Dynamics of the box-ball system with random initial conditions via Pitman's transformation

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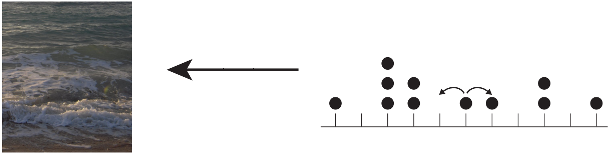
Background

<u>Statistical mechanics</u>

Goal : Derive macroscopic phenomena from microscopic systems

Typical microscopic model : stochastic interacting particle systems

- Exclusion process (SSEP,TASEP,ASEP...)
- Zero range process
- Interacting Brownian motions

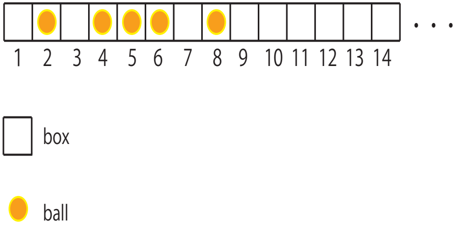


Question: What can we say about macroscopic properties of Box-Ball System (BBS) ?

Box-Ball System

Introduced in 1990 by Takahashi-Satsuma

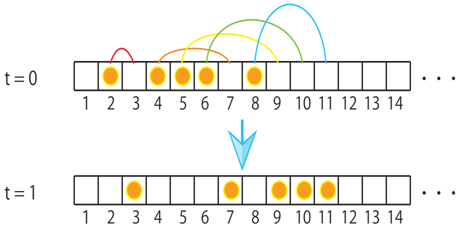
- Discrete time deterministic dynamics (Cell-Automaton)
- Finite number of balls



Box-Ball System

Def 1

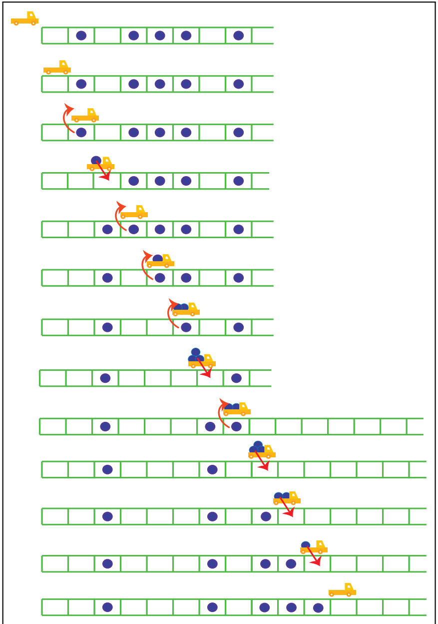
- Every ball moves exactly once in each evolution time step
- The leftmost ball moves first and the next leftmost ball moves next and so on...
- Each ball moves to its nearest right vacant box



Box-Ball System

Def 2

- A carrier moves from left to right
- The carrier picks up a ball when it finds a ball (The carrier can load any number of balls)
- The carrier puts down a ball when it comes to an empty box carrying at least one ball



Why BBS is interesting?

Key properties

- Solitonic behavior
- Integrable system (infinitely many conserved quantities)
- Initial value problem is solved by various methods

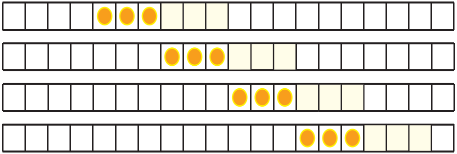
Connections to many physical models

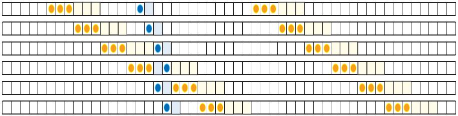
- Ultra-discretization of discrete KdV equation
- Crystallization of an integrable lattice model (six-vertex model)
- Ultra-discretization of Toda lattice
- Many variations of BBS have been also studied and known to have connections to variants of above models

Key property of BBS : Solitons

- $(1,0),(1,1,0,0),(1,1,1,0,0,0)\ldots$ are "Solitons"
- $(1,1,1,\ldots,1,0,0,0,\ldots,0)$: soliton of size *n*
- soliton of size *n* moves with speed *n*

