Linearisability and Integrability of Discrete Dynamical Systems from Cluster Algebras

Joe Pallister

November 12, 2018

Quiver Mutations

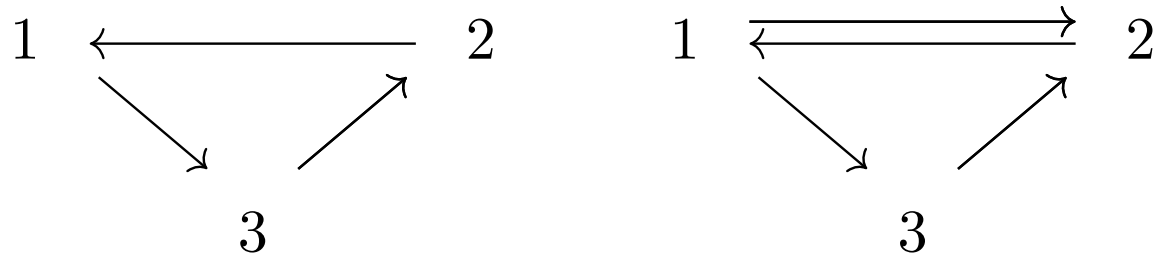
Let Q be a quiver without loops or 2 cycles. Quiver mutation at a vertex k, written μ_k, is defined as follows:

- For each length 2 path $i \to k \to j$ add a new arrow $i \to j$

Quiver Mutations

Let Q be a quiver without loops or 2 cycles. Quiver mutation at a vertex k, written μ_k, is defined as follows:

- For each length 2 path $i \to k \to j$ add a new arrow $i \to j$

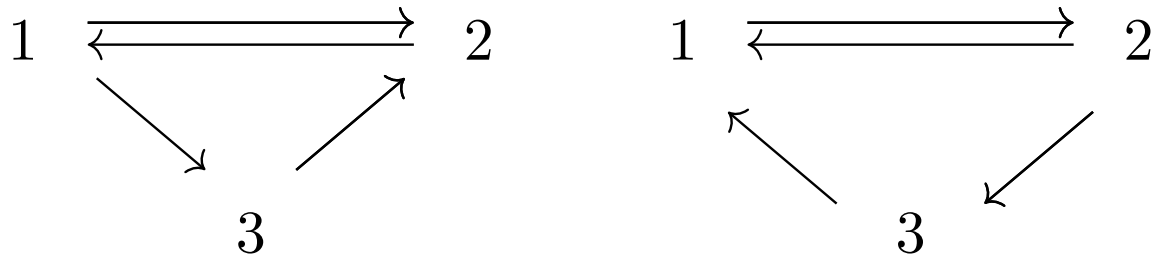


Quiver Mutations

- Reverse each arrow touching vertex k.

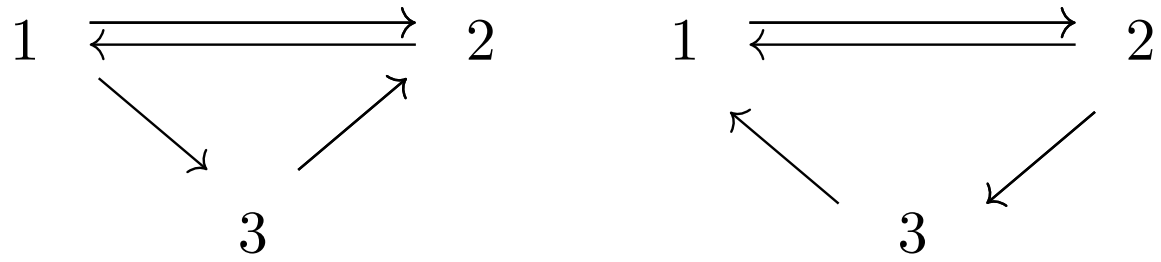
Quiver Mutations

- Reverse each arrow touching vertex k.



