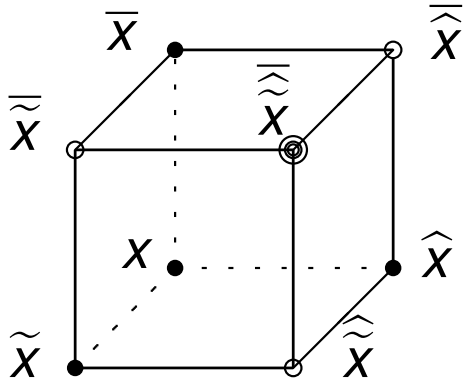
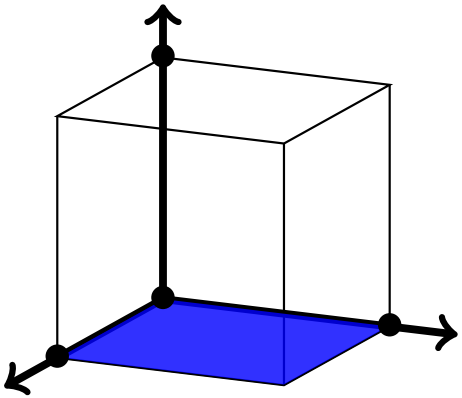
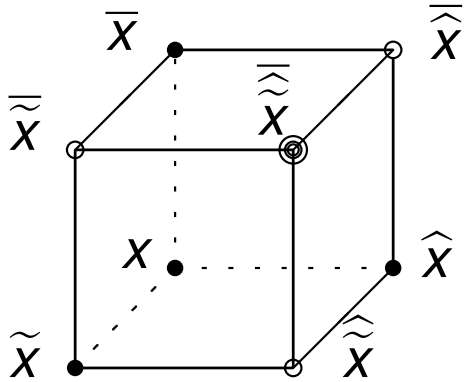




The equations on the faces of the consistency cube are:

Consistency-Around-a-Cube





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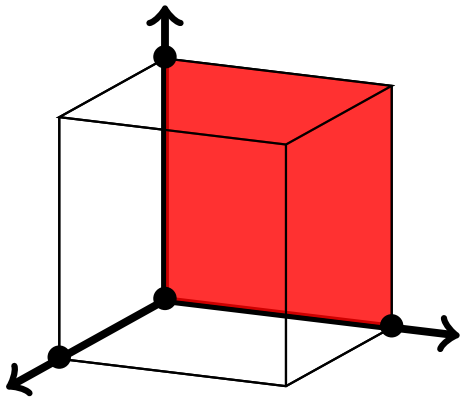

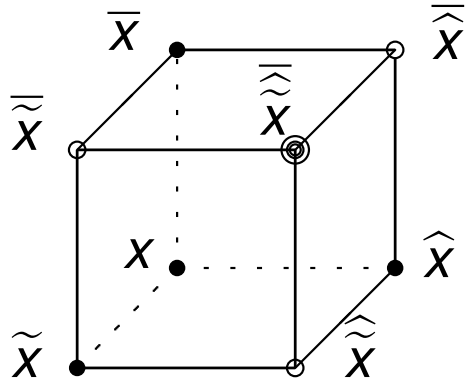

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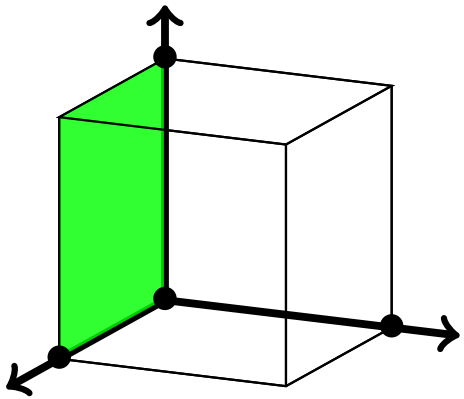

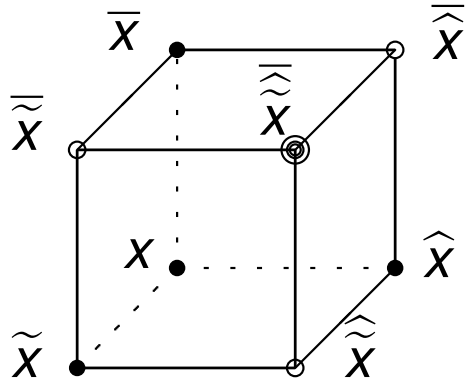

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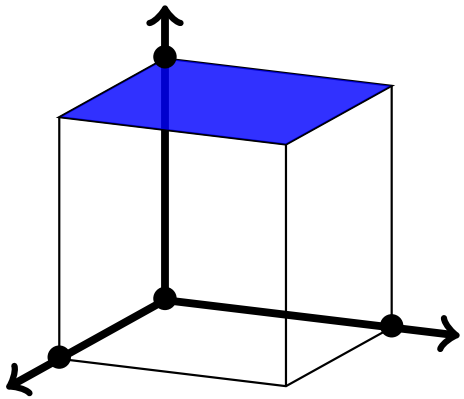