- Number of each type of solitons is conserved $\Rightarrow \exists$ Infinite number of conserved quantities
- The interaction between solitons are nonlinear
- Integrable system





Box-Ball System

- $\eta = (\eta_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}, \sum_{n \in \mathbb{N}} \eta_n < \infty$
- BBS (one time step) operator $T : \eta \to T\eta$
- W_n : the number of balls on the carrier as it passes location n ($W_0 := 0$)
- $T\eta_n = 0$ if $\eta_n = 1$
- $T\eta_n = 1$ if $\eta_n = 0$ and $W_{n-1} \geq 1$
- $W_n = \sum_{m=1}^{n} (\eta_m - T\eta_m)$
- $T\eta_n = \min\{1 - \eta_n, W_{n-1}\}$

Reversibility

- The time reversal of T, denoted by T^{-1} is obtained by the same rule of the dynamics but replace 'left' by 'right' in Def 1,2
- $T^{-1} = R \circ T \circ R$ where R is the operation of reversing the order of boxes

Our interest

As a dynamical system

- Invariant measures
- Ergodicity

As an interacting particle system

- Properties of invariant measures (parameter, translation invariance...)
- Examples of invariant measures
- Asymptotic behavior of the integrated particle current and the tagged particle (when the initial measure is random)
- Scaling limit (Box-Ball System on $\mathbb{R}$)

$\Rightarrow$ We need to define BBS on $\mathbb{Z}$ with infinitely many balls!

Remark

"ball" $\rightarrow$ "particle" from this slide

Previous result

Ferrari and his collaborators have been already studied BBS with infinitely many particles with random initial condition (2018.arxiv).

- Introduce BBS on $\mathbb{Z}$ and give a sufficient condition on initial configuration for the dynamics to be well-defined
- Show that the Bernoulli product measure with density less than $\frac{1}{2}$ is invariant under T
- Show that any invariant measure of T with density less than $\frac{1}{4}$ satisfying a nice mixing condition has a product decomposition of measures w.r.t the size of solitons

Our main result : Deterministic part

- Characterize $\mathcal{S}^T$, $\mathcal{S}^{T^{-1}}$: the domain of T, T^{-1}
- Characterize $\mathcal{S}^{rev}$: the space of "reversible" configurations $\{\eta\ ;\ TT^{-1}\eta = T^{-1}T\eta = \eta\}$
- Characterize $\mathcal{S}^{inv}$: the "invariant" space of configurations $\{\eta\ ;\ T^k\eta \in \mathcal{S}^{rev}\ \forall k \in \mathbb{Z}\}$

Remark

$\mathcal{S}^T \cap \mathcal{S}^{T^{-1}} \supsetneq \mathcal{S}^{rev} \supsetneq \mathcal{S}^{inv}$

- Construct a bijection between the initial configuration $(\eta_n)_{n\in\mathbb{Z}}$ and time series of the current at origin $(T^k W_0)_{k\in\mathbb{Z}}$

Our main result : Probabilistic part (random initial configuration)

BBS on $\mathbb{Z}$

- Some properties of invariant measures
- A sufficient condition to be invariant
- Three classes of probability measures satisfying this sufficient condition (including the product Bernoulli measures)
- A sufficient condition for a probability measure to make T be ergodic
- Ergodicity of the operator T for the above examples
- LLN, CLT and LDP for integrated currents of particles at origin for the above examples
- LLN, CLT and LDP for a tagged particle for some of the above examples

BBS on $\mathbb{R}$

- Introduce dynamics T for more general configurations in continuous state space
- Brownian motion with positive drift is invariant under T

Key Observation: BBS is Pitman's $2M - X$ transform

We introduce a path-encoding of the configuration. It reveals that the dynamics of BBS is exactly the Pitman's well-known $2M - X$ transformation.

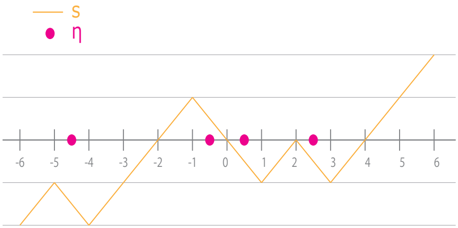
This observation enables us to study many new properties of BBS with infinite particles. It also gives a natural way to extend the dynamics on $\mathbb{R}$.

