# Another generalization of the box–ball system with many kinds of balls

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| $t$ | state | $Q_0^1$ | $Q_0^2$ | $Q_0^3$ | $E_0^1$ | $Q_1^1$ | $Q_1^2$ | $Q_1^3$ | $E_1^1$ | $Q_2^1$ | $Q_2^2$ | $Q_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ = 0: | `.1222333...11223...133............................` | 1 | 3 | 3 | 3 | 2 | 2 | 1 | 3 | 1 | 0 | 2 |
| $t$ = 1: | `.......1222333..1122..1333........................` | 1 | 3 | 3 | 2 | 2 | 2 | 0 | 2 | 1 | 0 | 3 |
| $t$ = 2: | `.............1222333112...12333...................` | 1 | 3 | 3 | 0 | 2 | 1 | 0 | 3 | 1 | 1 | 3 |
| $t$ = 3: | `...................122311223...123333.............` | 1 | 2 | 1 | 0 | 2 | 2 | 1 | 3 | 1 | 1 | 4 |
| $t$ = 4: | `.......................223..11123...1223333.......` | 0 | 2 | 1 | 2 | 3 | 1 | 1 | 3 | 1 | 2 | 4 |
| $t$ = 5: | `..........................223....11123....1223333.` | 0 | 2 | 1 | 4 | 3 | 1 | 1 | 4 | 1 | 2 | 4 |

Figure 1 Example of the **known** BBS with many kinds of balls and limited carrier capacity ($M = 3$, carrier capacity: 6)

| $t$ | state | $Q_0^1$ | $Q_0^2$ | $Q_0^3$ | $E_0^1$ | $Q_1^1$ | $Q_1^2$ | $Q_1^3$ | $E_1^1$ | $Q_2^1$ | $Q_2^2$ | $Q_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ = 0: | `.1222333...11223...133............................` | 1 | 3 | 3 | 3 | 2 | 2 | 1 | 3 | 1 | 0 | 2 |
| $t$ = 1: | `..1122233....11233..223...........................` | 2 | 3 | 2 | 4 | 2 | 1 | 2 | 2 | 0 | 2 | 1 |
| $t$ = 2: | `....1122233....12233112...........................` | 2 | 3 | 2 | 4 | 1 | 2 | 2 | 0 | 2 | 1 | 0 |
| $t$ = 3: | `......1122233...1122..1333........................` | 2 | 3 | 2 | 3 | 2 | 2 | 0 | 2 | 1 | 0 | 3 |
| $t$ = 4: | `........1122233...1133.2223.......................` | 2 | 3 | 2 | 3 | 2 | 0 | 2 | 1 | 0 | 3 | 1 |
| $t$ = 5: | `..........1122233...22311123......................` | 2 | 3 | 2 | 3 | 0 | 2 | 1 | 0 | 3 | 1 | 1 |
| $t$ = 6: | `............1122233.112...12333...................` | 2 | 3 | 2 | 1 | 2 | 1 | 0 | 3 | 1 | 1 | 3 |
| $t$ = 7: | `..............112223..1333.12223..................` | 2 | 3 | 1 | 2 | 1 | 0 | 3 | 1 | 1 | 3 | 1 |
| $t$ = 8: | `................112233.2223.11123.................` | 2 | 2 | 2 | 1 | 0 | 3 | 1 | 1 | 3 | 1 | 1 |
| $t$ = 9: | `..................1122311223...12333..............` | 2 | 2 | 1 | 0 | 2 | 2 | 1 | 3 | 1 | 1 | 3 |
| $t$ =10: | `....................112..122333.122233............` | 2 | 1 | 0 | 2 | 1 | 2 | 3 | 1 | 1 | 3 | 2 |
| $t$ =11: | `......................133.122223.111223...........` | 1 | 0 | 2 | 1 | 1 | 4 | 1 | 1 | 3 | 2 | 1 |
| $t$ =12: | `.......................223.111123...122333........` | 0 | 2 | 1 | 1 | 4 | 1 | 1 | 3 | 1 | 2 | 3 |
| $t$ =13: | `.......................112.....12333.1222233......` | 2 | 1 | 0 | 5 | 1 | 1 | 3 | 1 | 1 | 4 | 2 |
| $t$ =14: | `.........................133....12223.1112223.....` | 1 | 0 | 2 | 4 | 1 | 3 | 1 | 1 | 3 | 3 | 1 |
| $t$ =15: | `..........................223....11123...1122333..` | 0 | 2 | 1 | 4 | 3 | 1 | 1 | 3 | 2 | 2 | 3 |

Figure 2 Example of the **novel** BBS with many kinds of balls and limited number of rewrites ($M = 3$, $S^{(t)} = 2$ for all $t$)

## 1 Introduction

- A generalization of the BBS by Hatayama *et al*. [1] is known (Ex.: Figure 1)
- We propose **another** generalization of the BBS [2] (Ex.: Figure 2)

