Coprimeness property of the Toda type equations over multi-dimensional lattices

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Introduction

- What is an integrable fully-discrete systems?
- Examples of “Definitions”: Sufficient number of conserved quantities, existence of a Lax representation, multi-dimensional consistencies,…

⇒ We want to find easy detectors (criteria) for integrability.

Pioneering works:

- Singularity confinement --- Discrete analogue of the Painlevé test
- Algebraic entropy --- Complexity of iterated mappings

(cf.) Talk and poster on the dynamical degree by T. Mase and R. Willox.

Note: Other “detectors” include the Backlund transformation, the Devron property, growth of error-intervals in numerical computation, topological entropy, de-autonomisation, and so on.

Preliminaries

[Our initial aim] Construct an integrability "detector" which is universally applicable to fully-discrete systems.

- We have introduced the coprimeness (CP) condition as an algebraic re-interpretation to the singularity confinement test.
- CP condition has been found to be an useful integrability detector.

[Recent study] Many non-integrable (but interesting) coprimeness-preserving equations (CP eqs) have been discovered.

[Today's talk] We introduce several non-integrable coprimeness-preserving extensions to the known integrable dynamical systems:

e.g.) The Somos-4 sequence, and the 2dim discrete Toda lattice and its higher dimensional generalisation, higher dimensional HV equation.

(cf.) Integrable CP eqs: Poster on linearizable equations by R. Kamiya.

Fully-discrete equations

- Let us consider the discrete system on the integer lattice *L*.

$$x_h = \Phi\left(x_{h-v_1}, x_{h-v_2}, \cdots, x_{h-v_N}\right), \quad h \in L$$

- Here Φ is a rational function. Shifts $v_i \in L$ are independent on $\mathbb{Z}_{\geq 0}$

※Independentness is needed for the system to be an evolution equation.

- Set of initial variables H_0 is chosen for the evolution to be well-defined.

[e.g.]

Somos-4 sequence $x_{n+4} = \dfrac{x_{n+3}x_{n+1} + x_{n+2}^2}{x_n}, \; H_0 = \{x_0, x_1, x_2, x_3\}, L = \mathbb{Z}$

Discrete KdV equation

$$w_{n,m} = \frac{w_{n-1,m-1}}{1 - \delta w_{n-1,m-1}(w_{n-1,m} - w_{n,m-1})}, \; H_0 = \{w_{0,m}, w_{n,0} \mid m, n \in \mathbb{Z}_{\geq 0}\}, \; L = \mathbb{Z}^2$$

Singularity Confinement test (SC)

[Grammaticos, Ramani, Papageorgiou 1991]

- The equation passes the SC test, if the singularities (indeterminant points arising from div/0) are resolved after finite iterations and the information on the initial data are recovered.

[e.g.] One of the QRT mappings $x_{n+1} = \dfrac{x_n + 1}{x_{n-1} x_n^2}$ passes the SC test.

$$x_0 = a \to \quad x_1 = 0 \to \quad \infty \to \quad \text{indet} \quad \to \text{impossible}$$

introduce t $\Downarrow$

$$a \to \qquad t \to \qquad \frac{1+t}{at^2} \to \quad \frac{at(1+t+at^2)}{(1+t)^2} \to \frac{1+O(t)}{a(1+t+at^2)^2}$$

$t \to 0$ $\Downarrow$

$$a \to \qquad 0 \to \qquad \infty \to \qquad 0 \qquad \to \qquad a^{-1}$$

recovers information of a

Algebraic Entropy

- Algebraic entropy（AE） [Bellon-Viallet 1999]

[Def.] Algebraic entropy $\Lambda=\log\lambda$（$\lambda\geq 1$）of Φ is defined as:

$$\Lambda=\log\lambda=\lim_{n\to\infty}\frac{\log d_n}{n},\qquad d_n=\deg(\underbrace{\phi\circ\phi\circ\cdots\circ\phi}_{n})=\deg(\phi^n)$$

(λ is called the dynamical degree)

[Working definition]

- $\Lambda=0$ ⇒ the system is integrable
- $\Lambda>0$ ⇒ the system is non-integrable

[Note on multi-dimensional systems] AE is not defined because of more than one evolution directions.