Quiver Mutations

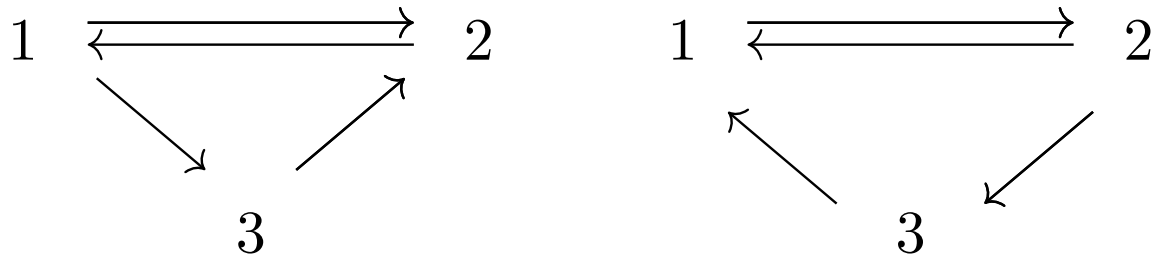
- Reverse each arrow touching vertex k.



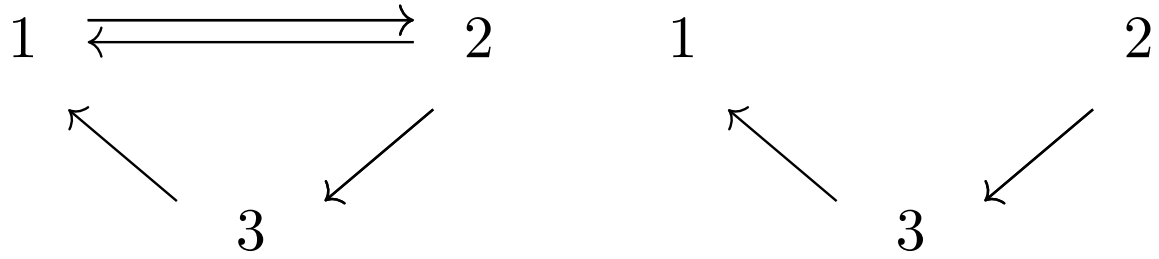
- Delete all 2 cycles that have appeared

Quiver Mutations

- Reverse each arrow touching vertex k.



- Delete all 2 cycles that have appeared



Cluster Mutations

- We consider indeterminates x_i (the initial variables) to be attached at vertex i.

Cluster Mutations

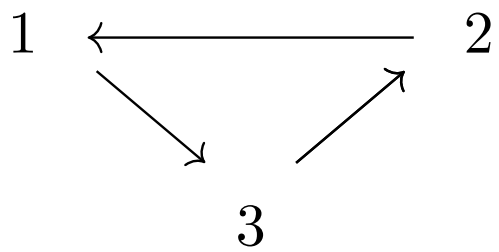
- We consider indeterminates x_i (the initial variables) to be attached at vertex i.
- Mutation at k will fix all variables x_i with $i \neq k$ but x_k becomes x'_k

$$x'_k = \frac{1}{x_k}\left(\prod_{j \to k} x_j + \prod_{j \leftarrow k} x_j\right)$$

Cluster Mutations

- We consider indeterminates x_i (the initial variables) to be attached at vertex i.
- Mutation at k will fix all variables x_i with $i \neq k$ but x_k becomes x'_k

$$x'_k = \frac{1}{x_k}\left(\prod_{j \to k} x_j + \prod_{j \leftarrow k} x_j\right)$$



$$x'_3 = \frac{1}{x_3}(x_1 + x_2)$$

Cluster Algebras

- After a mutation we get a new quiver and a new set of variables

Cluster Algebras

- After a mutation we get a new quiver and a new set of variables
- The cluster algebra $\mathcal{A}(Q)$ is the algebra generated by all cluster variables.

Cluster Algebras

- After a mutation we get a new quiver and a new set of variables
- The cluster algebra $\mathcal{A}(Q)$ is the algebra generated by all cluster variables.

Definition

The cluster algebra $\mathcal{A}(Q)$ is said to be finite type if the set of cluster variables is finite.

Finite Type

Theorem

[FZ03] A cluster algebra $\mathcal{A}(Q)$ is of finite type if and only if Q is mutation equivalent to an orientation of a simply laced Dynkin diagram.

