# Nonlinear $O(3)$ sigma model in discrete complex analysis


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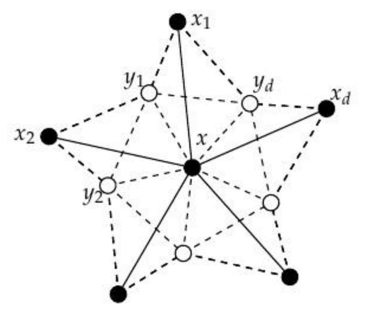
## 1. Our goal

Construction of a discrete version of the nonlinear O(3) sigma model on Discrete Riemann Surfaces Λ = Γ U Γ* in the sense of Mercat



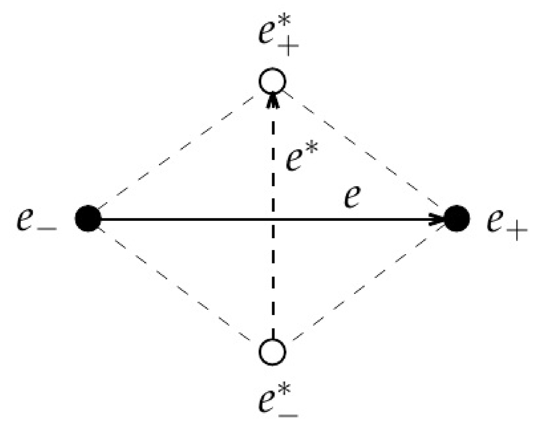
A discrete Riemann surface Λ = Γ U Γ*

## 2. Discrete holomorphic functions on Λ

$$f(e_+^*) - f(e_-^*) = i\rho(e)\big(f(e_+) - f(e_-)\big)$$

(A discrete version of the Cauchy-Riemann eqs. )

$$\rho(e) := \frac{\ell(e^*)}{\ell(e)}, \quad \text{Discrete conformal structure}$$



Edge e and its dual e*

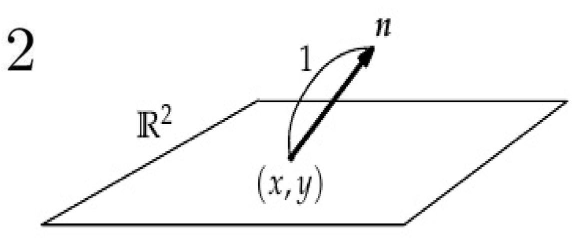
## 3. 2D nonlinear O(3) sigma Model (CP¹ Model)

- Energy

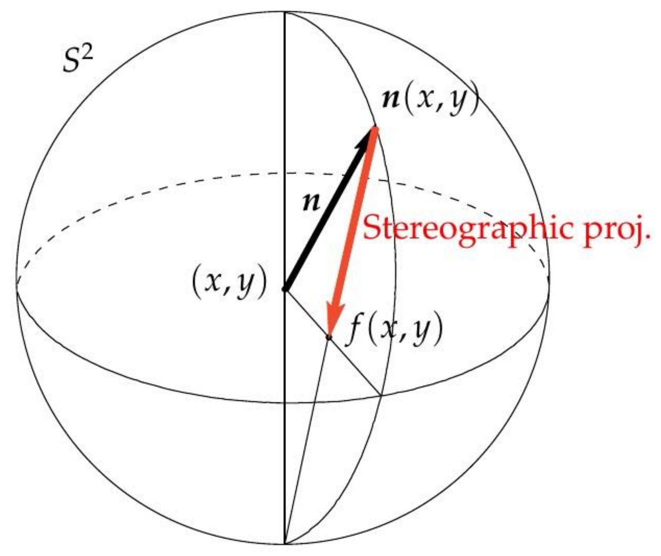
$$E = \frac{1}{2}\int_{\mathbb{R}^2} d^2x\, \partial^\mu \boldsymbol{n}\cdot\partial_\mu \boldsymbol{n}, \quad \mu = 1, 2$$