[Tentative def.] “Polynomial growth in every direction”=“integrable”

Coprimeness Property (CP)

[Def.] Discrete Dynamical system *F* satisfies the coprimeness property (CP) if the following condition is satisfied for any pair x_h, x_k :

- There exists a constant *D*>0 independent of h,k such that

$$\mathrm{dist}(h,k) \geq D \Rightarrow (x_h, x_k) = 1$$

Here, dist(,) is a metric on the lattice *L*.

[Note 1] x_* is a rational function of the initial variables.

[Note 2] Two Laurent polynomials F/M,G/N (M,N:monomials) are coprime in the sense of Laurent polynomials if *(F,G)*=1.

[Note 3] Two rational functions *F/H, G/K* are coprime if *(F,G)=(F,K)=(H,G)=(H,K)=1* in the sense of Laurent polynomials.

※Monomial common factors are allowed.

Coprimeness-preserving class

When a system satisfies the coprimeness condition it belong to the **coprimeness-preserving class (CP class)**.

[e.g.] Somos-4 recurrence $x_{n+4}x_n = x_{n+3}x_{n+1} + x_{n+2}^2$

(D. Gale 1991, Integer sequence No. A006720) belongs to the CP class.

Somos-4 is transformed into a nonlinear form (one of the QRT mappings)

$$u_{n+1} = \frac{u_n + 1}{u_{n-1}u_n^2}$$

via the transformation $u_n = \frac{x_{n+3}x_{n+1}}{x_{n+2}^2}$.

Coprimeness-preserving class

[Prop.] [JPA2014] For a system $u_{n+1} = \dfrac{u_n + 1}{u_{n-1} u_n^2}$

$$(u_n, u_m) = 1 \ \text{ if } \ |n - m| \geq 3.$$

※In the Def. of coprimeness we take

$$dist(u_n, u_m) = |n - m|, \ D = 3.$$

[Experiment] For initial values $(u_0, u_1) = (s, t)$

$$s, t, \frac{1+t}{st^2}, \frac{st(1+t+st^2)}{(1+t)^2}, \frac{(1+t)(1+2t+\cdots+s^2t^3)}{s(1+t+st^2)^2},$$

$$\frac{(1+t+st^2)(1+3t+\cdots+s^3t^4)}{t(1+2t+\cdots+s^2t^3)^2}, \frac{st(1+2t+\cdots+s^2t^3)(1+s+4t+\cdots+s^4t^6)}{(1+s+3t+\cdots+s^3t^4)^2}, \cdots$$

Extended Coprimeness Property

- We can define a coprimeness property over the initial variables & several “additional” functions.
- Let I be the set of initial variables, and let $R = \mathbb{Z}[I], E \subset R - \{0\}$.
- The system has the extended Laurent property if every iterate is in

$$R\left[f^{-1} \mid f \in E \right] =: R\left[E^{-1} \right]$$

- The system has the extended coprimeness property if every pair of iterates is coprime in the quotient field of $R\left[E^{-1} \right]$.

※If E={monomials of *R*}, we have the original Laurent and coprimeness properties.

※If E=*R*-{0}, $R\left[E^{-1} \right]$ is a field, and the coprimeness is trivially satisfied for any equation.

Example of Extended Coprimeness

- Generalized Somos-4 [Fomin-Zelevinsky 2002]

$$x_{n+4}x_n = x_{n+3}^m x_{n+1}^l + x_{n+2}^k \qquad \left(\mathrm{GCD}(k,l,m) = 1,2,4,8,\cdots\right)$$

※ We call it the CP-Somos-4.