Remark

Relation between (some versions of) Pitman's $2M - X$ transform and several important integrable systems and its application to stochastic models have been studied by O'Connell and his collaborators. Quantum Toda lattice, random polymers, random matrices, KPZ equation...so on.

Path encoding

- $\eta = (\eta_n)_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$
- $S = (S_n)_{n\in\mathbb{Z}} \in \mathfrak{S}^0$, $\mathfrak{S}^0 := \{S : \mathbb{Z} \to \mathbb{Z};\ S_0 = 0, |S_n - S_{n-1}| = 1\}$
- $\eta \leftrightarrow S$: $S_n - S_{n-1} = 1 - 2\eta_n$: One-to-one
- $\eta_n = 1$: particle $\leftrightarrow S_n - S_{n-1} = -1$: down jump
- $\eta_n = 0$: empty $\leftrightarrow S_n - S_{n-1} = 1$: up jump



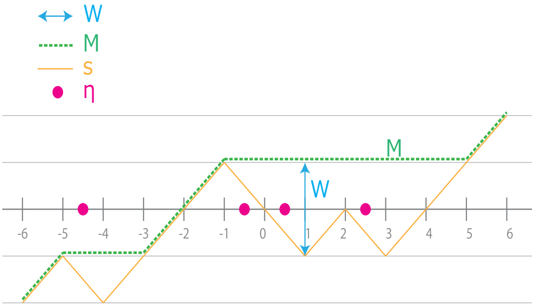
S: Path encoding of η

Past maximum and the carrier via path encoding

Suppose $\sum_{n \in \mathbb{Z}} \eta_n < \infty$

Lemma

$W_n = M_n - S_n$ where $M_n = \sup_{m \leq n} S_m$.



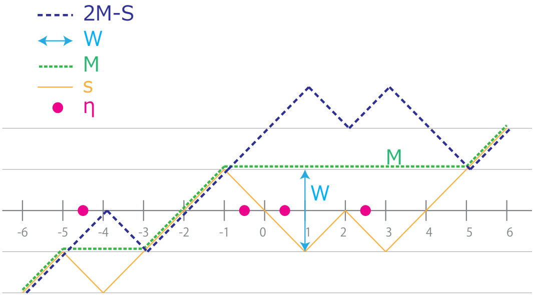
BBS is Pitman's transformation

Suppose $\sum_{n\in\mathbb{Z}} \eta_n < \infty$.

$S \to TS$ is the reflection with respect to the past maximum :

Lemma

$TS_n = 2M_n - S_n - 2M_0$ *where* TS *is the path encoding of* $T\eta$.



Known results for Pitman's transformation for one-sided stochastic processes

- $(S_n)_{n\in\mathbb{Z}_+}$: Simple random walk (SRW) $\rightarrow$ $(TS_n)_{n\in\mathbb{Z}_+}$: SRW conditioned to be non-negative
- $(S_x)_{x\geq 0, x\in\mathbb{R}}$: Brownian motion (BM) $\rightarrow$ $(TS_x)_{x\geq 0, x\in\mathbb{R}}$: 3-dimensional Bessel Process

* Many variations of the results are also known

Domain and definition of T

The dynamics can be generalized straightforwardly to the configuration with infinitely many particles : $\sum_{n\in\mathbb{Z}} \eta_n = \infty$.

$$\mathcal{S}^T := \{S \in \mathcal{S}^0 \,;\, \limsup_{n\to-\infty} S_n < \infty\} = \{S \in \mathcal{S}^0 \,;\, M_0 < \infty\}$$

For $S \in \mathcal{S}^T$, we define

$$(TS)_n = 2M_n - S_n - 2M_0$$

or equivalently,

$$(T\eta)_n = \min\{1 - \eta_n, W_{n-1}\}$$

where $W_n = M_n - S_n$.

- We prove $\mathcal{S}^T$ is a "true" domain of T.