$$\boldsymbol{n} := (n^1, n^2, n^3), \quad |\boldsymbol{n}| = 1$$

$S^2$ : Target space




- Def. $f :=$ Stereographic proj. $\circ\, \boldsymbol{n}$




$$f : \mathbb{R}^2 \xrightarrow{\boldsymbol{n}} S^2 \xrightarrow{\text{Str. proj.}} \mathbb{R}^2 \cong \mathbb{C},$$

$$(x, y) \mapsto \frac{n^1(x,y) + in^2(x,y)}{1 + n^3(x,y)}$$

- An inequality for E

$$E = \left\|\frac{df \mp i * df}{1+|f|^2}\right\|^2 \pm 4\pi N \geq 4\pi|N|$$

- Topological quantum number $N \in \mathbb{Z} \cong \pi_2(S^2)$

$$N = \frac{1}{4\pi}\int_{\mathbb{R}^2} d^2x\, \boldsymbol{n}\cdot(\partial_x \boldsymbol{n} \times \partial_y \boldsymbol{n}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \boldsymbol{n}^* \mathrm{dvol}_{S^2}$$

- Saturation of the inequality

$$E = 4\pi|N|: \text{ BPS states}$$

⟺ (A)SD eqs. $df \mp i * df = 0$ : $f$ is (anti-)holomorphic.

⟹ The EL (Euler-Lagrange) eqs. are fulfilled.

- Example:

For $f(z) = z$ we have N=1 O(3) instanton solution.

$$\boldsymbol{n} = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2}\right)$$

## 4. Mercat's discrete norm and inner product

- Norm $\|\alpha\|^2 := (\alpha, \alpha)$ α: 1-form on Λ
- Inner product

$$(\alpha, \beta) := \iint \alpha \wedge *\bar{\beta} = \sum_{e\in\Lambda_1} \rho(e)\left(\int_e \alpha\right)\left(\int_e \bar{\beta}\right)$$

α, β: 1-forms on Λ
$\Lambda_1$: Edges of Λ

## 5. Line integral and an inequality

- Line integral

$$\int_e f\cdot\alpha := \frac{f(e_+) + f(e_-)}{2}\int_e \alpha$$

f: 0-form on Λ,
α: 1-form on Λ

- An inequality for $E^{\text{disc.}}$

$$E^{\text{disc.}} = \left\|\frac{df \mp i * df}{1+|f|^2}\right\|^2 \pm \mathcal{A}(f)^{\text{disc.}} \geq |\mathcal{A}(f)^{\text{disc.}}|$$

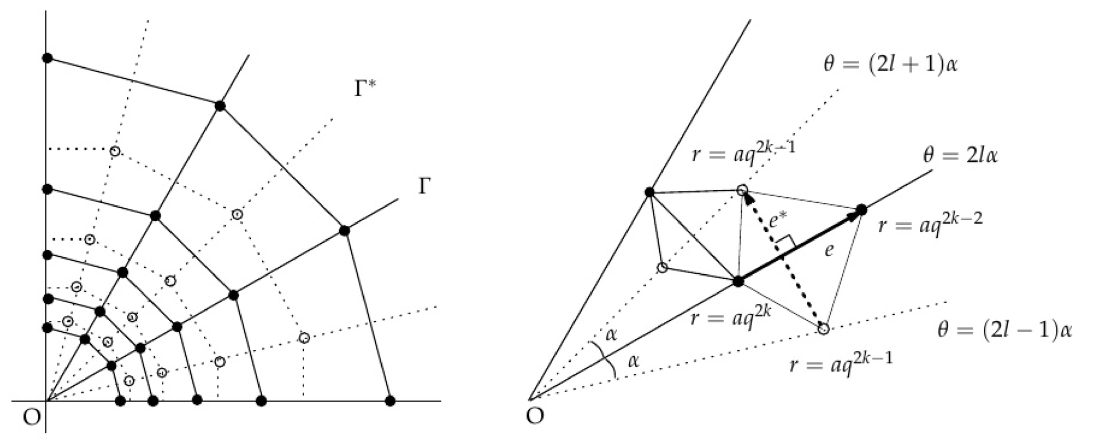
with

$$E^{\text{disc.}} := \sum_{e\in\Gamma_1} \nu(e_+, e_-, e_+^*, e_-^*)\Big(\rho(e)\big|f(e_+) - f(e_-)\big|^2 + \rho(e^*)\big|f(e_+^*) - f(e_-^*)\big|^2\Big),$$