[Prop.] It has irreducible Laurentness and the coprimeness. [JMP2015]

- Nonlinear form of the CP-Somos-4:

$$\frac{\left(u_{n+4}-1\right)\left(u_n-1\right)}{\left(u_{n+2}-1\right)^k} = \frac{u_{n+3}^m u_{n+1}^l}{u_{n+2}^k}$$

- Correspondence: $u_{n+2} = \dfrac{x_{n+4}x_n}{x_{n+2}^k}$ (Different from the relation between Somos-4 and a QRT system)

Nonlinear form of CP-Somos-4

- Nonlinear form $$\frac{(u_{n+4}-1)(u_n-1)}{(u_{n+2}-1)^k}=\frac{u_{n+3}^m u_{n+1}^l}{u_{n+2}^k}\cdots\cdots\dagger$$

is constructed from the reduction of 2-dim discrete Toda.

- By $u_{n+2}=\frac{x_{n+4}x_n}{x_{n+2}^k}$ it corresponds with CP-Somos-4 non-trivially:

[Sketch]

$$\dagger \Leftrightarrow F_n=\frac{x_{n+4}x_n-x_{n+2}^k}{x_{n+3}^m x_{n+1}^l},\quad \frac{F_{n+4}F_n}{F_{n+2}^k}=1$$

- F_n is explicitly written by F_0, F_1, F_2, F_3.
- Non-autonomous system $x_{n+4}x_n=F_n\cdot x_{n+3}^m x_{n+1}^l+x_{n+2}^k$ includes the CP-Somos-4.
- It is well-defined over the initial variables $\{F_0, F_1, F_2, F_3, x_4, x_5, x_6, x_7\}$.

Nonlinear form of CP-Somos-4

- [Key Lemma] Each iterate x_n (non-auto system) has the extended Laurent property as

$$x_n \in \mathbb{Z}\left[E'^{-1}\right] := \mathbb{Z}\left[g^{-1} \mid g \in E'\right]$$

$$E' = \left\{x_4, x_5, x_6, x_7, x_4x_0 - x_2^k, x_5x_1 - x_3^k, x_6x_2 - x_4^k, x_7x_3 - x_5^k\right\}$$

  Moreover, it is irreducible and pairwise coprime.

- [Thm.] Each iterate u_n has the extended coprimeness property as

$$\text{If } n \neq n' \bmod 2 \text{ or } |n - n'| \geq 5, \text{then}, (u_n, u_{n'}) = 1 \text{ in } Q:$$

$$Q = (\text{Quotient field of } \mathbb{Z}\left[E^{-1}\right]),$$

$$E = \left\{u_2, u_3, u_4, u_5, (u_2 - 1), (u_3 - 1), (u_4 - 1), (u_5 - 1)\right\}.$$

(Sketch) Use the key Lemma and $u_2 - 1 = \dfrac{x_4x_0 - x_2^k}{x_2^k}$.

Coprimeness-preserving extension to the two-dimensional discrete Toda (CP-2dim-dToda)

[Thm.][JMP2017] Let us study

$$\tau_{t+1,n,m+1}\tau_{t-1,n+1,m} = \left(\tau_{t,n+1,m}\right)^{k_1}\left(\tau_{t,n,m+1}\right)^{k_2} + \left(\tau_{t,n,m}\right)^{l_1}\left(\tau_{t,n+1,m+1}\right)^{l_2}$$

$$GCD(k_1,k_2,l_1,l_2) = 1,2,4,8,\cdots$$

Each iterate is an irreducible Laurent polynomial in

$$\tau_{t,n,m} \in \mathbb{Z}\left[\tau_{0,n,m}^{\pm}, \tau_{1,n,m}^{\pm}\right]$$

Moreover, two distinct iterates are pairwise coprime.

※ Original 2dim-dToda if superscripts are all 1.

※ Initial variables are all the points on the n-m planes with $t=0,1$.