### ⋙ Rule

- There is an array of infinite boxes. Several boxes hold one ball.
- Each ball has an index (from 1 to $M$). We regard empty boxes as balls with index 0 (periods . indicates 0 in the figures). Note that we regard the indexes cyclically: $k \equiv k \bmod (M+1)$.
- The **known** rule (Ex.: Figure 1):
  - A carrier of balls moves from left to right. The carrier can hold a finite number of balls.
  - When the carrier passes each box, the carrier exchanges the ball with index $k$ in the box with a ball whose index is the smallest among the indexes of the balls, whose indexes are larger than $k$, in the carrier (ex.: if $k = 1$ and $M = 3$, then the order is $2 < 3 < 0 < 1$).
- The **novel** rule (Ex.: Figure 2):
  - A machine moves from left to right. The carrier has a state called "the numbers of rewrites" $(V^{(0)}, V^{(1)}, \dots, V^{(M)})$. Initial state at time $t$ is $(V^{(0)}, V^{(1)}, \dots, V^{(M)}) = (0, S^{(t)}, \dots, S^{(t)})$.
  - When the machine passes each box, if the index of the ball in the box is $k$: if $V^{(k)} \ge 1$ then rewrites the index $k$ as $k-1$ and sets the state $V^{(k)} \leftarrow V^{(k)} - 1$, $V^{(k-1)} \leftarrow V^{(k-1)} + 1$; if $V^{(k)} = 0$ then does nothing.

There are two types of the time evolution equations corresponding to the novel BBS as follows.

## 2 Reduced nonautonomous ultradiscrete KP lattice

$$U_n^{(k,t+1)} = U_n^{(k,t)} - X_n^{(k,t)} + X_n^{(k+1,t)}, \quad V_{n+1}^{(k,t)} = V_n^{(k,t)} - X_n^{(k,t)} + X_n^{(k+1,t)},$$
$$X_n^{(k,t)} = \min(U_n^{(k,t)}, V_n^{(k,t)}),$$

where $U_n^{(k+M+1,t)} = U_n^{(k,t)}$ and $V_n^{(k+M+1,t)} = V_n^{(k,t)}$ for all $k$, $t$ and $n$, with the boundary conditions for $n \to -\infty$:

$$U_n^{(k,t)} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k = 1, 2, \dots, M, \end{cases} \qquad V_n^{(k,t)} = \begin{cases} 0 & \text{if } k = 0, \\ S^{(t)} & \text{if } k = 1, 2, \dots, M. \end{cases}$$

- $U_n^{(k,t)} \in \{0, 1\}$: the number of balls with index $k$ in the $n$th box at time $t$;
- $V_n^{(k,t)}$: "the number of rewrites of index $k$" of the machine arriving at the $n$th box at time $t$;
- $X_n^{(k,t)}$: 1 if $U_n^{(k,t)} = 1$ and $V_n^{(k,t)} \ge 1$, otherwise 0.

An $N$-soliton solution is derived from Baker–Akhiezer functions via reduction and ultradiscretization procedure:

$$U_n^{(k,t)} = \begin{cases} F_n^{(M,t)} - F_{n+1}^{(M,t)} + F_{n+1}^{(0,t)} - F_n^{(0,t)} + 1 & \text{if } k = 0, \\ F_n^{(k-1,t)} - F_{n+1}^{(k-1,t)} + F_{n+1}^{(k,t)} - F_n^{(k,t)} & \text{if } k = 1, 2, \dots, M, \end{cases}$$
$$V_n^{(k,t)} = \begin{cases} F_n^{(M,t)} - F_n^{(M,t+1)} + F_n^{(0,t+1)} - F_n^{(0,t)} & \text{if } k = 0, \\ F_n^{(k-1,t)} - F_n^{(k-1,t+1)} + F_n^{(k,t+1)} - F_n^{(k,t)} + S^{(t)} & \text{if } k = 1, 2, \dots, M, \end{cases}$$
$$X_n^{(k,t)} = F_n^{(k-1,t)} - F_{n+1}^{(k-1,t)} + F_{n+1}^{(k-1,t+1)} - F_n^{(k-1,t+1)},$$

where

$$F_n^{(k,t)} = \min\left(0, \min_{\substack{0 \le r_0 < r_1 < \dots < r_{m-1} \le N-1 \\ m = 1, 2, \dots, N}} \left(\sum_{i=0}^{m-1} \tilde{\Psi}_{r_i,n}^{(k,t)}(i)\right)\right),$$
$$\tilde{\Psi}_{r,n}^{(k,t)}(i) := \Theta_r + \min_{l=0,1,\dots,t}\left(\sum_{j=1}^{(k'+l+i) \bmod M} \zeta_{r,j} + \left(\left\lfloor \frac{k'+l+i}{M} \right\rfloor + i\right) Z_r + \tilde{S}_{t-l}^{(t)}\right) - n \min(Z_r, 1).$$