Path spaces

We define $T^{-1}S := 2I - S - 2I_0$ for $S \in \mathcal{S}^{T^{-1}}$ where $I_n = \inf_{m \geq n} S_m$,

$$\mathcal{S}^{T^{-1}} := \{S \in \mathcal{S}^0 \ ; \ \liminf_{n\to\infty} S_n > -\infty\} = \{S \in \mathcal{S}^0 \ ; \ I_0 \in \mathbb{R}\}.$$

$TT^{-1}S = S$ or $T^{-1}TS = S$ does not necessarily hold!

$$\mathcal{S}^{rev} := \{S \in \mathcal{S}^T \cap \mathcal{S}^{T^{-1}}; T^{-1}TS = S, TT^{-1}S = S\} \subsetneq \mathcal{S}^T \cap \mathcal{S}^{T^{-1}}.$$

$T(\mathcal{S}^{rev}) \subset \mathcal{S}^{rev}$ or $T^{-1}(\mathcal{S}^{rev}) \subset \mathcal{S}^{rev}$ does not necessarily hold!

$$\mathcal{S}^{inv} := \{S \in \mathcal{S}^0 \ ; \ T^k S \in \mathcal{S}^{rev} \ \forall k \in \mathbb{Z}\} \subsetneq \mathcal{S}^{rev}.$$

- We characterize $\mathcal{S}^{rev}$ and $\mathcal{S}^{inv}$ explicitly.

Sub-critical and critical boundary conditions

$$\mathcal{S}^{sub-critical} := \mathcal{S}^{inv} \cap \{\lim_{n\to\pm\infty} S_n = \pm\infty\} \supset \{S \in \mathcal{S}^0 \ ; \ \lim_{n\to\pm\infty} \frac{S_n}{n} = c\}$$

for any $0 < c \leq 1$.

$S \in \mathcal{S}^{sub-critical}$ implies the density of particles is asymptotically less than $\frac{1}{2}$ as $n \to \pm\infty$.

$$\mathcal{S}^{critical} := \mathcal{S}^{inv} \cap \{\lim_{n\to\pm\infty} S_n \neq \pm\infty\}.$$

$S \in \mathcal{S}^{critical}$ implies the density of particles is asymptotically $\frac{1}{2}$ as $n \to \pm\infty$.

Particle current at origin

W_0 : the number of particles moved by the carrier from $\{... -2, -1, 0\}$ to $\{1, 2, \dots\}$ on the first evolution of the BBS

$T^{k-1}W_0$: the number of particles moved by the carrier from $\{... -2, -1, 0\}$ to $\{1, 2, \dots\}$ for the k-th evolution.

- Q 1 : How the property of the current sequence $(T^k W_0)_{k\in\mathbb{Z}}$ reflects the property of the whole configuration $(\eta_n)_{n\in\mathbb{Z}}$
- Q 2 : Asymptotic behavior of the integrated current $C_k := \sum_{\ell=1}^{k} T^\ell W_0$.

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Theorem

$(T^k W_0)_{k \in \mathbb{Z}}$ and $(\eta_n)_{n \in \mathbb{Z}}$ are one-to-one on an enough large subspace of $\mathcal{S}^{inv}$.

Random initial configuration

From now on, we consider $\eta = (\eta_n)$ is a random configuration.

- We always assume $\eta \in \mathcal{S}^{rev}$, a.s.

Remark

If $\eta = (\eta_n)$ is stationary ergodic under the space shift, then $\frac{S_n}{n} \to 1 - 2\rho$ $(n \to \pm\infty)$, $\mathbf{P}$-a.s. where $\rho = \mathbf{P}(\eta_0 = 1)$.
In particular, if $\rho < \frac{1}{2}$, $\eta \in \mathcal{S}^{inv}$ $\mathbf{P}$-a.s. (and so $\eta \in \mathcal{S}^{rev}$ $\mathbf{P}$-a.s.).
And also, if $\rho > \frac{1}{2}$, then $S \notin \mathcal{S}^{T}$ $\mathbf{P}$-a.s.

Gibbs measures

We expect there are invariant measures given as Gibbs measures parametrized by $(\beta_k)_{k\geq 0}$

$$\frac{1}{Z}\exp\Big(\sum_{k=0}^{\infty}\beta_k f_k(\eta)\Big)P(d\eta)$$

where P is the reference measure under which η is the i.i.d. sequence with density $\frac{1}{2}$, $f_0(\eta)=\sum_{n\in\mathbb{Z}}\eta_n$ is the total number of particles, and $f_k(\eta)$ is the total number of solitons with size greater than or equal to k. Note that $f_1(\eta)=\sum_{n\in\mathbb{Z}}1_{\{\eta_n=1,\eta_{n+1}=0\}}$.

