Toda and q-Toda equations for Nekrasov partition functions

Kanehisa Takasaki

SIDE13, Hakata, November 13, 2018

1. Introduction
2. Deriving q-Toda equations from Toda hierarchy
3. Toda-like equations for $U(1)$ Nekrasov functions
4. Dual partition function of 5D $SU(N)$ theory
5. Conclusion

1. Introduction

Background:

Recent studies on isomonodromic tau functions

— Conformal blocks, Nekrasov functions, etc.

— Conjectures on Toda-like bilinear equations

This talk:

Previous results 1), 2) and outlook on

— Origin of q-Toda equations in 2D Toda hierarchy

— Toda-like equation for $U(1)$ Nekrasov functions

— Dual partition function of 5D $SU(N)$ theory

1) K.T., J. Geom. Phys. **59** (2009), 1244–1257, arXiv:0903.2607.

2) T. Maeda, T. Nakatsu, K.T. and T. Tamakoshi, JHEP **03** (2005), 056, arxiv:hep-th/0412327.

2. Deriving q-Toda equations from Toda hierarchy

Ref) K.T., J. Geom. Phys. **59** (2009), 1244–1257, arXiv:0903.2607.

Tau functions of 2D Toda hierarchy

$$\tau(s,\boldsymbol{t},\bar{\boldsymbol{t}}) = \sum_{\lambda,\bar{\lambda}\in\mathcal{P}} S_\lambda(\boldsymbol{t}) S_{\bar{\lambda}}(\bar{\boldsymbol{t}}) g_{s,\lambda\bar{\lambda}},$$

$$s\in\mathbb{Z},\ \boldsymbol{t}=(t_1,t_2,\ldots),\ \bar{\boldsymbol{t}}=(\bar{t}_1,\bar{t}_2,\ldots),$$

where

$$S_\lambda(\boldsymbol{t}) = \det(S_{\lambda_i-i+j}(\boldsymbol{t}))_{i,j=1}^\infty,$$

$$\sum_{n=0}^\infty S_n(\boldsymbol{t}) z^n = \exp\left(\sum_{k=1}^\infty t_k z^k\right),$$

$$g_{s,\lambda\bar{\lambda}} = \langle s,\lambda|g|s,\lambda\rangle,\ g\in\widehat{GL}(\infty)$$

2D difference Toda equation

$\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}})$ satisfies the bilinear equation

$$\tau(s, \boldsymbol{t} - [x], \bar{\boldsymbol{t}})\tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}} - [y]) - \tau(s, \boldsymbol{t}, \bar{\boldsymbol{t}})\tau(s, \boldsymbol{t} - [x], \bar{\boldsymbol{t}} - [y])$$
$$+xy\tau(s+1, \boldsymbol{t}, \bar{\boldsymbol{t}})\tau(s-1, \boldsymbol{t} - [x], \bar{\boldsymbol{t}} - [y]) = 0,$$

where

$$[x] = \left(x, \frac{x^2}{2}, \dots, \frac{x^k}{k}, \dots \right).$$

This is a difference analogue of the 2D Toda equation.

Key to q-difference analogue

Define a specialized tau function as

$$\mathcal{T}(s,x,y)=\tau(s,[x]_q,[y]_{\bar{q}}),$$

$$[x]_q=\left(\frac{x}{1-q},\frac{x^2}{2(1-q^2)},\dots,\frac{x^k}{k(1-q^k)},\dots\right),$$

$$[y]_{\bar{q}}=\left(\frac{x}{1-\bar{q}},\frac{x^2}{2(1-\bar{q}^2)},\dots,\frac{x^k}{k(1-\bar{q}^k)},\dots\right).$$