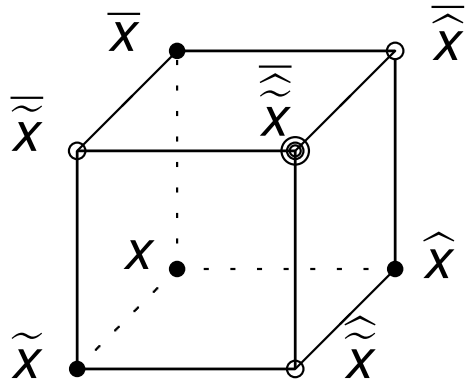

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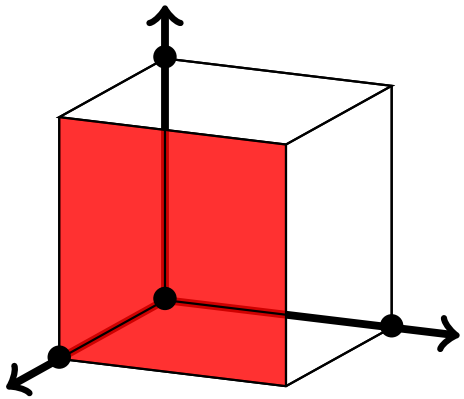

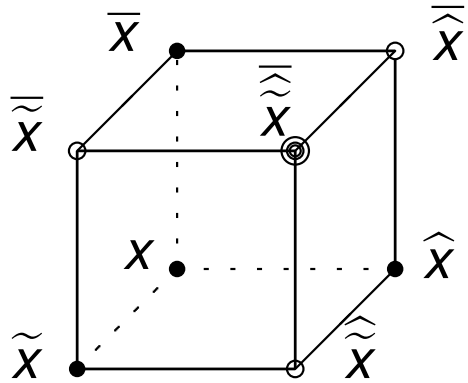

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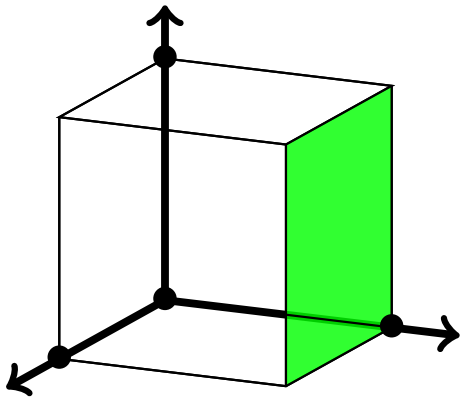

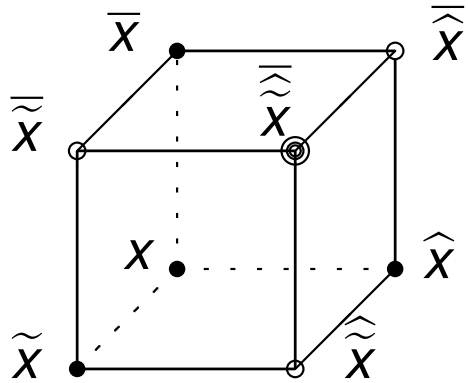

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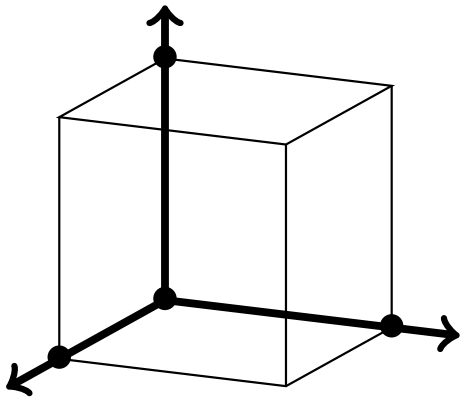

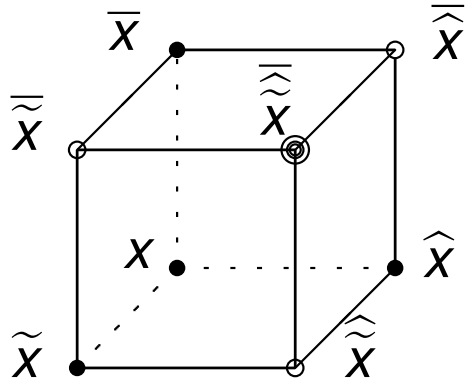

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CAC: the three computed values for $\overline{\overline{\overline{x}}}$ must be the same.

Search: The class of equations

The following assumptions are made about the equations:

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3. The equations are irreducible.
4. Equations Q_{12}, Q_{23} and Q_{31} may be different.
5. On parallel planes all quad equations are identical.
6. The equations are homogeneous quadratic.

Assumptions 1,2,3 are standard, 6 is specific to this work.

Assumption 4 means we are considering triplets of equations (cf. Boll's work, but he does not have assumption 5)

The typical equation is

$$Q_{12}(x,\widetilde{x},\widehat{x},\widehat{\widetilde{x}};p,q) := x\widetilde{x}\,c_1+\widetilde{x}\widehat{x}c_2+\widehat{x}\widehat{\widetilde{x}}\,c_3+x\widehat{\widetilde{x}}\,c_4+x\widehat{x}c_5+\widetilde{x}\widehat{\widetilde{x}}c_6 = 0,$$

c_j may depend on the lattice variables p, q.
Similarly for $Q_{23}(x,\widehat{x},\overline{x},\widehat{\overline{x}};q,r)$ and $Q_{31}(x,\overline{x},\widetilde{x},\widetilde{\overline{x}};r,p)$ by cyclic permutation $p \to q \to r$, $\widetilde{x} \to \widehat{x} \to \overline{x}$.

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The triplets of equations are named by numbers indicating which c_i are nonzero. If $|c_i| = 1$ if $c_i \neq 0$ then
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A detailed description of the search has been given in preprint **arXiv:1806.08511**.