General properties of invariant measures

Theorem (Invariant measures have some spacial homogeneity)

Suppose $T\eta \overset{d}{=} \eta$. *Then, the followings hold.*
(i) $S \in \mathcal{S}^{sub-critical} \cup \mathcal{S}^{critical}$, $\mathbf{P}$*-a.s.*
(ii) There exists a constant $\rho \in [0, \frac{1}{2}]$ *such that*

$$\mathbf{P}(\eta_n = 1) = \rho, \qquad \forall n \in \mathbb{Z}.$$

Moreover, $\rho = \frac{1}{2}$ *if and only if* $S \in \mathcal{S}_{critical}$, $\mathbf{P}$*-a.s.*
(iii) If $S \in \mathcal{S}_{critical}$, $\mathbf{P}$*-a.s., then* $\eta \overset{d}{=} 1 - \eta$

Remark

There exists an invariant measure which is not translation (shift) invariant.

Duality between particle configuration and current at origin

Let θ be the canonical shift on $\mathbb{Z}_+^{\mathbb{Z}}$:

$$\theta : \mathbb{Z}_+^{\mathbb{Z}} \to \mathbb{Z}_+^{\mathbb{Z}} : \qquad (\theta Z)_k = Z_{k+1}.$$

- Q 1 : How the property of the current sequence $(T^k W_0)_{k\in\mathbb{Z}}$ reflects the property of the configuration $(\eta_n)_{n\in\mathbb{Z}}$

Theorem

Suppose $S \in \mathcal{S}^{sub-critical}$, $\mathbf{P}$-a.s.
The distribution of η is invariant and ergodic under T if and only if $((T^k W)_0)_{k\in\mathbb{Z}}$ is stationary and ergodic under θ.

A sufficient condition to be invariant under T

Theorem

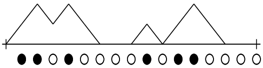
Any two of the three following conditions imply the third:

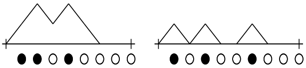
$$\overleftarrow{\eta} \overset{d}{=} \eta, \qquad \bar{W} \overset{d}{=} W, \qquad T\eta \overset{d}{=} \eta$$

where $\overleftarrow{\eta}$ *is the reversed configuration and* $\bar{W}$ *is the reversed carrier process given as*

$$\overleftarrow{\eta}_n = \eta_{-(n-1)}, \qquad \bar{W}_n = W_{-n}.$$

Any one of the three conditions does not imply the others.





Invariant measure: Example 1 (i.i.d.)

$\eta = (\eta_n)$: i.i.d Bernoulli random variables with $P(\eta_0 = 1) = p < \frac{1}{2}$

Then, since W is a reflected simple random walk on $\mathbb{Z}_{\geq 0}$

$$\overleftarrow{\eta} \overset{d}{=} \eta, \qquad \bar{W} \overset{d}{=} W$$

are satisfied and so

$$T\eta \overset{d}{=} \eta.$$

The measure is given as a Gibbs measure

- $\beta_0 = \log(\frac{1-p}{p})$, $\beta_k = 0$ $(k \geq 1)$.

Remark

Ferrari et al. had already shown the invariance under T for this example.

Invariant measure: Example 2 (η :Markov)

$\eta = (\eta_n)$: a two-sided stationary Markov chain on $\{0, 1\}$ with transition matrix

$$\begin{pmatrix} 1 - p_0 & p_0 \\ 1 - p_1 & p_1 \end{pmatrix}$$

where $p_0 \in (0, 1)$, $p_1 \in [0, 1)$ satisfy $p_0 + p_1 < 1$.

Then, it is clear that $\overleftarrow{\eta} \stackrel{d}{=} \eta$.

Proposition (Hambly-Martin-O'connell, 2001)

Under this condition, $\bar{W} \stackrel{d}{=} W$

And so

$$T\eta \stackrel{d}{=} \eta.$$

The measure is given as a Gibbs measure

- $\beta_0 = \log(\frac{1-p_0}{p_1})$, $\beta_1 = \log(\frac{p_1(1-p_0)}{p_0(1-p_1)})$, $\beta_k = 0$ $(k \geq 2)$.