Key relations

$$[qx]_q=[x]_q-[x],\quad [\bar{q}y]_{\bar{q}}=[y]_{\bar{q}}-[y],$$

$$\mathcal{T}(s,qx,y)=\tau(s,[x]_q-[x],[y]_{\bar{q}}),$$

$$\mathcal{T}(s,x,\bar{q}y)=\tau(s,[x]_q,[y]_{\bar{q}}-[y]).$$

q-difference analogue of 2D Toda equation

$\mathcal{T}(s,x,y) = \tau(s,[x]_q,[y]_{\bar{q}})$ satisfies the q-difference equation

$$\mathcal{T}(s,qx,y)\mathcal{T}(s,x,\bar{q}y) - \mathcal{T}(s,x,y)\mathcal{T}(s,qx,\bar{q}y)$$
$$+xy\mathcal{T}(s+1,x,y)\mathcal{T}(s-1,qx,\bar{q}y) = 0.$$

This is a q-difference analogue (Kajiwara & Satsuma 1991) of the 2D Toda equation.

Remarks on Schur functions

1) $S_\lambda(t_1, t_2, \ldots)$ can be converted to the Schur function $s_\lambda(x_1, x_2, \ldots)$ in Macdonald's book by substituting

$$t_k = \frac{1}{k}\sum_{i\geq 1} x_i^k, \quad k = 1, 2, \ldots.$$

2) $S_\lambda([x]_q)$ is related to the principal specialization of $s_\lambda(x_1, x_2, \ldots)$:

$$S_\lambda([x]_q) = s_\lambda(x, xq, xq^2, \ldots) = x^{|\lambda|} s_\lambda(1, q, q^2, \ldots).$$

Thus $\mathcal{T}(s, x, y)$ is a double sum of the form

$$\mathcal{T}(s, x, y) = \sum_{\lambda, \bar{\lambda} \in \mathcal{P}} s_\lambda(1, q, q^2, \ldots) s_{\bar{\lambda}}(1, \bar{q}, \bar{q}^2, \ldots) x^{|\lambda|} y^{|\bar{\lambda}|} g_{s, \lambda\bar{\lambda}}.$$

Further specialization of tau function

If g takes the diagonal form

$$g = \exp\left(\sum_{n=-\infty}^{\infty} \log g_n :\psi_{-n}\psi_n^*:\right) \longleftrightarrow \mathrm{diag}(g_n)_{n=-\infty}^{\infty},$$

the matrix elements of g can be expressed as

$$\langle s, \lambda | g | s, \bar{\lambda}\rangle = \delta_{\lambda\bar{\lambda}} g_{s,\lambda}, \quad g_{s,\lambda} = \prod_{i=1}^{\infty} \frac{g_{\lambda_i - i + 1 + s}}{g_{-i+1}},$$

and $\mathcal{T}(s, x, y)$ becomes a single sum

$$\mathcal{T}(s, x, y) = \sum_{\lambda \in \mathcal{P}} s_\lambda(1, q, q^2, \ldots) s_\lambda(1, \bar{q}, \bar{q}^2, \ldots)(xy)^{|\lambda|} g_{s,\lambda}.$$

Further specialization of tau function (cont'd)

In this case,

$$\mathcal{T}(s,x,y) = \mathcal{T}(s,xy),$$

$$\mathcal{T}(s,z) = \sum_{\lambda\in\mathcal{P}} s_\lambda(1,q,q^2,\ldots)s_\lambda(1,\bar{q},\bar{q}^2,\ldots)z^{|\lambda|}g_{s,\lambda}.$$

$\mathcal{T}(s,z)$ satisfies a 1D reduction of the 2D q-difference Toda equation:

$$\mathcal{T}(s,qz)\mathcal{T}(s,\bar{q}z) - \mathcal{T}(s,z)\mathcal{T}(s,q\bar{q}z)$$
$$+z\mathcal{T}(s+1,z)\mathcal{T}(s-1,q\bar{q}z) = 0.$$

3. Toda-like equations for $U(1)$ Nekrasov functions

5D $U(1)$ Nekrasov function with external potential

$$Z(s,Q) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|+s(s+1)/2} e^{\Phi(s,\lambda)}.$$