Some result

The result include Q3($\delta = 0$), Q1($\delta = 0$), H3($\delta = 0$) along with their asymmetric reductions.

There are also more exotic triplets with CAC property, including

$$\begin{aligned} c_6(p,q)\widehat{\widetilde{x}}\widetilde{x} + c_3(p,q)\widehat{x}\widehat{\widetilde{x}} + c_1(p,q)x\widetilde{x} + c_5(p,q)x\widehat{x} &= 0, \\ r(x\widehat{x} + \overline{x}\widehat{\overline{x}}) + x\widehat{\overline{x}} + \overline{x}\widehat{x} &= 0, \\ r(x\widetilde{x} + \overline{x}\widetilde{\overline{x}}) + x\widetilde{\overline{x}} + \overline{x}\widetilde{x} &= 0. \end{aligned}$$

with code name {53,15,58}. The triply shifted variable is

$$\widehat{\widetilde{\overline{x}}} = -\overline{x}\,\frac{c_1(p,q)\widetilde{x} + c_5(p,q)\widehat{x}}{c_6(p,q)\widetilde{x} + c_3(p,q)\widehat{x}}\,,$$

which clearly has the tetrahedron property.

Two more examples: $\{63, 10, 10\}$

$$
\begin{aligned}
x\widetilde{x}\, c_1(p,q) + \widetilde{x}\widehat{x} c_2(p,q) + \widehat{x}\widehat{\widetilde{x}}\, c_3(p,q) + x\widehat{\widetilde{x}}\, c_4(p,q) + x\widehat{x} c_5(p,q) + \widetilde{x}\widehat{\widetilde{c}}_6(p,q) &= 0,\\
\widehat{x}\overline{x} - x\widehat{\overline{x}} &= 0,\\
\overline{x}\widetilde{x} - x\widetilde{\overline{x}} &= 0.
\end{aligned}
$$

$$
\widetilde{\overline{x}} = -\frac{c_5(p,q)x\overline{x}\widehat{x} + c_1(p,q)x\widetilde{x}\overline{x} + c_2(p,q)\widetilde{x}\widehat{x}\overline{x}}{c_4(p,q)x^2 + c_6(p,q)x\widetilde{x} + c_3(p,q)x\widehat{x}}\,.
$$

$\{10, 58, 15\}$

$$
\begin{aligned}
\widehat{x}\widetilde{x} - x\widehat{\widetilde{x}} &= 0,\\
x\overline{x}c_5(q,r) + x\widehat{\overline{x}}c_4(q,r) + \overline{x}\widehat{x}c_2(q,r) + \widehat{\overline{x}}\widehat{x}c_6(q,r) &= 0,\\
x\overline{x}c_1(r,p) + x\widetilde{\overline{x}}c_4(r,p) + \overline{x}\widetilde{x}c_2(r,p) + \widetilde{\overline{x}}\widetilde{x}c_3(r,p) &= 0.
\end{aligned}
$$

$$
\widetilde{\overline{x}} = \overline{x}\,\frac{(c_1(r,p)x + c_2(r,p)\widetilde{x})(c_5(q,r)x + c_2(q,r)\widehat{x})}{(c_4(r,p)x + c_3(r,p)\widetilde{x})(c_4(q,r)x + c_6(q,r)\widehat{x})}.
$$

Gauge invariance, rotations and reflections

The equations can be changed by a gauge transformation provided that the transformation does not break the uniformity of the 3D lattice. That is, it should not introduce non-autonomous terms, except possibly as overall multipliers.

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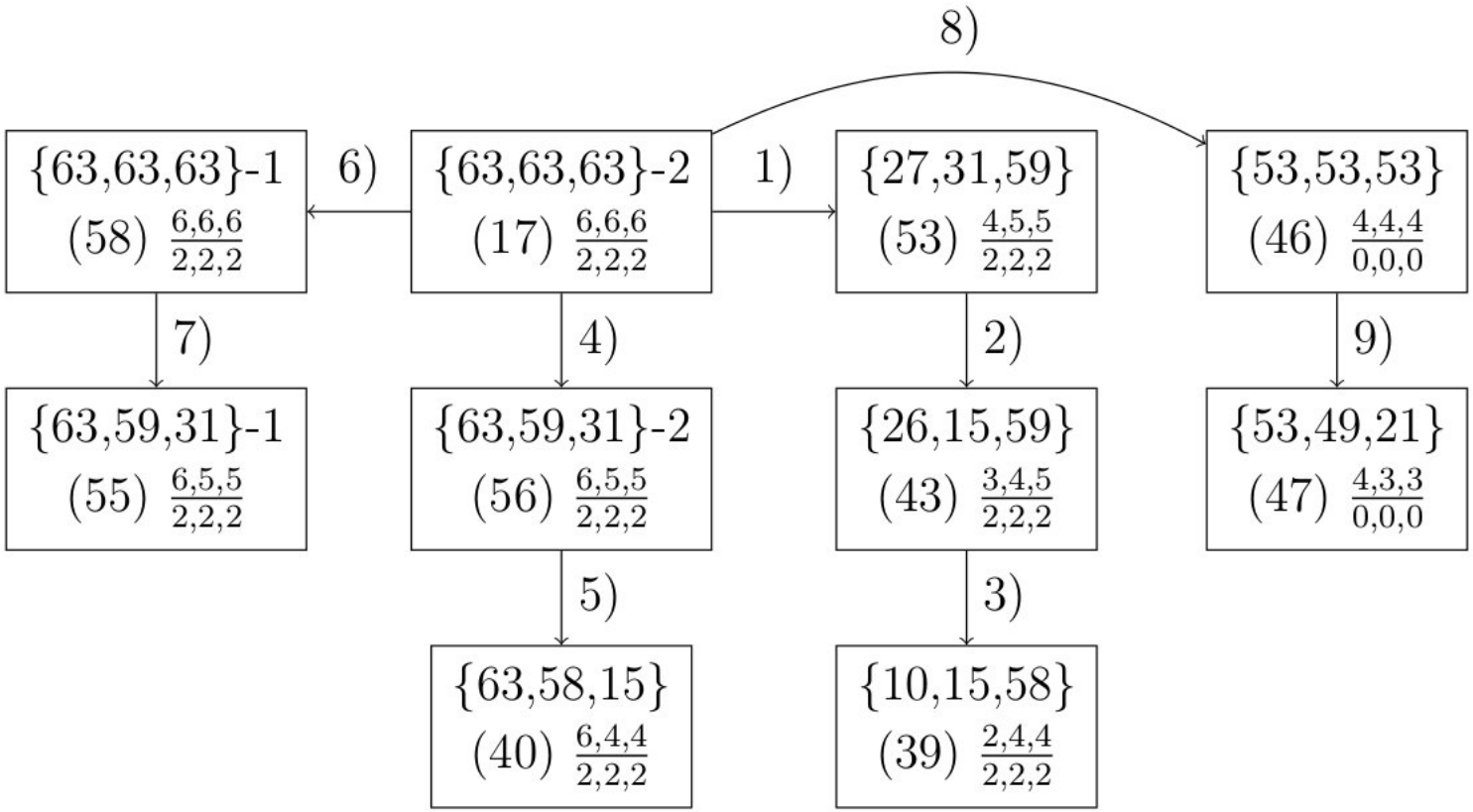
For a given triplet other related triplets can be obtained by rotations and reflections, they form the orbit of the triplet.