If $p_0 = p_1 = p$, then (η_n) is i.i.d with parameter p.

Invariant measure: Example 3 (Bounded solitons)

Fix any $K \in \mathbb{Z}_+$.
$\eta = (\eta_n)$: a sequence of i.i.d. Bernoulli random variables with parameter $p \in (0,1)$ conditioned on $\{\sup_{n \in \mathbb{Z}} W_n \le K\}$
Since $\mathbf{P}(\sup_{n \in \mathbb{Z}} W_n \le K) = 0$, we need a limiting operation to define the measure precisely.
For this case, we prove that

$$\overleftarrow{\eta} \stackrel{d}{=} \eta, \qquad \bar{W} \stackrel{d}{=} W$$

are satisfied and so

$$T\eta \stackrel{d}{=} \eta.$$

The measure is given as a Gibbs measure formally

- $\beta_0 = \log(\frac{1-p}{p})$, $\beta_1 = \cdots = \beta_K = 0$, $\beta_k = \infty$ $(k \ge K+1)$.

Current at origin

$T^{k-1}W_0$: the number of particles moved by the carrier from $\{... -2, -1, 0\}$ to $\{1, 2, \dots\}$ for the k-th evolution.

- Q 2 : Asymptotic behavior of the integrated current $C_k := \sum_{\ell=1}^{k} T^\ell W_0$.

Theorem

For Example 1: $(T^k W_0)_{k\in\mathbb{Z}}$ *is an i.i.d. sequence with Geometric distribution with parameter* $\frac{1-2p}{1-p}$.
For Example 2: $(T^k\eta_0, T^k W_0)_{k\in\mathbb{Z}}$ *is a stationary two-sided Markov chain on* $\{0,1\}\times\mathbb{Z}_+$.
For Example 3: $(T^k W_0)_{k\in\mathbb{Z}}$ *is a stationary two-sided Markov chain on* $\{0,1,\dots,K\}$.

Corollary

For Examples 1,2 and 3, the distribution of η *is ergodic under* T.

LLN, CLT and LDP for the integrated current

Theorem

If $(\eta_n)_{n\in\mathbb{Z}}$ is given by one of the three examples, whose path-encoding is supported on $\mathcal{S}_{sub-critical}$, then

(i)

$$\frac{C_k}{k} \to \mathbf{E} W_0, \quad k \to \infty \quad a.s.$$

(ii)

$$\frac{C_k - k\mathbf{E} W_0}{\sqrt{\sigma^2 k}} \to N(0,1) \quad k \to \infty$$

in distribution, where

$$\sigma^2 := \mathrm{Var}\,(W_0) + 2\sum_{k=1}^{\infty} \mathrm{Cov}\left(W_0, \left(T^k W\right)_0\right) \in (0,\infty).$$

(iii) $(C_k)_k$ satisfies LDP.

Asymptotic behavior of a tagged particle : Def 1

Consider the dynamics given by Def 1.

Remark

The order of particles is preserved.

- $X_0 := \min\{n \geq 1; \eta_n = 1\}$: the position of the tagged particle at time 0
- X_k : the position of the tagged particle at time k

Theorem

Suppose η is i.i.d product Bernoulli sequence with density $p < \frac{1}{2}$.

$$\frac{X_k}{k} \to \frac{\mu_p}{p} = \frac{1}{1-2p} =: v_p \quad a.s.$$

Remark

$v_p = \frac{\mu_p}{p}$. *Each soliton moves at its own speed, but every particle moves asymptotically at the same speed.*

Generalization

We give a Pitman's transformation type expression for

- BBS with finite capacity of box and carrier
- Multi-color BBS
- Ultra-discrete KdV equation
- Ultra-discrete Toda equation
- Discrete KdV equation

Using this expression, we obtain

- "infinite particles version" of these systems
- an explicit class of invariant measures for these systems
- asymptotic behavior of "current" under these invariant measures

Future work

- Properties of "random solitons" on $\mathbb{Z}$ and on $\mathbb{R}$
- Scaling limit in non-equilibrium states
- Connections to other stochastic models (random polymers, random matrices, KPZ equation,...) and integrable systems (KdV equation, Toda lattice,...)