Q is a new parameter, $q^{-\rho}$ is the special point

$$q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{i-1/2}, \ldots)$$

and $\Phi(s,\lambda)$ is an external potential of the form

$$\Phi(s,\lambda) = \sum_{i=1}^{\infty} \xi_{\lambda_i - i + s + 1} - \sum_{i=1}^{\infty} \xi_{-i+1}.$$

Special value of Schur function

$s_\lambda(q^{-\rho})$ is a slight modification of the principal specialization:

$$s_\lambda(q^{-\rho}) = q^{|\lambda|/2} s_\lambda(1, q, q^2, \ldots) = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j)\in\lambda}(q^{-h(i,j)/2} - q^{h(i,j)/2})},$$

where

$$\kappa(\lambda) = \sum_{i\geq 1} \lambda_i(\lambda_i - 2i + 1) = 2\sum_{(i,j)\in\lambda}(j-i),$$

$$h(i,j) = \lambda_i + {}^{\mathrm{t}}\lambda_j - i - j + 1 \quad \text{(hook length)}$$

Relation to $\mathcal{T}(s,z)$

$$\mathcal{T}(s,z) = \sum_{\lambda\in\mathcal{P}} s_\lambda(1,q,q^2,\ldots)s_\lambda(1,\bar{q},\bar{q}^2,\ldots)z^{|\lambda|}g_{s,\lambda}$$

$$\Big\downarrow \quad \bar{q}=q,\ z=q,\ g_n = Q^n e^{\xi_n}$$

$$Z(s,Q) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|+s(s+1)/2} e^{\Phi(s,\lambda)}$$

In particular,

$$Z(s,qQ) = q^{s(s+1)/2}\mathcal{T}(s,q^2),$$

$$Z(s,q^2Q) = q^{s(s+1)}\mathcal{T}(s,q^3).$$

Toda-like equation for $Z(s,Q)$

The bilinear equation

$$\mathcal{T}(s,qz)\mathcal{T}(s,\bar{q}z) - \mathcal{T}(s,z)\mathcal{T}(s,q\bar{q}z)$$
$$+z\mathcal{T}(s+1,z)\mathcal{T}(s-1,q\bar{q}z) = 0.$$

for $\mathcal{T}(s,z)$ turns into a bilinear equation for $Z(s,Q)$:

$Z(s,Q)$ satisfies the Toda-like equation

$$Z(s,qQ)^2 - Z(s,q^2Q)Z(s,Q)$$
$$+q^{2s+1}Z(s+1,Q)Z(s-1,q^2Q) = 0.$$

4D limit of 5D $U(1)$ Nekrasov function

$$Z(s,Q) = \sum_{\lambda\in\mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|+s(s+1)/2} e^{\Phi(s,\lambda)}$$

$$\Big\downarrow \; q = e^{-R\hbar},\ Q = (R\Lambda)^2,\ R \to 0$$

$$Z_{4\mathrm{D}}(s,\Lambda) = \sum_{\lambda\in\mathcal{P}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \left(\frac{\Lambda}{\hbar}\right)^{2|\lambda|+s(s+1)} e^{\Phi_{4\mathrm{D}}(s,\lambda)}$$

(4D $U(1)$ Nekrasov function)

$\dim\lambda$ is the number of standard tableaux of shape λ, and

$$\frac{\dim\lambda}{|\lambda|!} = S_\lambda(1,0,0,\ldots) = \frac{1}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

4D limit of Toda-like equation

$$\frac{Z(s,q^2Q)}{Z(s,Q)} - \left(\frac{Z(s,qQ)}{Z(s,Q)}\right)^2 = q^{2s+1}\frac{Z(s-1,q^2Q)Z(s+1,Q)}{Z(s,Q)^2}$$

$$\Big\downarrow \; q = e^{-R\hbar},\ Q = (R\Lambda)^2,\ R \to 0$$

$$\left(\Lambda\frac{d}{d\Lambda}\right)^2 \log Z_{\mathrm{4D}}(s,\Lambda) = 4\frac{Z_{\mathrm{4D}}(s-1,\Lambda)Z_{\mathrm{4D}}(s+1,\Lambda)}{Z_{\mathrm{4D}}(s,\Lambda)^2}$$

The Toda-like equation of Nekrasov and Okounkov for $Z_{\mathrm{4D}}(s,\Lambda)$ (Nekrasov & Okounkov 2003) can be thus recovered.