It is only necessary to consider one representative of an orbit.

Reductions, part 1

The results can be arranged into groups whose entries are related by reductions. For example:



{63,63,63}-2 is Q3($\delta = 0$), {63,63,63}-1 is Q1($\delta = 0$) and {53,53,53} is H3($\delta = 0$). All others are asymmetric reductions.

Reductions, part 1, an example

Starting with {63,63,63}-2, which is Q3

$$
\begin{aligned}
(q^2-p^2)(x\widehat{\widetilde{x}}+\widehat{x}\widetilde{x})-p(q^2-1)(x\widehat{x}+\widehat{\widetilde{x}}\widetilde{x})+(p^2-1)q(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}}) &= 0, \qquad (17a)\\
(r^2-q^2)(x\widehat{\overline{x}}+\overline{x}\widehat{x})-q(r^2-1)(x\overline{x}+\widehat{\overline{x}}\widehat{x})+(q^2-1)r(x\widehat{x}+\overline{x}\widehat{\overline{x}}) &= 0, \qquad (17b)\\
(p^2-r^2)(x\widetilde{\overline{x}}+\widetilde{x}\overline{x})-r(p^2-1)(x\widetilde{x}+\widetilde{\overline{x}}\overline{x})+(r^2-1)p(x\overline{x}+\widetilde{x}\widetilde{\overline{x}}) &= 0. \qquad (17c)
\end{aligned}
$$

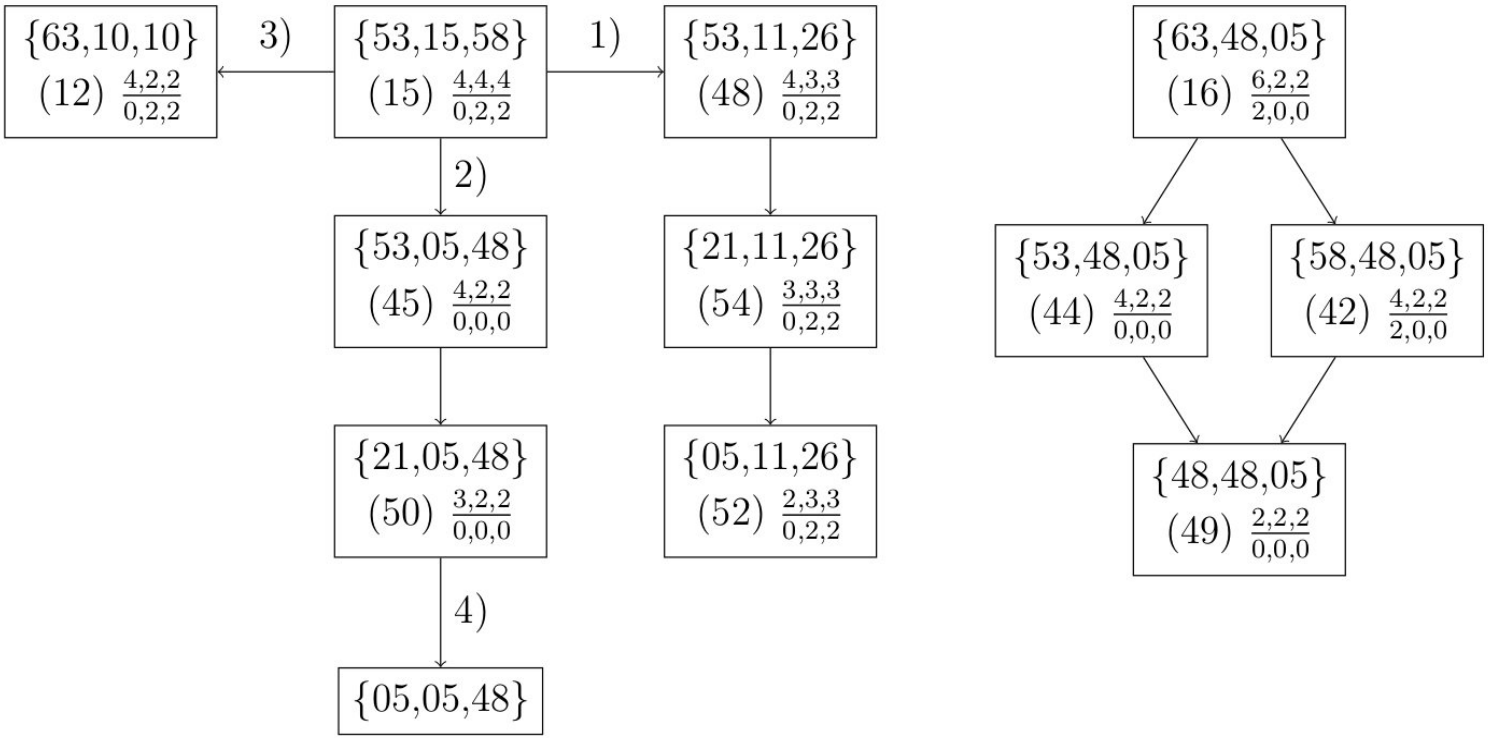
Scale $\overline{x} \to \overline{x}/r$ and then take the leading term as $r \to \infty$.
Also change signs of p, q. Then get