Finite Type

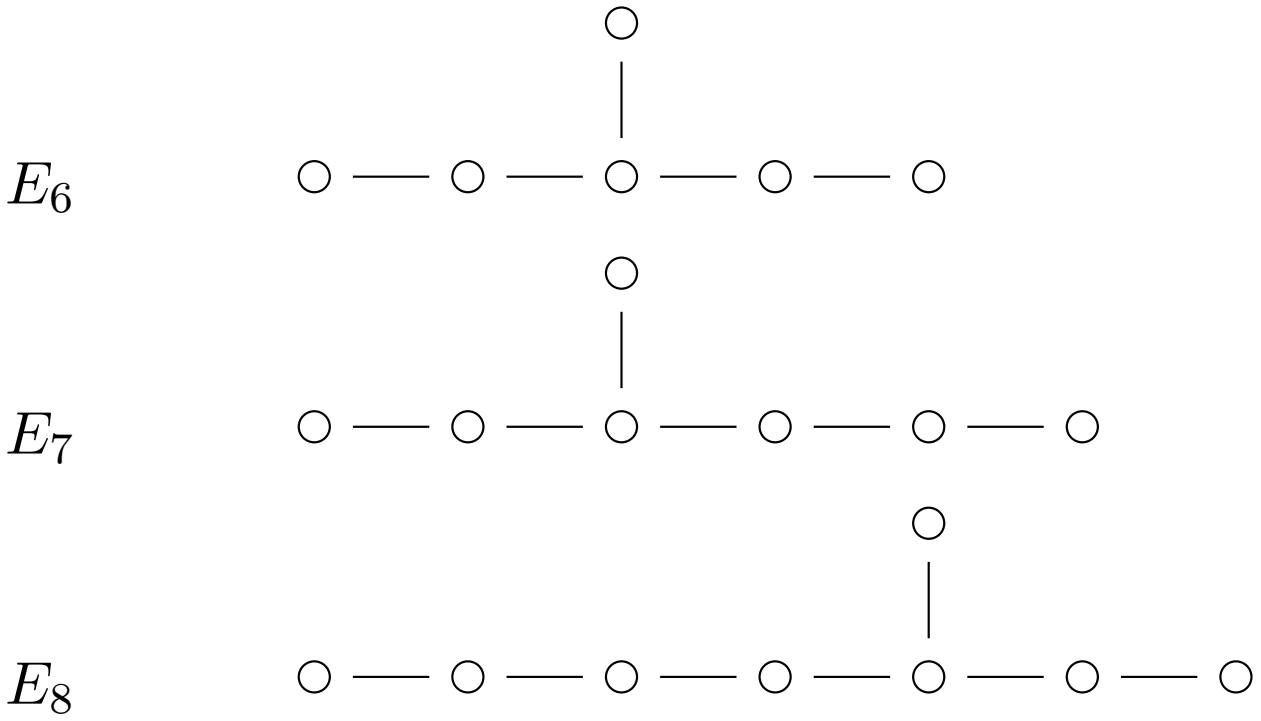
Theorem

[FZ03] A cluster algebra $\mathcal{A}(Q)$ is of finite type if and only if Q is mutation equivalent to an orientation of a simply laced Dynkin diagram.

A_N

D_N

Finite Type



Frieze Sequences

- We perform the mutation $\mu := \mu_{N+1}\mu_N \dots \mu_1$ such that we only ever mutate at a sink

Frieze Sequences

- We perform the mutation $\mu := \mu_{N+1}\mu_N \dots \mu_1$ such that we only ever mutate at a sink
- We will flip each arrow twice, recovering Q

Frieze Sequences

- We perform the mutation $\mu := \mu_{N+1}\mu_N \dots \mu_1$ such that we only ever mutate at a sink
- We will flip each arrow twice, recovering Q
- This allows us to apply μ repeatedly. For each vertex the mutation relations will look the same

Frieze Sequences

- We perform the mutation $\mu := \mu_{N+1}\mu_N \dots \mu_1$ such that we only ever mutate at a sink
- We will flip each arrow twice, recovering Q
- This allows us to apply μ repeatedly. For each vertex the mutation relations will look the same
- This gives a system of $N+1$ recurrences, one for each vertex, called the frieze sequence for Q