4. Dual partition function of $SU(N)$ theory

Ref) T. Maeda, T. Nakatsu, K.T. & T. Tamakoshi, JHEP **03** (2005), 056, arxiv:hep-th/0412327.

External potential in $U(1)$ Nekrasov function

$$Z(s, Q) = \sum_{\lambda \in \mathcal{P}} s_\lambda(q^{-\rho})^2 Q^{|\lambda|+s(s+1)/2} e^{\Phi(s,\lambda)},$$

$$\Phi(s, \lambda) = \sum_{i=1}^{\infty} \xi_{\lambda_i - i + s + 1} - \sum_{i=1}^{\infty} \xi_{-i+1}.$$

(The definition of $\Phi(s, \lambda)$ is effectively a finite sum. The subtraction terms is related to normal ordering of a fermion bilinear.)

Decomposition under external periodic potential

If the external potential is N-periodic, i.e.,

$$\xi_{n+N} = \xi_n,$$

the sum over the single partition λ can be decomposed to a sum over the N-tuple $\vec{p} = (p_1, \ldots, p_N)$ of integers (charges) and the N-tuple $\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ of partitions (colored partitions):

$$Z(s, Q) = \sum_{\vec{p} \in \mathbb{Z}^N,\ p_1 + \cdots + p_N = 0} e^{\vec{p} \cdot \vec{\xi}(s)} Q^{s(s+1)/2} Z(\vec{p}, Q),$$

$$\vec{\xi}(s) = (\xi_{1+s}, \ldots, \xi_{N+s}), \quad Z(\vec{p}, Q) = \sum_{\vec{\lambda} \in \mathcal{P}^N} \cdots .$$

Decomposition under periodic external potential (cont'd)

The summands of $Z(\vec{p}, Q)$ are factorized to a $\vec{\lambda}$-independent factor $C(\vec{p}, Q)$ (perturbative part) and a sum of $\vec{\lambda}$-dependent factors $Z^{\text{inst}}_{\vec{\lambda}}(\vec{p}, Q)$ (instanton part):

$$Z(\vec{p}, Q) = C(\vec{p}, Q) \sum_{\vec{\lambda} \in \mathcal{P}^N} Z^{\text{inst}}_{\vec{\lambda}}(\vec{p}, Q).$$

$Z^{\text{inst}}_{\vec{\lambda}}(\vec{p}, Q)$ turns out to be the 5D $SU(N)$ Nekrasov function with Chern-Simons corrections. In summary:

> Under the N-periodic external potential, $Z(s, Q)$ becomes a dual partition function of 5D $SU(N)$ theory in the sense of Nekrasov and Okounkov.

$N = 2$: charges and colored partitions

Write the components of $\vec{p}$ and $\vec{\lambda}$ as

$$\vec{p} = (p, -p), \quad \vec{\lambda} = (\mu, \bar{\mu})$$

and identify the triple $(p, \mu, \bar{\mu})$ with the partition λ:

$$\lambda = (p, \mu, \bar{\mu}).$$

1) The triple is obtained by splitting the Maya diagram of λ (as a subset of $\mathbb{Z}$) into the subsets of odd and even integers:

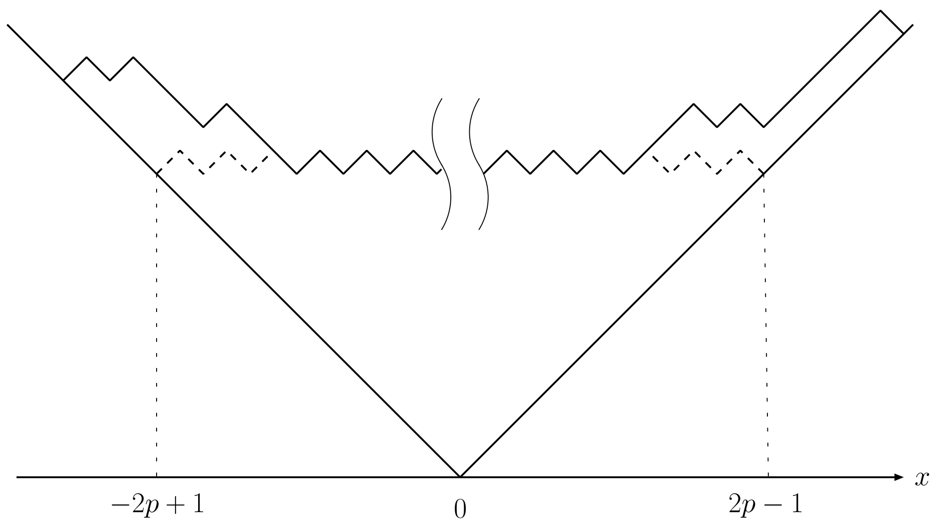
$$\{\lambda_i - i + 1\}_{i=1}^{\infty} = \{2(\mu_i - i + p) + 1\}_{i=1}^{\infty} \cup \{2(\bar{\mu}_i - i - p) + 2\}_{i=1}^{\infty}.$$

$N = 2$: charges and colored partitions (cont'd)

2) $(p, \emptyset, \emptyset)$ represents a staircase Young diagram:

$$(p, \emptyset, \emptyset) = \begin{cases} (2p-1, 2p-2, \ldots, 1) & \text{if } p > 0, \\ \emptyset & \text{if } p = 0, \\ (2|p|, 2|p|-1, \ldots, 1) & \text{if } p < 0. \end{cases}$$

3) The staircase Young diagram is the core of λ, and the colored partition $(\mu, \bar{\mu})$ represents the quotient of λ by the core.



$N=2$: Factorization of weights

$$Q^{|\lambda|}s_\lambda(q^{-\rho})^2 = Q^{|(p,\emptyset,\emptyset)|}s_{(p,\emptyset,\emptyset)}(q^{-\rho})^2 \times Q^{|(p,\mu,\bar{\mu})|-|(p,\emptyset,\emptyset)|}\frac{s_{(p,\mu,\bar{\mu})}(q^{-\rho})^2}{s_{(p,\emptyset,\emptyset)}(q^{-\rho})^2}.$$

The two parts on the right side give the perturbative part $C(p,Q)$ and the instanton part $Z^{\mathrm{inst}}_{\mu\bar{\mu}}(p,Q)$:

$$C(p,Q) = Q^{|(p,\emptyset,\emptyset)|}s_{(p,\emptyset,\emptyset)}(q^{-\rho})^2,$$

$$Z^{\mathrm{inst}}_{\mu\bar{\mu}}(p,Q) = Q^{|(p,\mu,\bar{\mu})|-|(p,\emptyset,\emptyset)|}\frac{s_{(p,\mu,\bar{\mu})}(q^{-\rho})^2}{s_{(p,\emptyset,\emptyset)}(q^{-\rho})^2}.$$

$N = 2$: Computation of $C(p, Q)$

$$C(p,Q) = \begin{cases} \displaystyle\prod_{k=1}^{2p-1} \frac{Q^k}{(Q_F^{1/2} q^k - Q_F^{-1/2} q^{-k})^{2k}} & \text{if } p > 0, \\ 1 & \text{if } p = 0, \\ \displaystyle\prod_{k=1}^{2|p|} \frac{Q^k}{(Q_F^{-1/2} q^k - Q_F^{1/2} q^{-k})^{2k}} & \text{if } p < 0 \end{cases}$$

where $Q_F = q^{1-4p}$.