Here, $k' := k \bmod (M+1)$, $Z_r \ge 0$ and $\Theta_r$ are the parameters determine the size and phase of the soliton, respectively, $\zeta_{r,1}, \zeta_{r,2}, \dots, \zeta_{r,M}$ are nonnegative parameters satisfying $\sum_{j=1}^{M} \zeta_{r,j} = Z_r$, and

$$\tilde{S}_i^{(t)} := \begin{cases} 0 & \text{if } i = 0, \\ \min\limits_{0 \le j_0 < j_1 < \dots < j_{i-1} \le t-1} \left(\sum_{l=0}^{i-1} S^{(j_l)}\right) & \text{if } i = 1, 2, \dots, t, \\ +\infty & \text{otherwise.} \end{cases}$$

## 3 Reduced nonautonomous ultradiscrete 2D Toda lattice

Or, so-called **nonautonomous ultradiscrete hungry Toda lattice**:

$$\begin{aligned} \tilde{Q}_n^{(k,t)} &= \min(Q_n^{(k,t)}, A_n^{(k,t)}), & k &= 1, \dots, M, \\ \tilde{Q}_n^{(M+1,t)} &= \min(B_n^{(1,t)}, E_n^{(1,t)}), \\ Q_n^{(k,t+1)} &= Q_n^{(k,t)} - \tilde{Q}_n^{(k,t)} + \tilde{Q}_n^{(k+1,t)}, & k &= 1, \dots, M, \\ E_n^{(1,t+1)} &= E_n^{(1,t)} - \tilde{Q}_n^{(M+1,t)} + \tilde{Q}_{n+1}^{(1,t)}, \\ A_{n+1}^{(k,t)} &= A_n^{(k,t)} - \tilde{Q}_n^{(k,t)} + \tilde{Q}_n^{(k+1,t)}, & k &= 1, \dots, M, \\ B_{n+1}^{(1,t)} &= B_n^{(1,t)} - \tilde{Q}_n^{(M+1,t)} + \tilde{Q}_{n+1}^{(1,t)}, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} A_0^{(k,t)} &= S^{(t)}, & k &= 1, \dots, M, \\ B_0^{(1,t)} &= \min(Q_0^{(1,t)}, S^{(t)}), \quad E_N^{(1,t)} = +\infty. \end{aligned}$$

- $N$: the number of solitons;
- $Q_n^{(k,t)}$: the number of balls with index $k$ in the $n$th soliton at time $t$;
- $E_n^{(1,t)}$: the number of the $n$th block of empty boxes at time $t$;
- $A_n^{(k,t)}$: "the number of rewrites of index $k$" of the machine arriving at the $n$th soliton at time $t$;
- $B_n^{(1,t)}$: "the number of rewrites of index 0" of the machine arriving at the $n$th block of empty boxes at time $t$;

Particular solutions are derived from **biorthogonal polynomials** as

$$\begin{aligned} Q_n^{(k,t)} &= T_n^{(k,t)} - T_{n+1}^{(k,t)} + T_{n+1}^{(k+1,t)} - T_n^{(k+1,t)}, \\ E_n^{(1,t)} &= T_{n+2}^{(1,t)} - T_{n+1}^{(1,t)} + T_n^{(M+1,t)} - T_{n+1}^{(M+1,t)}, \\ A_n^{(k,t)} &= T_n^{(k,t)} - T_n^{(k+1,t)} + T_n^{(k+1,t+1)} - T_n^{(k,t+1)} + S^{(t)}, \\ B_n^{(1,t)} &= T_n^{(M+1,t)} - T_{n+1}^{(1,t)} + T_{n+1}^{(1,t+1)} - T_n^{(M+1,t+1)}, \end{aligned}$$

where

$$T_n^{(k,t)} = \min_{\substack{0 \le r_0 < r_1 < \dots < r_{n-1} \le N-1 \\ 0 \le c_0 < c_1 < \dots < c_{n-1} \le t+n-1}} \left(\sum_{j=0}^{n-1}\left(W_{r_j}^{(k+c_j)} + \frac{k + Mj + c_j}{M} Z_{r_j}\right) + \hat{S}^{(t)}_{(t-c_0, t+1-c_1, \dots, t+n-1-c_{n-1})'}\right),$$
$$\hat{S}^{(t)}_{(\lambda_0, \lambda_1, \dots, \lambda_{n-1})'} := \min_Y \left(\sum_{j=0}^{t-1} y_j S^{(j)}\right).$$

Here, $W_r^{(k)}$ are constants satisfying $W_r^{(k+M)} = W_r^{(k)}$, $\min\limits_Y$ indicates the minimum value over all semistandard Young tableaux $Y$ of the partition $(\lambda_0, \lambda_1, \dots, \lambda_{n-1})'$, and $y_0, y_1, \dots, y_{n-1}$ are the weights of $Y$.

**References**

[1] G. Hatayama, K. Hikami, R. Inoue, A. Kuniba, T. Takagi and T. Tokihiro, *The $A_M^{(1)}$ automata related to crystals of symmetric tensors*, J. Math. Phys. **42** (2001), 274–308.

[2] K. Maeda, *Another generalization of the box–ball system with many kinds of balls*, J. Integrable Syst. **3** (2018), xyy007.