{63,59,31}-2

$$
\begin{aligned}
(p^2-q^2)(x\widehat{\widetilde{x}}+\widehat{x}\widetilde{x})+(p^2-1)q(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})-p(q^2-1)(x\widehat{x}+\widehat{\widetilde{x}}\widetilde{x}) &= 0, \qquad (56a)\\
(q^2-1)x\widehat{x}+q(x\overline{x}+\widehat{\overline{x}}\widehat{x})+x\widehat{\overline{x}}+\overline{x}\widehat{x} &= 0, \qquad (56b)\\
(p^2-1)x\widetilde{x}+p(x\overline{x}+\widetilde{\overline{x}}\widetilde{x})+x\widetilde{\overline{x}}+\overline{x}\widetilde{x} &= 0, \qquad (56c)
\end{aligned}
$$

Reductions, part 2

Two further chains of reductions, here the entries contain some arbitrary functions.



Definition of Bäcklund transformation

We can arrange the 6 equations of the cube as follows ($y = \overline{x}$):

$$Q_{12}(y, \widetilde{y}, \widehat{y}, \widehat{\widetilde{y}}; p, q) = 0, \tag{2a}$$

$$Q_{23}(x, \widehat{x}, y, \widehat{y}; q, r) = 0, \quad Q_{23}(\widetilde{x}, \widehat{\widetilde{x}}, \widetilde{y}, \widehat{\widetilde{y}}; q, r) = 0 \tag{2b}$$

$$Q_{31}(x, y, \widetilde{x}, \widetilde{y}; r, p) = 0, \quad Q_{31}(\widehat{x}, \widehat{y}, \widehat{\widetilde{x}}, \widehat{\widetilde{y}}; r, p) = 0, \tag{2c}$$

$$Q_{12}(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0, \tag{2d}$$

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$$Q_{12}(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0, \tag{2d}$$

This provides a Bäcklund transformation (BT) if

- After eliminating $\widetilde{y}, \widehat{y}, \widehat{\widetilde{y}}$ from three of the equations in (2b),(2c), the fourth equation factorizes as

$$(\text{something depending on y}) \times Q_{12}(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0$$

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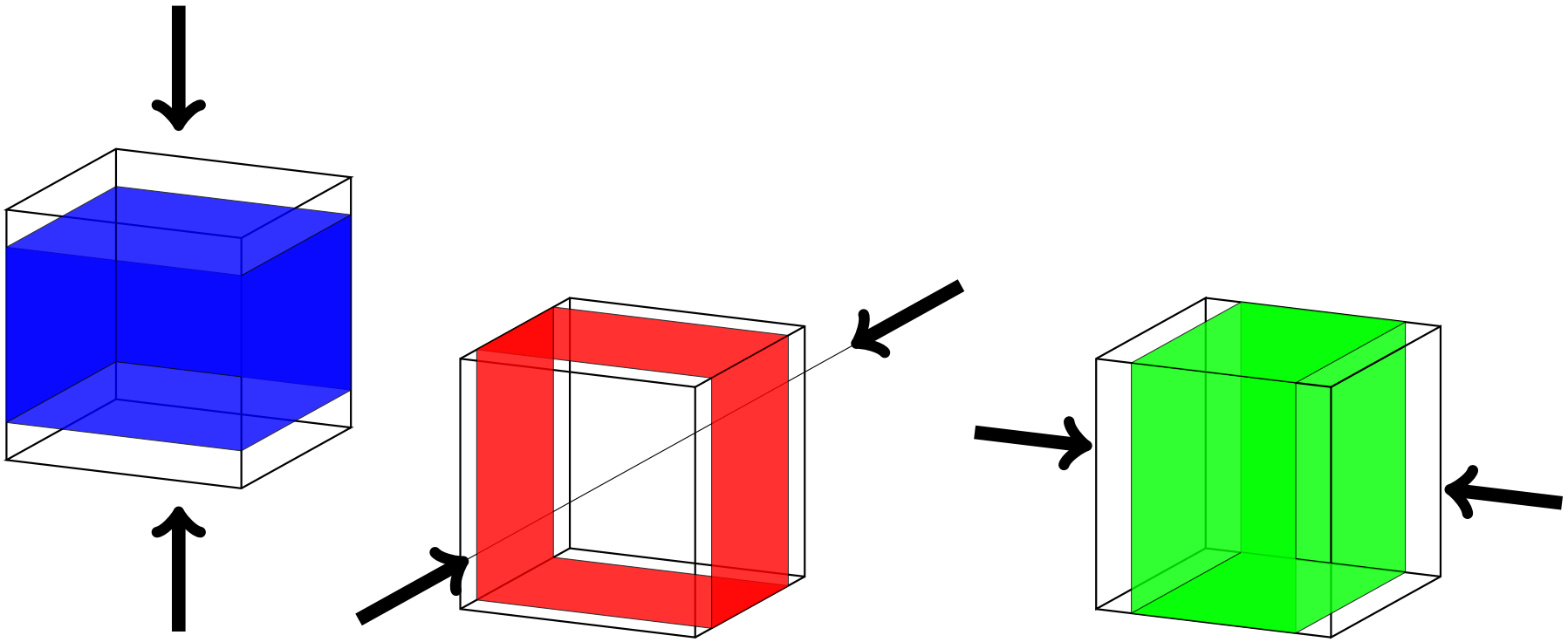
- Similarly after eliminating $\widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}$ we get $Q_{12}(y, \widetilde{y}, \widehat{y}, \widehat{\widetilde{y}}; p, q)$.