Frieze Sequences

- At vertex k

$$X_{n+1}^k = \frac{1}{X_n^k}\left(1 + \prod_{i \to k} X_n^i \prod_{i \leftarrow k} X_{n+1}^i\right)$$

Frieze Sequences

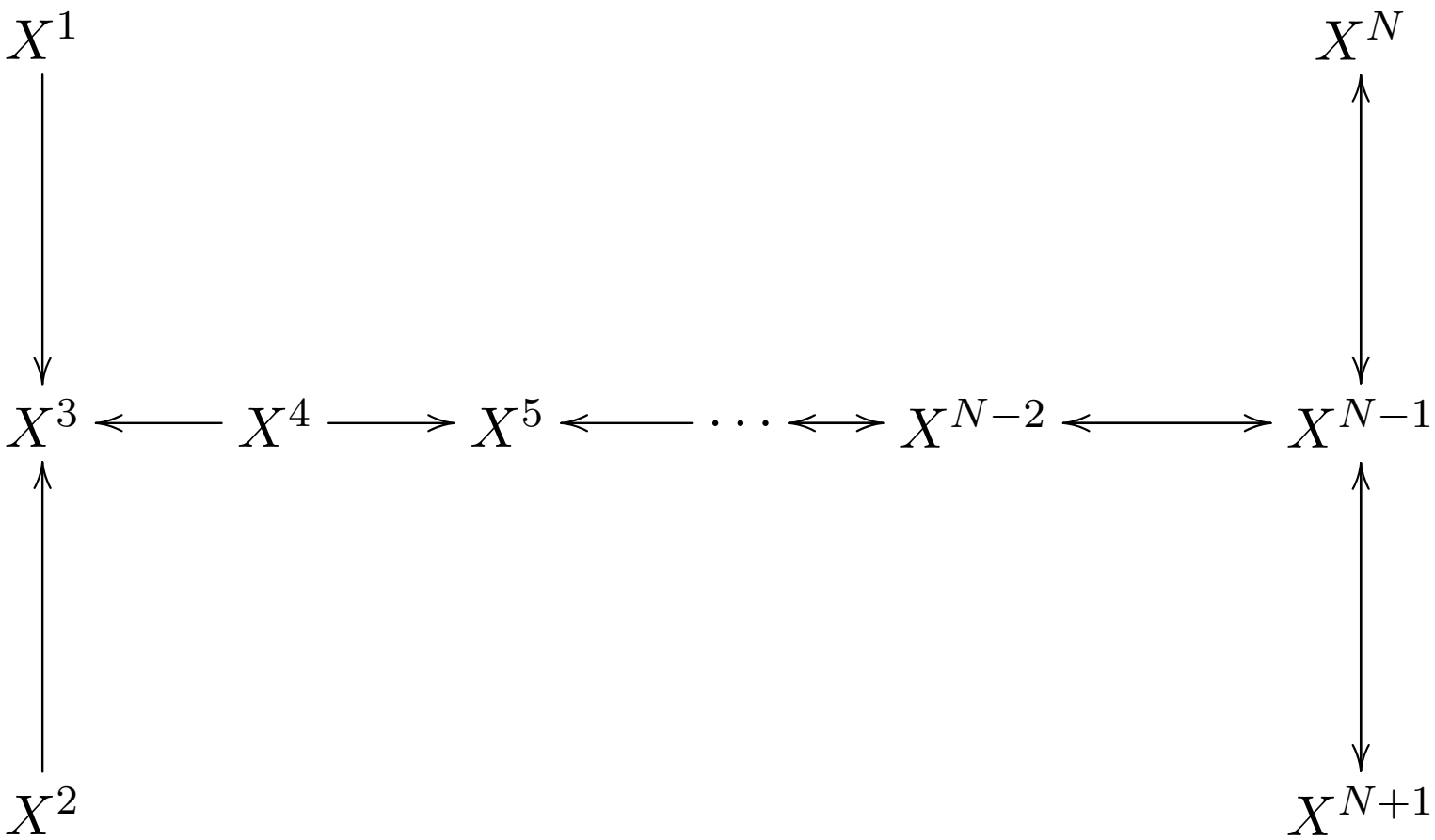
- At vertex k

$$X_{n+1}^k = \frac{1}{X_n^k}\left(1 + \prod_{i \to k} X_n^i \prod_{i \leftarrow k} X_{n+1}^i\right)$$