A final expression should be written in terms of the q-Barnes finction?

$N=2$: Computation of $Z^{\text{inst}}_{\mu\bar{\mu}}(p,Q)$

An intermediate stage ($Q_F = q^{1-4p}$):

$$Z^{\text{inst}}_{\mu\bar{\mu}}(p,Q) = Q^{2|\mu|+2|\bar{\mu}|} Q_F^{|\mu|-|\bar{\mu}|} q^{-2\kappa(\mu)-2\kappa(\bar{\mu})}$$
$$\times \prod_{(i,j)\in\mu} \frac{1}{(q^{-h_\mu(i,j)} - q^{h_\mu(i,j)})^2}$$
$$\times \prod_{(i,j)\in\bar{\mu}} \frac{1}{(q^{-h_{\bar{\mu}}(i,j)} - q^{h_{\bar{\mu}}(i,j)})^2}$$
$$\times \prod_{i,j=1}^{\infty} \frac{(Q_F^{1/2} q^{-(\mu_i-\bar{\mu}_j-i+j)} - Q_F^{-1/2} q^{\mu_i-\bar{\mu}_j-i+j})^2}{(Q_F^{1/2} q^{i-j} - Q_F^{-1/2} q^{j-i})^2}.$$

$N=2$: Computation of $Z^{\text{inst}}_{\mu\bar{\mu}}(p,Q)$ (cont'd)

$$Z^{\text{inst}}_{\mu\bar{\mu}}(p,Q) = Q^{2|\mu|+2|\bar{\mu}|} Q_F^{|\mu|-|\bar{\mu}|} q^{-2\kappa(\mu)-2\kappa(\bar{\mu})}$$
$$\times N_{\mu\mu}(q^2) N_{\bar{\mu}\bar{\mu}}(q^2) N_{\mu\bar{\mu}}(q^2, Q_F) N_{\bar{\mu}\mu}(q^2, Q_F^{-1})$$

where $Q_F = q^{1-4p}$, and $N_{\bullet\bullet}$'s are the Nekrasov factors

$$N_{\lambda\lambda}(q) = \prod_{(i,j)\in\lambda} \frac{1}{(1-q^{-h_\lambda(i,j)})(1-q^{h_\lambda(i,j)})},$$

$$N_{\lambda\mu}(q,u) = \prod_{(i,j)\in\lambda} \frac{1}{1-uq^{-\lambda_i - {}^{\mathrm{t}}\mu_j + i + j - 1}}$$
$$\times \prod_{(i,j)\in\mu} \frac{1}{1-uq^{\mu_i + {}^{\mathrm{t}}\lambda_j - i - j + 1}}.$$

5. Conclusion

1) Toda-like equations for $U(1)$ Nekrasov functions

The 5D $U(1)$ Nekrasov function $Z(s,Q)$ satisfies a Toda-like q-difference equation. This equation can be derived from the 2D Toda hierarchy by a few steps of specialization. The 4D version $Z_{4\mathrm{D}}(s,\Lambda)$ satisfies a Toda-like differential equation.

2) Dual partition function of $SU(N)$ theory

If the coefficients ξ_n of the external potential $\Phi(s,\lambda)$ are N-periodic, $Z(s,Q)$ becomes a dual partition function of 5D $SU(N)$ theory. A drawback is that some parameters therein are specialized to discrete values (associated with $\vec{p} \in \mathbb{Z}^N$). To achieve a full generality, we have to find a suitable deformation of the weights $s_\lambda(q^{-\rho})^2$.

3) Including matter hypermultiplets

This issue is not very serious. Contribution of matter fields can be included as corrections of the external potential.

4) Discrepancy with isomonodromic tau functions

Our 5D Nekrasov functions looks slightly different from the known isomonodromic tau functions because of Chern-Simons corrections.

Thank you
for your attention!