Three ways to generate equations by BT:

- top and bottom from left-back-right-front
- front and back from bottom-left-top-right
- left and right from bottom-back-top-front



Recall:

• After eliminating $\widetilde{y}, \widehat{y}, \widehat{\widetilde{y}}$ from three of the equations in (2b),(2c), the fourth equation factorizes as

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Recall:

• After eliminating $\widetilde{y}, \widehat{y}, \widehat{\widetilde{y}}$ from three of the equations in (2b),(2c), the fourth equation factorizes as

$$(\text{something depending on y}) \times Q_{12}(x, \widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}; p, q) = 0$$

Since we have the CAC property, everything should work as described.

Problem: The factor "(something depending on y)" may vanish identically and therefore no equation Q_{12} is produced.

Example $\{X, 10, 10\}$ (recall $y = \overline{x}$):

$$\begin{aligned} \widehat{x}y - x\widehat{y} = 0, \quad \widehat{\widetilde{x}}\widetilde{y} - \widetilde{x}\widehat{\widetilde{y}} &= 0, \\ \widetilde{x}y - x\widetilde{y} = 0, \quad \widehat{\widetilde{x}}\widehat{y} - \widehat{x}\widehat{\widetilde{y}} &= 0. \end{aligned}$$

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We find that

$$\widehat{y} = \widehat{x}\frac{y}{x}, \quad \widetilde{y} = \widetilde{x}\frac{y}{x}, \quad \widehat{\widetilde{y}} = \widehat{\widetilde{x}}\frac{y}{x},$$

and this is enough to satisfy all 4 equations.
Thus **no equation for x is generated** and we can say BT fails (equivalently: the Lax pair is fake).

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This can be predicted: With these side equations CAC is possible with bottom-top equations containing **free functions.** There is no way the simple side equations can generate them.

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This can be predicted: With these side equations CAC is possible with bottom-top equations containing **free functions**. There is no way the simple side equations can generate them.
The mere possibility of free functions implies BT fails.

BT results

Eqn	Top	Bottom	Left	Right	Back	Front	param
63,63,63-1 (58)	1	1	1	1	1	1	p, q, r
63,59,31-1 (55)	2_3	2_1	1	1	1	1	p, q
63,63,63-2 (17)	1	1	1	1	1	1	p, q, r
63,59,31-2 (56)	1	1	1	1	1	1	p, q
63,58,15 (40)	2*	2*	1	1	1	1	p, q
27,31,59 (53)	1	1	1	1	1	1	r
26,15,59 (43)	1	1	2*	2*	1	1	r
10,15,58 (39)	0	0	2*	2*	2*	2*	r
53,53,53 (46)	1	1	1	1	1	1	p, q, r
53,49,21 (47)	1	1	1	1	1	1	p, q

This table gives the number of equations produced by BTs.

| 63,58,15 (40) | 2* | 2* | 1 | 1 | 1 | 1 | p, q |

$$
\begin{aligned}
p(1-q^2)(x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}})-q(1-p^2)(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})+(p^2-q^2)(x\widehat{\widetilde{x}}+\widehat{x}\widetilde{x}) &= 0,\\
q(x\overline{x}+\widehat{x}\widehat{\overline{x}})+x\widehat{\overline{x}}+\overline{x}\widehat{x} &= 0,\\
p(x\overline{x}+\widetilde{x}\widetilde{\overline{x}})+x\widetilde{\overline{x}}+\overline{x}\widetilde{x} &= 0.
\end{aligned}
$$

$$
\widetilde{\widehat{x}}=\frac{\overline{x}(-p\widehat{x}+q\widetilde{x})}{p\widetilde{x}-q\widehat{x}}.
$$

$$\left| \begin{array}{cccccccc} 63,58,15 & (40) & 2^* & 2^* & 1 & 1 & 1 & 1 & p,q \end{array} \right|$$

$$\begin{aligned} p(1-q^2)(x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}})-q(1-p^2)(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})+(p^2-q^2)(x\widehat{\widetilde{x}}+\widehat{x}\widetilde{x}) &= 0,\\ q(x\overline{x}+\widehat{x}\widehat{\overline{x}})+x\widehat{\overline{x}}+\overline{x}\widehat{x} &= 0,\\ p(x\overline{x}+\widetilde{x}\widetilde{\overline{x}})+x\widetilde{\overline{x}}+\overline{x}\widetilde{x} &= 0. \end{aligned}$$

$$\widehat{\widetilde{x}}=\frac{\overline{x}(-p\widehat{x}+q\widetilde{x})}{p\widetilde{x}-q\widehat{x}}.$$