Theorem

[KS11] [ARS10] The frieze sequences associated to a quiver Q satisfy linear relations if and only if Q is Dynkin ADE or Affine $\tilde{A}\tilde{D}\tilde{E}$

$\tilde{D}$ type



$\tilde{D}$ type

- We will mutate at each sink then each source. These give, for example

$$X^3_{n+1} = \frac{1 + X^1_n X^2_n X^4_n}{X^3_n} \qquad X^5_{n+1} = \frac{1 + X^4_n X^6_n}{X^5_n}$$

$\tilde{D}$ type

- We will mutate at each sink then each source. These give, for example

$$X^3_{n+1} = \frac{1 + X^1_n X^2_n X^4_n}{X^3_n} \qquad X^5_{n+1} = \frac{1 + X^4_n X^6_n}{X^5_n}$$

$$X^1_{n+1} = \frac{1 + X^3_{n+1}}{X^1_n} \qquad X^4_{n+1} = \frac{1 + X^3_{n+1} X^5_{n+1}}{X^4_n}$$

$\tilde{D}$ type

- We will mutate at each sink then each source. These give, for example

$$X^3_{n+1} = \frac{1 + X^1_n X^2_n X^4_n}{X^3_n} \qquad X^5_{n+1} = \frac{1 + X^4_n X^6_n}{X^5_n}$$

$$X^1_{n+1} = \frac{1 + X^3_{n+1}}{X^1_n} \qquad X^4_{n+1} = \frac{1 + X^3_{n+1} X^5_{n+1}}{X^4_n}$$

- This construction gives the frieze sequence for $\tilde{D}_N$

Linearisability of the $\tilde{D}_9$ quiver

- We shall demonstrate the situation with an example where $N = 9$

Linearisability of the $\tilde{D}_9$ quiver

- We shall demonstrate the situation with an example where $N = 9$
- We look at the determinant of the following matrix

$$M := \begin{bmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

$$M := \begin{bmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

$$M := \begin{bmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \end{bmatrix}$$

- The top left 2×2 matrix has determinant -1 by the relations on the previous slide

Linearisability of the $\tilde{D}_9$ quiver

$$M := \begin{bmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \end{bmatrix}$$

- The top left 2×2 matrix has determinant -1 by the relations on the previous slide
- By construction, each 2×2 has determinant -1

Linearisability of the $\tilde{D}_9$ quiver

Theorem (Dodgson condensation)

Let $M = (m_{ij})$ be a 3×3 matrix. Then

$$|M| = \frac{|NW||SE| - |NE||SW|}{m_{22}}$$

where, for example, NW denotes the connected 2×2 matrix in the upper left of M

Linearisability of the $\tilde{D}_9$ quiver

Theorem (Dodgson condensation)

Let $M = (m_{ij})$ be a 3×3 matrix. Then

$$|M| = \frac{|NW||SE| - |NE||SW|}{m_{22}}$$

where, for example, NW denotes the connected 2×2 matrix in the upper left of M

- M has a kernel vector $(1, -J_n, 1)^T$

Linearisability of the $\tilde{D}_9$ quiver

We add a row to M

$$\begin{bmatrix} X^1_{n+1} X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

Such that each new 2×2 has determinant -1

$$\begin{bmatrix} X^1_{n+1} X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

With some thought we see that this new row has the same kernel vector

$$\begin{bmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \end{bmatrix} \begin{bmatrix} 1 \\ -J_n \\ 1 \end{bmatrix} = 0$$

Linearisability of the $\tilde{D}_9$ quiver

We add rows in this way until the 2×2 with μ_8 appears

$$\begin{bmatrix} X^1_{n+1} X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\ X^6_{n+3} & X^7_{n+3} & X^8_{n+2} \\ X^7_{n+4} & X^8_{n+3} & X^9_{n+3} X^{10}_{n+3} \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