$$\mathcal{A}(f)^{\text{disc.}} := -2\,\mathrm{Im}\sum_{e\in\Gamma_1} \nu(e_+, e_-, e_+^*, e_-^*)\,(f(e_+) - f(e_-))\,(\overline{f(e_+^*) - f(e_-^*)}).$$

$$\nu(e_+, e_-, e_+^*, e_-^*) := \left(\frac{\lambda(e_+) + \lambda(e_-)}{2}\right)^2 + \{e \to e^*\}, \qquad \lambda(z) := (1 + |f(z)|^2)^{-1}$$

## 6. Calculation of discrete area $\mathcal{A}(f)^{\text{disc.}}$



The first quadrant of a polar ("spider-web") lattice and two neighbouring kites

Continuum limit $M \to \infty,\ q \to 1-0,$ with $\rho = \frac{2q\sin(\pi/M)}{1-q^2}$ fixed,

$$\mathcal{A}(Cz^{\pm 1})^{\text{disc.}}_{(M,q)} = M\sin\frac{\pi}{M}\cdot\left(\frac{4q}{1+q^2} + |C|^2(q^{-2}+q^2+4)(q^{-1}-q)\sum_{n\in\mathbb{Z}}\frac{q^{2n}}{(1+|C|^2q^{2n})^2}\right)$$

$M\sin\frac{\pi}{M} \to \pi$, $\frac{4q}{1+q^2} \to 2$, $(q^{-2}+q^2+4) \to 6$, $(q^{-1}-q)\sum_{n\in\mathbb{Z}}\frac{q^{2n}}{(1+|C|^2q^{2n})^2} \to 1/|C|^2$

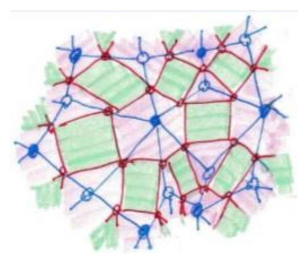
$\to 2\cdot 4\pi$

Where $f(z) = Cz^{\pm 1}$ are discrete holomorphic functions and we have used the following Jackson integral

$$(1-q)\sum_{n\in\mathbb{Z}}\frac{q^{2n}}{(1+|C|^2q^{2n})^2} = \int_0^\infty \frac{r}{(1+|C|^2r^2)^2}d_q r \xrightarrow{q\to 1-0} \int_0^\infty \frac{r}{(1+|C|^2r^2)^2}dr = \frac{1}{2|C|^2}$$

## 7. Work in progress

- Geometrical meaning of the area $\mathcal{A}(f)^{\text{disc.}}$. Topological?
- (A)SD eqs. $\overset{?}{\Longrightarrow}$ The EL eqs.
- Bobenko and Günther's Medial graphs X



F(X) ≅ V(Λ) U V(◇), ◇ := Λ*
≅ {$F_v$} U {$F_Q$},
$F_v$: Polygon around the vertex v ∈ Λ,
$F_Q$: Parallelogram in the quadrilateral Q ∈ Λ


**References**

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[2] Bobenko, A. I. & Günther, F. (2017) Discrete Riemann surfaces based on quadrilateral cellular decompositions. *Advances in Mathematics*, **311**, 885-932; arXiv:1511.00652v2 [math.CV].
[3] Kamata, M., Sekiguchi, M. & Tadokoro, U., arXiv: hep-th, 1602.08923v2.