BT with back-left-front-right equations yield:

- if eliminate $\widetilde{y},\widehat{y},\widehat{\widetilde{y}}$ get (bottom equation)$\cdot y\cdot(x\widehat{\widetilde{x}}-\widehat{x}\widetilde{x})$.
- if eliminate $\widetilde{x},\widehat{x},\widehat{\widetilde{x}}$ get (top equation)$\cdot x\cdot(y\widehat{\widetilde{y}}-\widehat{y}\widetilde{y})$.

The extra factor corresponds to $\{10,58,15\}$.

| 63,59,31-1 (55) | 2_3 | 2_1 | 1 | 1 | 1 | 1 | p, q |

$$
\begin{aligned}
(q-p)(x\widehat{\widetilde{x}}+\widehat{x}\widetilde{x})-p(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})+q(x\widehat{x}+\widehat{\widetilde{x}}\widetilde{x}) &= 0,\\
qx\widehat{x}+(x\overline{x}+\overline{\widehat{x}}\widehat{x})+(x\overline{\widehat{x}}+\overline{x}\widehat{x}) &= 0,\\
px\widetilde{x}+(x\overline{x}+\overline{\widetilde{x}}\widetilde{x})+(x\overline{\widetilde{x}}+\widetilde{x}\overline{x}) &= 0,
\end{aligned}
$$

$$
\overline{\widehat{\widetilde{x}}}=\overline{x}+\frac{(p-q)\widehat{x}\widetilde{x}}{\widehat{x}-\widetilde{x}}.
$$

| 63,59,31-1 (55) | 2_3 | 2_1 | 1 | 1 | 1 | 1 | p, q |

$$
\begin{aligned}
(q-p)(x\widehat{\widetilde{x}}+\widehat{x}\widetilde{x})-p(x\widetilde{x}+\widehat{x}\widehat{\widetilde{x}})+q(x\widehat{x}+\widehat{\widetilde{x}}\widetilde{x}) &= 0,\\
qx\widehat{x}+(x\overline{x}+\overline{\widetilde{x}}\widehat{x})+(x\overline{\widehat{x}}+\overline{x}\widehat{x}) &= 0,\\
px\widetilde{x}+(x\overline{x}+\overline{\widetilde{x}}\widetilde{x})+(x\overline{\widetilde{x}}+\overline{x}\widetilde{x}) &= 0,
\end{aligned}
$$

$$
\overline{\widehat{\widetilde{x}}}=\overline{x}+\frac{(p-q)\widehat{x}\widetilde{x}}{\widehat{x}-\widetilde{x}}.
$$

BT with back-left-front-right equations yield:

- if eliminate $\widetilde{y}, \widehat{y}, \widehat{\widetilde{y}}$ get
  (bottom equation) $\cdot(x\widehat{x}\widehat{\widetilde{x}}+x\widehat{x}\widetilde{x}+x\widehat{\widetilde{x}}\widetilde{x}+\widehat{x}\widehat{\widetilde{x}}\widetilde{x})$
- if eliminate $\widetilde{x}, \widehat{x}, \widehat{\widetilde{x}}$ get (top equation)$\cdot x^2\cdot(y+\widetilde{y}+\widehat{y}+\widehat{\widetilde{y}})$.

Eqn	Top	Bottom	Left	Right	Back	Front	TET
53,15,58 (15)	0	0	1	1	1	1	yes
53,05,48 (45)	0	0	1	1	1	1	yes
21,05,48 (50)	0	0	1	1	1	1	yes
53,11,26 (48)	0	0	1	1	1	1	yes
21,11,26 (54)	0	0	1	1	1	1	yes
05,11,26 (52)	0	0	1	1	2^*	2^*	yes
63,48,05 (16)	0	0	1	1	1	1	no
58,48,05 (42)	0	0	$2c^1$	$2c^1$	0	0	no
53,48,05 (44)	0	0	1	1	1	1	yes
48,48,05 (49)	0	0	2	2	0	0	yes
10,15,58 (39)	0	0	2^*	2^*	2^*	2^*	yes

Eqn	Top	Bottom	Left	Right	Back	Front	TET
63,10,10	0	0	1	1	1	1	no
62,10,10	0	0	1	1	1	1	no
58,10,10	0	0	$2c^2$	$2c^2$	0	0	no
42,10,10	0	0	1	1	0	0	no
10,10,10	0	0	0	0	0	0	no
43,10,10	0	0	1	1	1	1	no
61,10,10	0	0	1	1	1	1	no
60,10,10	0	0	1	1	1	1	no
56,10,10	0	0	$2c^1$	$2c^1$	0	0	no
53,10,10	0	0	1	1	1	1	yes
52,10,10	0	0	1	1	1	1	yes
05,10,10	0	0	0	0	2	2	yes

(a) eqBT for equations of type {X,10,10}.