- Out of the blue

$$J_n = \frac{X^1_{n+1}X^2_{n+1} + X^4_n}{X^3_{n+1}} = \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n} = \frac{X^1_{n-1}X^2_{n-1} + X^4_n}{X^3_n}$$

Linearisability of the $\tilde{D}_9$ quiver

- Out of the blue

$$J_n = \frac{X^1_{n+1}X^2_{n+1} + X^4_n}{X^3_{n+1}} = \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n} = \frac{X^1_{n-1}X^2_{n-1} + X^4_n}{X^3_n}$$

- Giving

$$\begin{bmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \end{bmatrix} \begin{bmatrix} 1 \\ -J_n \\ 1 \end{bmatrix} = 0$$

Linearisability of the $\tilde{D}_9$ quiver

This matrix we can stick to the top of M

$$\begin{bmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\ X^6_{n+3} & X^7_{n+3} & X^8_{n+2} \\ X^7_{n+4} & X^8_{n+3} & X^9_{n+3}X^{10}_{n+3} \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

From similar identities we add rows at the bottom

$$\begin{bmatrix} X_n^4 & X_n^3 & X_{n-1}^1 X_{n-1}^2 \\ X_{n+1}^1 & X_n^2 & X_{n-1}^1 \\ X_{n+1}^1 X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\ \vdots & \vdots & \vdots \\ X_{n+4}^7 & X_{n+3}^8 & X_{n+3}^9 X_{n+3}^{10} \\ X_{n+5}^{10} & X_{n+4}^9 & X_{n+3}^{10} \\ X_{n+5}^9 X_{n+5}^{10} & X_{n+4}^8 & X_{n+4}^7 \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

Now we can add rows using mutation relations again. Here we see μ_8

$$\begin{bmatrix} X_n^4 & X_n^3 & X_{n-1}^1 X_{n-1}^2 \\ X_{n+1}^1 & X_n^2 & X_{n-1}^1 \\ X_{n+1}^1 X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\ \vdots & \vdots & \vdots \\ X_{n+5}^{10} & X_{n+4}^9 & X_{n+3}^{10} \\ X_{n+5}^9 X_{n+5}^{10} & X_{n+4}^8 & X_{n+4}^7 \\ X_{n+5}^8 & X_{n+5}^7 & X_{n+4}^6 \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

Continuing...

$$\begin{bmatrix} X_n^4 & X_n^3 & X_{n-1}^1 X_{n-1}^2 \\ X_{n+1}^1 & X_n^2 & X_{n-1}^1 \\ X_{n+1}^1 X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\ \vdots & \vdots & \vdots \\ X_{n+5}^8 & X_{n+5}^7 & X_{n+4}^6 \\ X_{n+6}^7 & X_{n+5}^6 & X_{n+5}^5 \\ X_{n+6}^6 & X_{n+6}^5 & X_{n+5}^4 \\ X_{n+7}^5 & X_{n+6}^4 & X_{n+6}^3 \\ X_{n+7}^4 & X_{n+7}^3 & X_{n+6}^2 X_{n+6}^1 \end{bmatrix}$$

Linearisability of the $\tilde{D}_9$ quiver

This is enough

$$\begin{bmatrix} X_n^4 & X_n^3 & X_{n-1}^1 X_{n-1}^2 \\ X_{n+1}^1 & X_n^2 & X_{n-1}^1 \\ X_{n+1}^1 X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\ \vdots & \vdots & \vdots \\ X_{n+6}^6 & X_{n+6}^5 & X_{n+5}^4 \\ X_{n+7}^5 & X_{n+6}^4 & X_{n+6}^3 \\ X_{n+7}^4 & X_{n+7}^3 & X_{n+6}^2 X_{n+6}^1 \end{bmatrix}$$

Linear Relations with Periodic Coefficients

- From the kernel for these two rows we observe that

$$J_n = \frac{X_n^4 + X_{n-1}^1 X_{n-1}^2}{X_n^3} = \frac{X_{n+7}^4 + X_{n+6}^1 X_{n+6}^2}{X_{n+7}^3}$$

  hence J_n is period 7.

