Commuting Transfer Matrix Method in Integrable Systems

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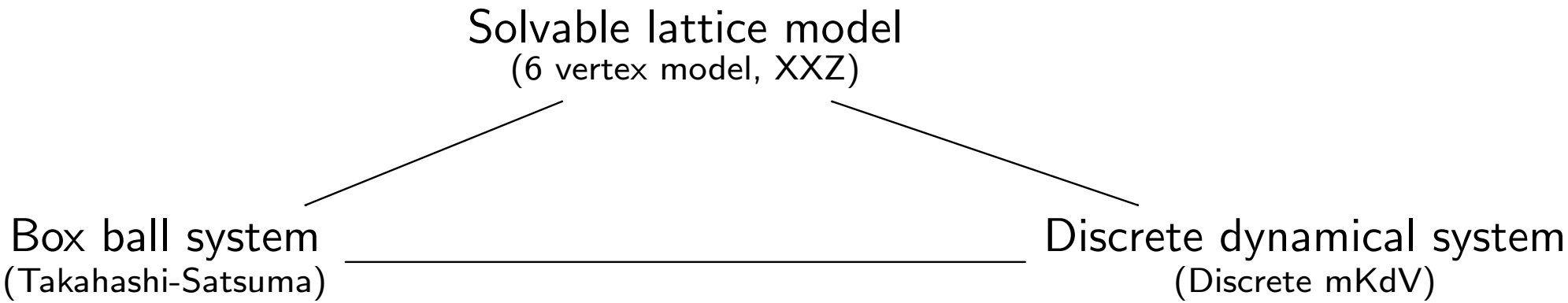
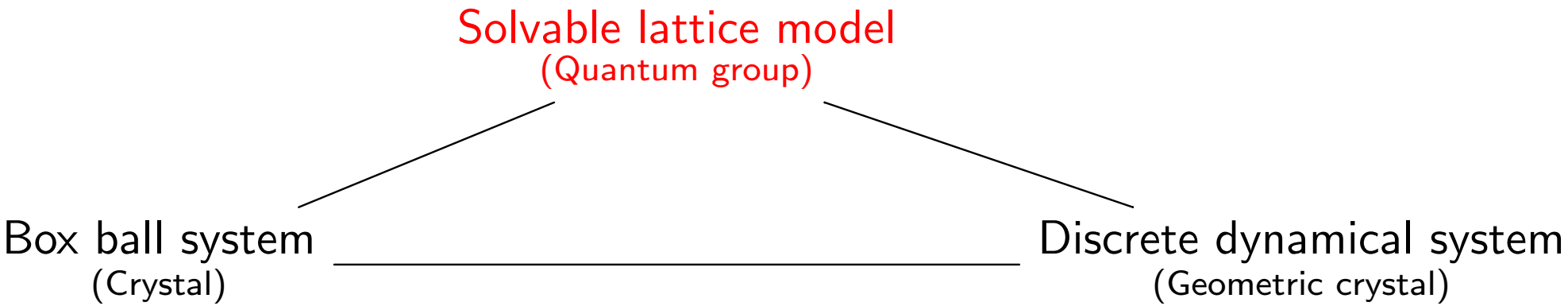
Trinity



proposed by Y. Yamada around 2002.

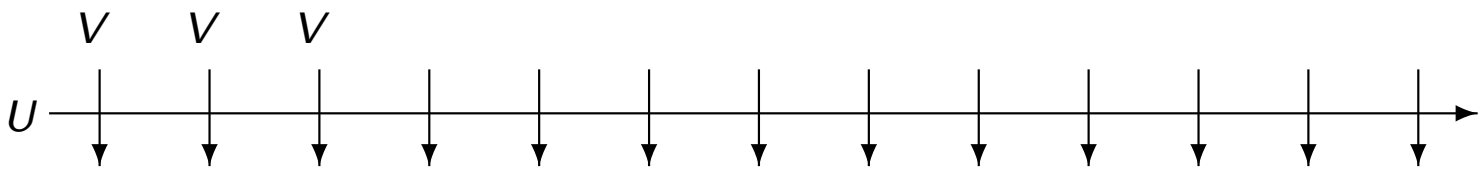
Solvable lattice model



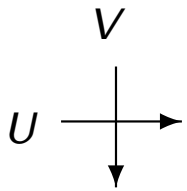
review a transfer matrix, R-matrix, the Yang-Baxter equation.