Eqn	Top	Bottom	Left	Right	Back	Front	TET
10,58,15	$2c^4$	$2c^4$	0	0	0	0	no
10,58,07	$2c^3$	$2c^3$	0	0	0	0	no
10,58,11	$2c^3$	$2c^3$	0	0	0	0	no
10,26,11	1	1	0	0	0	0	no
10,50,11	$2c^2$	$2c^2$	0	0	0	0	no
10,50,07	1	1	0	0	0	0	no
10,58,05	$2c^2$	$2c^2$	0	0	0	0	no
10,56,05	1	1	0	0	0	0	no
10,26,05	$2c^1$	$2c^1$	0	0	0	0	no
10,48,05	0	0	0	0	0	0	no

(b) eqBT for sub-cases of {10,58,15}.

Results for some more trivial triplets with CAC.

$$\left| \; 58{,}48{,}05\ (42) \; \right| \; 0 \; \left| \; 0 \; \right| \; 2c^1 \; \left| \; 2c^1 \; \right| \; 0 \; \left| \; 0 \; \right|$$

$$\begin{aligned} c_4(p,q)(\sigma_1 \widetilde{x}\widehat{x} + x\widehat{\widetilde{x}}) + c_5(p,q)(\sigma_2 x\widehat{x} + \widetilde{x}\widehat{\widetilde{x}}) &= 0, \\ c_5(q,r)x\overline{x} + c_6(q,r)\widehat{\widetilde{x}}\overline{\widehat{x}} &= 0, \\ x\overline{x} - \sigma_1\sigma_2\widetilde{x}\overline{\widetilde{x}} &= 0, \end{aligned}$$

$$\overline{\widetilde{x}} = \frac{x\overline{x}\, c_5(q,r)}{\widehat{x}\, c_6(q,r)} \frac{c_4(p,q)x + c_5(p,q)\widetilde{x}}{c_4(p,q)\sigma_2\widetilde{x} + \sigma_1 c_6(p,q)x}.$$

| 58,48,05 (42) | 0 | 0 | $2c^1$ | $2c^1$ | 0 | 0 |

$$
\begin{aligned}
c_4(p,q)(\sigma_1\widetilde{x}\widehat{x}+x\widehat{\widetilde{x}})+c_5(p,q)(\sigma_2 x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}}) &= 0,\\
c_5(q,r)x\overline{x}+c_6(q,r)\widehat{\widetilde{x}}\overline{\widetilde{x}} &= 0,\\
x\overline{x}-\sigma_1\sigma_2\widetilde{x}\overline{\widetilde{x}} &= 0,
\end{aligned}
$$

$$
\overline{\widetilde{\widehat{x}}}=\frac{x\overline{x}\,c_5(q,r)}{\widehat{x}\,c_6(q,r)}\,\frac{c_4(p,q)x+c_5(p,q)\widetilde{x}}{c_4(p,q)\sigma_2\widetilde{x}+\sigma_1 c_6(p,q)x}.
$$

Set $\sigma_1=\sigma_2=1$. Try to generate left and right equations.
BT with bottom-back-top-front equations yields ($y=\widehat{x}$) :

- eliminate y: $(x\overline{x}-\widetilde{x}\overline{\widetilde{x}})\cdot[c_4(p,q)(x\overline{\widetilde{x}}+\overline{x}\widetilde{x})+c_5(p,q)(x\overline{x}+\overline{\widetilde{x}}\widetilde{x})]$
- eliminate x: $(y\overline{y}-\widetilde{y}\overline{\widetilde{y}})\cdot[c_4(p,q)(y\overline{\widetilde{y}}+\overline{y}\widetilde{y})+c_5(p,q)(y\overline{y}+\overline{\widetilde{y}}\widetilde{y})]$

| 58,48,05 (42) | 0 | 0 | $2c^1$ | $2c^1$ | 0 | 0 |

$$
\begin{aligned}
c_4(p,q)(\sigma_1\widetilde{x}\widehat{x}+x\widehat{\widetilde{x}})+c_5(p,q)(\sigma_2 x\widehat{x}+\widetilde{x}\widehat{\widetilde{x}}) &= 0,\\
c_5(q,r)x\overline{x}+c_6(q,r)\widehat{\widetilde{x}}\overline{\widetilde{x}} &= 0,\\
x\overline{x}-\sigma_1\sigma_2\widetilde{x}\overline{\widetilde{x}} &= 0,
\end{aligned}
$$

$$
\overline{\widetilde{x}}=\frac{x\overline{x}\,c_5(q,r)}{\widehat{x}\,c_6(q,r)}\,\frac{c_4(p,q)x+c_5(p,q)\widetilde{x}}{c_4(p,q)\sigma_2\widetilde{x}+\sigma_1 c_6(p,q)x}.
$$

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- eliminate x: $(y\overline{y}-\widetilde{y}\overline{\widetilde{y}})\cdot[c_4(p,q)(y\overline{\widetilde{y}}+\overline{y}\widetilde{y})+c_5(p,q)(y\overline{y}+\overline{\widetilde{y}}\widetilde{y})]$

The extra factor corresponds to $\{58,48,15\}$ but has wrong parameter dependence.

Conclusions

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Conclusions

1. The CAC condition is only weakly associated with integrability, it is definitely not sufficient.
2. Multidimensional consistency is necessary in some form.
3. Our results suggest that the existence of a unique BT in all directions of the cube is rather strongly associated with integrability, while non-uniqueness may be a signal of partial integrability.
4. Integrability may be lost during reductions.

More in preprint `arXiv:1806.08511`.