Linear Relations with Periodic Coefficients

- From the kernel for these two rows we observe that

$$J_n = \frac{X_n^4 + X_{n-1}^1 X_{n-1}^2}{X_n^3} = \frac{X_{n+7}^4 + X_{n+6}^1 X_{n+6}^2}{X_{n+7}^3}$$

  hence J_n is period 7.

- A friendlier expression is

$$J_n = \frac{X_{n+1}^1 + X_{n-1}^1}{X_n^2}$$

Linear Relations with Periodic Coefficients

- From the kernel for these two rows we observe that

$$J_n = \frac{X_n^4 + X_{n-1}^1 X_{n-1}^2}{X_n^3} = \frac{X_{n+7}^4 + X_{n+6}^1 X_{n+6}^2}{X_{n+7}^3}$$

  hence J_n is period 7.

- A friendlier expression is

$$J_n = \frac{X_{n+1}^1 + X_{n-1}^1}{X_n^2}$$

- Giving a linear relation with periodic coefficients

$$X_{n+1}^1 - J_n X_n^2 + X_{n-1}^1 = 0$$

Linear Relations with Constant Coefficients

- Observe that

$$\begin{bmatrix} X^1_{n-1} & X^2_n \\ X^1_{n+6} & X^2_{n+7} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & J_n \end{bmatrix} = \begin{bmatrix} X^2_n & X^1_{n+1} \\ X^2_{n+7} & X^1_{n+8} \end{bmatrix}$$

Linear Relations with Constant Coefficients

- Observe that

$$\begin{bmatrix} X^1_{n-1} & X^2_n \\ X^1_{n+6} & X^2_{n+7} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & J_n \end{bmatrix} = \begin{bmatrix} X^2_n & X^1_{n+1} \\ X^2_{n+7} & X^1_{n+8} \end{bmatrix}$$

- We define the matrices on the LHS as Ψ_n and L_n respectively. Hence

$$\Psi_n L_n = \sigma \Psi_{n+1}$$

where $\sigma = (12)$.

Linear Relations with Constant Coefficients

- Hence for

$$M_n := L_n L_{n+1} \dots L_{n+13}$$

we have

$$\Psi_n M_n = \Psi_{n+14}$$

and that the trace of M_n is constant, which we define as $\mathcal{K}$

Linear Relations with Constant Coefficients

- Hence for

$$M_n := L_n L_{n+1} \dots L_{n+13}$$

we have

$$\Psi_n M_n = \Psi_{n+14}$$

and that the trace of M_n is constant, which we define as $\mathcal{K}$

- By Cayley-Hamilton

$$X_{n+28}^2 - \mathcal{K} X_{n+14}^2 + X_n^2 = 0$$

Linear Relations with Constant Coefficients

Theorem

The frieze sequences for the extending vertices for the $\tilde{A}\tilde{D}\tilde{E}$ quivers satisfy the linear relations

$$X_{n+2b} - \mathcal{K}X_{n+b} + X_n = 0$$

where $\mathcal{K}$ and b are constants that depend on the quiver.

Theorem

For the $\tilde{A}\tilde{D}\tilde{E}$ frieze sequences we have quantities with the following periods

Quiver	*Period*
$\tilde{A}_{p,q}$	p q
$\tilde{D}_N$	$N-2$ 2 2
$\tilde{E}_6$	3 3 2
$\tilde{E}_7$	4 3 2
$\tilde{E}_8$	5 3? 2?

Poisson Structure

- Let $B = (b_{ij})$ be the matrix such that b_{ij} is the number of arrows from i to j in Q.

Poisson Structure

- Let $B = (b_{ij})$ be the matrix such that b_{ij} is the number of arrows from i to j in Q.
- We have a log-canonical 2-form

$$\omega = \sum_{i<j} b_{ij} d\log(x_i) \wedge d\log(x_j)$$

Poisson Structure

- Let $B = (b_{ij})$ be the matrix such that b_{ij} is the number of arrows from i to j in Q.
- We have a log-canonical 2-form

$$\omega = \sum_{i<j} b_{ij} d\log(x_i) \wedge d\log(x_j)$$

- Under a mutation μ_k:

$$\mu_k(\omega) = \sum_{i<j} b'_{ij} d\log(x'_i) \wedge d\log(x'_j)$$

Poisson Structure

- This 2-form is closed and is nondegenerate if and only if B is. In the $\tilde{D}$ case B is degenerate.