Cp-2dim-dToda (Nonlinear form)

The equation we study (shifted as $1+V_* = U_*$ for a technical reason):

$$\frac{(U_{t+1,n,m}-1)(U_{t-1,n,m}-1)}{\left(U_{t,n+\frac{1}{2},m-\frac{1}{2}}-1\right)^{k_1}\left(U_{t,n-\frac{1}{2},m+\frac{1}{2}}-1\right)^{k_2}} = \frac{U^{l_1}_{t,n-\frac{1}{2},m-\frac{1}{2}}U^{l_2}_{t,n+\frac{1}{2},m+\frac{1}{2}}}{U^{k_1}_{t,n+\frac{1}{2},m-\frac{1}{2}}U^{k_2}_{t,n-\frac{1}{2},m+\frac{1}{2}}}.$$

Related to the ”tau-function” form:

$$\begin{cases}\tau_{t+1,n,m}\tau_{t-1,n,m} = \left(\tau_{t,n+1/2,m-1/2}\right)^{k_1}\left(\tau_{t,n-1/2,m+1/2}\right)^{k_2} + F_{t-1,n,m}\left(\tau_{t,n-1/2,m-1/2}\right)^{l_1}\left(\tau_{t,n+1/2,m+1/2}\right)^{l_2},\\ F_{t+1,n,m}F_{t-1,n,m} = F^{k_1}_{t,n+1/2,m-1/2}F^{k_2}_{t,n-1/2,m+1/2}.\end{cases}$$

Correspondence:

$$U_{t,n,m} = \frac{\tau_{t+1,n,m}\tau_{t-1,n,m}}{\tau^{k_1}_{t,n+1/2,m-1/2}\tau^{k_2}_{t,n-1/2,m+1/2}}.$$

Main result on Cp-2dim-dToda (nonlinear form) —Extended Coprimeness Property—

- [Key Lemma] The tau-function form has the irreducible Laurent and the coprimeness properties in $S := \mathbb{Z}[E^{-1}]$.

Here, $$E = \left\{ \begin{matrix} \tau_{2,n,m}, \tau_{3,n,m}, (\tau_{2,n,m}\tau_{0,n,m} - \tau_{1,n+1/2,m-1/2}^{k_1}\tau_{1,n-1/2,m+1/2}^{k_2}), \\ (\tau_{3,n,m}\tau_{1,n,m} - \tau_{2,n+1/2,m-1/2}^{k_1}\tau_{2,n-1/2,m+1/2}^{k_2}) \end{matrix} \right\}_{n,m}.$$

- [Thm.] [JPA2018-2]

Two iterates $U_{t,n,m}, U_{t',n',m'}$ of the nonlinear form is in the quotient field of $T := \mathbb{Z}[G]$ and is pairwise coprime on *T* if the following condition is satisfied:

$$|t - t'| > 2 \text{ or } (n', m') \neq (n, m), (n-1, m+1), (n+1, m-1).$$

Here, $$G = \left\{ U_{0,n,m}, U_{1,n,m}, \left(U_{0,n,m} - 1\right), \left(U_{1,n,m} - 1\right) \right\}_{n,m}.$$

Higher-dimensional coprimeness-preserving systems

（1）(a+b)-dimensional “discrete Toda lattice”

(non-integrable except for a=b=1, k=m=1)

$$\tau_{t+1,\mathbf{n}}\tau_{t-1,\mathbf{n}} = \prod_{i=1}^{a}\left(\tau_{t,\mathbf{n}+\mathbf{e}_i}\right)^{k_i}\left(\tau_{t,\mathbf{n}-\mathbf{e}_i}\right)^{m_i} + \prod_{i=a+1}^{a+b}\left(\tau_{t,\mathbf{n}+\mathbf{e}_i}\right)^{k_i}\left(\tau_{t.\mathbf{n}-\mathbf{e}_i}\right)^{m_i}$$

$$\mathbf{n} = \left(n_1, n_2, \cdots, n_{a+b}\right), \mathbf{e}_i : \text{unit vector}$$

（2）Higher dimensional analogue of the Hietarinta-Viallet equation (also related to CP-discrete KdV eq. when d=1)

$$x_{t+1,\mathbf{n}} = -x_{t-1,\mathbf{n}} + \sum_{i=1}^{d}\left(\frac{a_i}{\left(x_{t,\mathbf{n}+\mathbf{e}_i}\right)^{k_i}} + \frac{b_i}{\left(x_{t,\mathbf{n}-\mathbf{e}_i}\right)^{m_i}}\right) \qquad (k_i, m_i \in 2\mathbb{N})$$