Transfer matrix, R-matrix

Transfer matrix $T : V^{\otimes L} \to V^{\otimes L}$



R-matrix $R : U \otimes V \to V \otimes U$



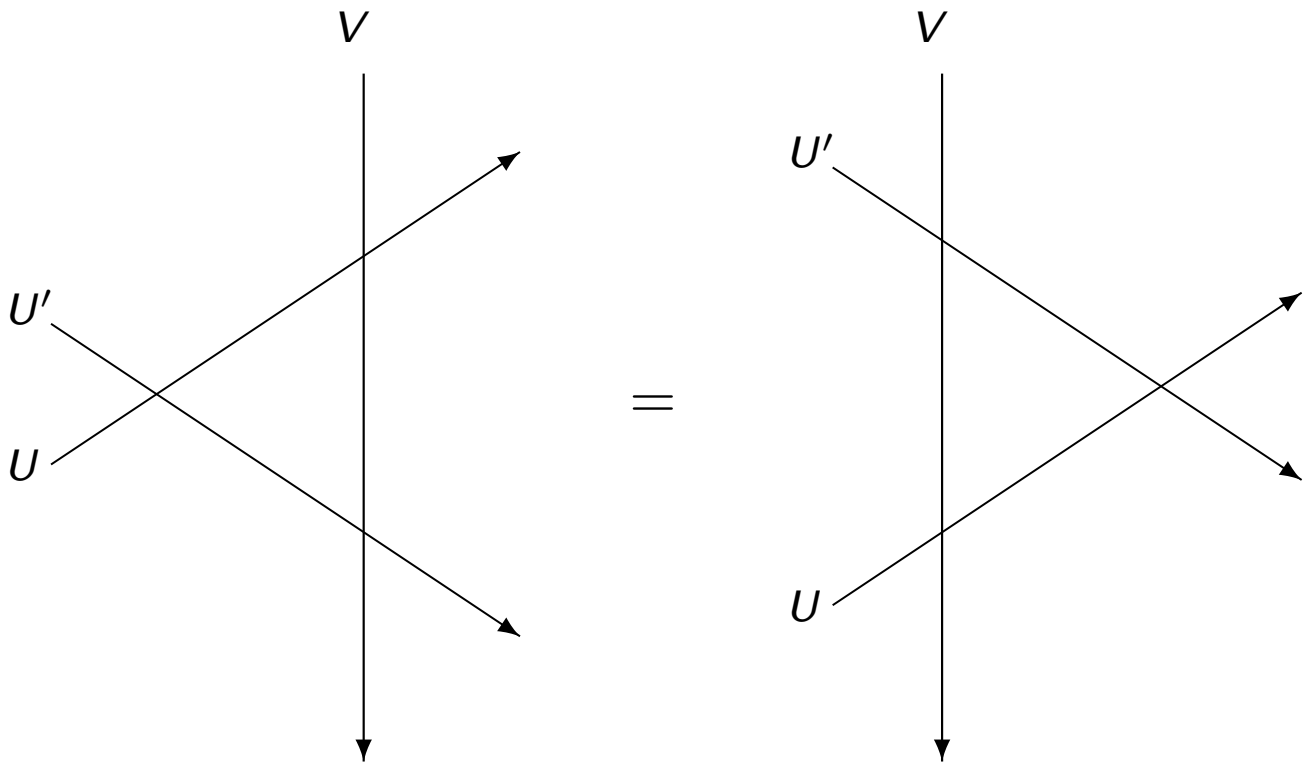
If you have commuting family of transfer matrices, our life become easier.

$$T_{U'} T_U = T_U T_{U'}$$

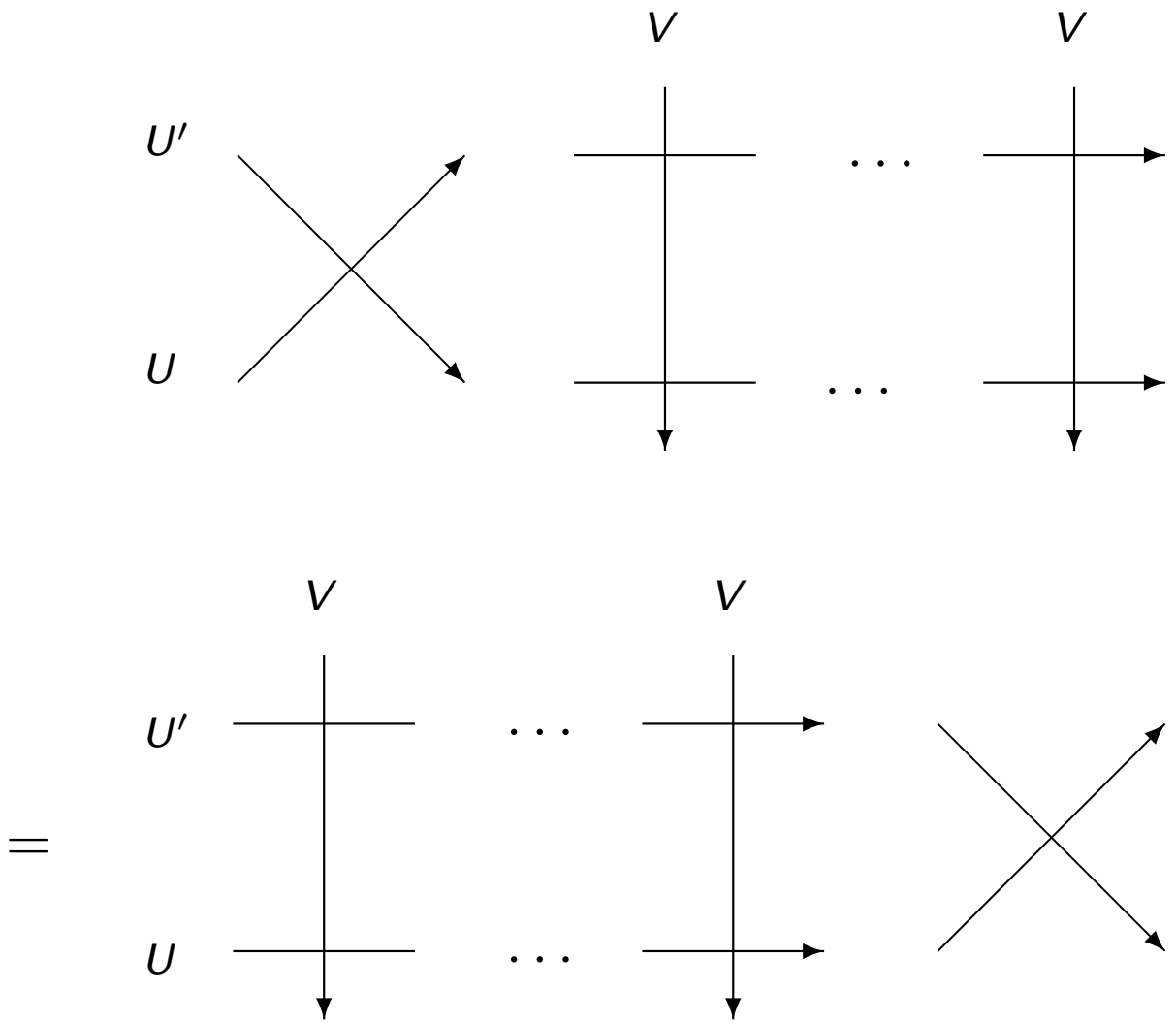
for any U, U' belonging to a family of vector spaces.