Poisson Structure

- This 2-form is closed and is nondegenerate if and only if B is. In the $\tilde{D}$ case B is degenerate.
- It is a theorem of [FH14] that we can project to a lower dimensional space with a log-canonical, non-degenerate two form with coefficients $\hat{B}$.

Poisson Structure

- This 2-form is closed and is nondegenerate if and only if B is. In the $\tilde{D}$ case B is degenerate.
- It is a theorem of [FH14] that we can project to a lower dimensional space with a log-canonical, non-degenerate two form with coefficients $\hat{B}$.
- Defining $C := \hat{B}^{-1}$ will give coefficients for a log canonical Poisson bracket

Poisson Structure for $\tilde{D}_9$

- In our case with $N = 9$ we have reduced coordinates

$$X^1X^2, X^3, X^4, \ldots, X^8, X^9X^{10}$$

and $\hat{B}$ is

$$\begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & -1 & & & & & \\ & 1 & 0 & 1 & & & & \\ & & -1 & 0 & -1 & & & \\ & & & 1 & 0 & 1 & & \\ & & & & -1 & 0 & -1 & \\ & & & & & 1 & 0 & 1 \\ & & & & & & -1 & 0 \end{bmatrix}$$

Poisson Structure for $\tilde{D}_9$

- The J variables form a Poisson subalgebra

$$\{J_i, J_j\} = (-1)^{i+j+1} J_i J_j - \delta_{i,j-1} + \delta_{i,j+1}$$

Poisson Structure for $\tilde{D}_9$

- The J variables form a Poisson subalgebra

$$\{J_i, J_j\} = (-1)^{i+j+1} J_i J_j - \delta_{i,j-1} + \delta_{i,j+1}$$

- We observe that this is a sum of a quadratic and a constant bracket

Poisson Structure for $\tilde{D}_N$ with Odd N

Theorem

For the $\tilde{D}_N$ quiver with odd N the J_i generate a Poisson subalgebra such that

$$\{J_i, J_j\} = (-1)^{i+j+1} J_i J_j - \delta_{i,j-1} + \delta_{i,j+1}$$

Integrability for $\tilde{D}_N$ with Odd N

- From this we can construct a bi-Hamiltonian ladder of functions h_i starting with the Casimir for $\{-,-\}_0$ and ending with the Casimir for $\{-,-\}_2$, which are easily found to be

$$h_1 = \sum J_i \qquad h_{(N-1)/2} = \prod J_i$$

Integrability for $\tilde{D}_N$ with Odd N

- From this we can construct a bi-Hamiltonian ladder of functions h_i starting with the Casimir for $\{-,-\}_0$ and ending with the Casimir for $\{-,-\}_2$, which are easily found to be

$$h_1 = \sum J_i \qquad h_{(N-1)/2} = \prod J_i$$

- It is a theorem of [Mag78] that the h_i provide enough commuting first integrals for Liouville integrability of the $N-2$ dimensional system

Ibrahim Assem, Christophe Reutenauer, and David Smith, *Friezes.*, Adv. Math. **225** (2010), no. 6, 3134–3165 (English).

Allan P Fordy and Andrew Hone, *Discrete integrable systems and poisson algebras from cluster maps*, Communications in Mathematical Physics **325** (2014), no. 2, 527–584.

Sergey Fomin and Andrei Zelevinsky, *Cluster algebras ii: Finite type classification*, Inventiones mathematicae **154** (2003), no. 1, 63–121.

Bernhard Keller and Sarah Scherotzke, *Linear recurrence relations for cluster variables of affine quivers*, Advances in Mathematics **228** (2011), no. 3, 1842–1862.

Franco Magri, *A simple model of the integrable hamiltonian equation*, Journal of Mathematical Physics **19** (1978), no. 5, 1156–1162.