Result on the CP Class (1)

- [Thm.] [JPA-Special issue 2018, arXiv:1803.06774] The equation

$$\tau_{t+1,\mathbf{n}}\tau_{t-1,\mathbf{n}} = \prod_{i=1}^{a}\left(\tau_{t,\mathbf{n}+\mathbf{e}_i}\right)^{k_i}\left(\tau_{t,\mathbf{n}-\mathbf{e}_i}\right)^{m_i} + \prod_{i=a+1}^{a+b}\left(\tau_{t,\mathbf{n}+\mathbf{e}_i}\right)^{k_i}\left(\tau_{t.\mathbf{n}-\mathbf{e}_i}\right)^{m_i}$$

has the Laurent property. Moreover, if $GCD\left(k_i, l_i \mid i = 1, \cdots, a+b\right)$

is a non-negative power of 2, it satisfies the irreducibility and the coprimeness.

Here, the initial variables are $\left\{\tau_{0,\mathbf{n}}, \tau_{1,\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^{a+b}\right\}$, and it evolves towards the positive direction of t.

[Thm. 2] Its evolution is expressed as a mutation of the LP algebra.

(Details are omitted. Mainly done by N. Okubo.)

Result on the CP Class (2)

- [Thm.] Let us study $x_{t+1,\mathbf{n}} = -x_{t-1,\mathbf{n}} + \sum_{i=1}^{d} \left(\frac{a_i}{\left(x_{t,\mathbf{n}+\mathbf{e}_i}\right)^{k_i}} + \frac{b_i}{\left(x_{t,\mathbf{n}-\mathbf{e}_i}\right)^{m_i}} \right) \qquad (k_i, m_i \in 2\mathbb{N})$

with a certain condition on the indices *k*, *m*. (details are omitted here)

Its two iterates $x_{t,\mathbf{n}}, x_{t',\mathbf{n}'}$ are coprime in $\mathbb{Q}\left(\left\{x_{0,\mathbf{m}}, x_{1,\mathbf{m}}\right\}_{m\in Z^d}, \{a_i, b_i\}\right)$,

If $|t-t'|>2$ or $\mathrm{dist}(\mathbf{n},\mathbf{n}')>2$.

- Sketch: Use the Laurent property and the irreducibility of the "tau-function" form.

[Lemma]

By the transformation $x_{t,\mathbf{n}} = \frac{f_{t+1,\mathbf{n}} f_{t-1,\mathbf{n}}}{\prod_{i=1}^{d} f_{t,\mathbf{n}+\mathbf{e}_i}^{k_i} f_{t,\mathbf{n}-\mathbf{e}_i}^{l_i}}$ we obtain the tau-function form of the equation w.r.t. f_*. Each iterate f_* satisfies the irreducibility.

Result on the CP Class (2)

- One of the reductions of (2):

$$x_n + x_{n-p-q} = \frac{a}{x_{n-q}^k} + \frac{b}{x_{n-p}^k} \quad (k : even).$$

- An extension of the Hietarinta-Viallet equation to a multi-term recurrence.

[Prop.](in prep.) Its dynamical degree is the maximal real root of

$$\lambda^{p+q} - k\lambda^p - k\lambda^q + 1 = 0.$$

[Cor.] When p=1, q=2, k=2, we recover the original H-V equation.

※The root of a “characteristic” polynomial related to the singularity structure gives the dynamical degree.

Conclusion and Problems

[Conclusion]

① Defined the coprimeness and the extended coprimeness properties.

② Constructed a coprimeness-preserving non-integrable equation related to the two dimensional Toda.

③ Constructed its generalisations to equations over a higher-dimensional lattice.

[Future Problems]

① Relations of the coprimeness with various notions of entropy.
(e.g.) Diophantine entropy & algebraic entropy (growth of complexity of an iteration) vs. topological entropy & Lyapunov exponent (growth of a small perturbation).

② Integrated method of proving the coprimeness.

③ Possible application to coprime sequences of integers in number theory.

Thank you for your attention!!

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