Yang-Baxter equation

$$R_{U'V}R_{UV}R_{UU'} = R_{UU'}R_{UV}R_{U'V}$$



Commutativity of transfer matrices



Choices of U and V — Kirillov-Reshetikhin module

We have many choices of U and V to play our game with.

$\mathfrak{g}$: affine Lie algebra

$U_q(\mathfrak{g})$: quantum algebra associated to $\mathfrak{g}$

$W_s^{(r)}(u)$ ($r \in I \setminus \{0\}, s \in \mathbb{Z}_{\geq 1}, u$ paremeter):
finite-dimensional $U_q(\mathfrak{g})$-module called Kirillov-Reshetikhin module

$T_s^{(r)}(u) = T_{W_s^{(r)}(u)}$ satisfy certain algebraic relation called T-system.

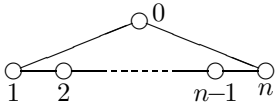
$$T_s^{(r)}(u+1)\,T_s^{(r)}(u-1) = T_{s+1}^{(r)}(u)\,T_{s-1}^{(r)}(u) + \prod_{r' \sim r} T_s^{(r')}(u)$$

for simply laced types.

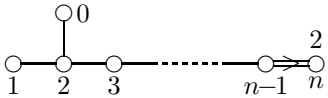
Table 1: Dynkin diagrams for $X_N^{(r)}$. The enumeration of the nodes with $I = \{0, 1, \ldots, n\}$ is specified under or the right side of the nodes. In addition, the numbers t_i (resp. $t_i^\vee$) defined in (2.1) are attached *above* the nodes for $r = 1$ (resp. $r > 1$) if and only if $t_i \neq 1$ (resp. $t_i^\vee \neq 1$).

$A_1^{(1)}$:

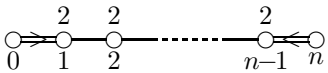
$A_n^{(1)}$: ($n \geq 2$)



$B_n^{(1)}$: ($n \geq 3$)

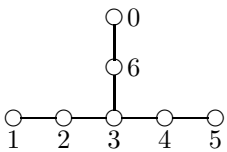


$C_n^{(1)}$: ($n \geq 2$)

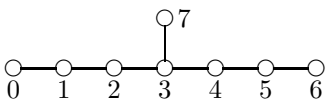


$D_n^{(1)}$: ($n \geq 4$)

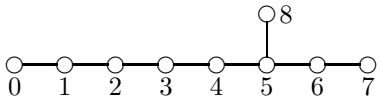
$E_6^{(1)}$:



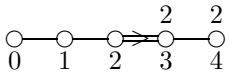
$E_7^{(1)}$:



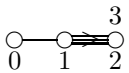
$E_8^{(1)}$:



$F_4^{(1)}$:

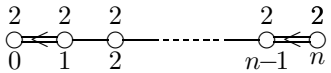


$G_2^{(1)}$:

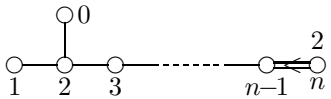


$A_2^{(2)}$:

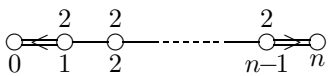
$A_{2n}^{(2)}$: ($n \geq 2$)



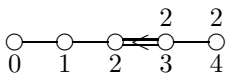
$A_{2n-1}^{(2)}$: ($n \geq 3$)



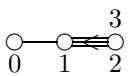
$D_{n+1}^{(2)}$: ($n \geq 2$)



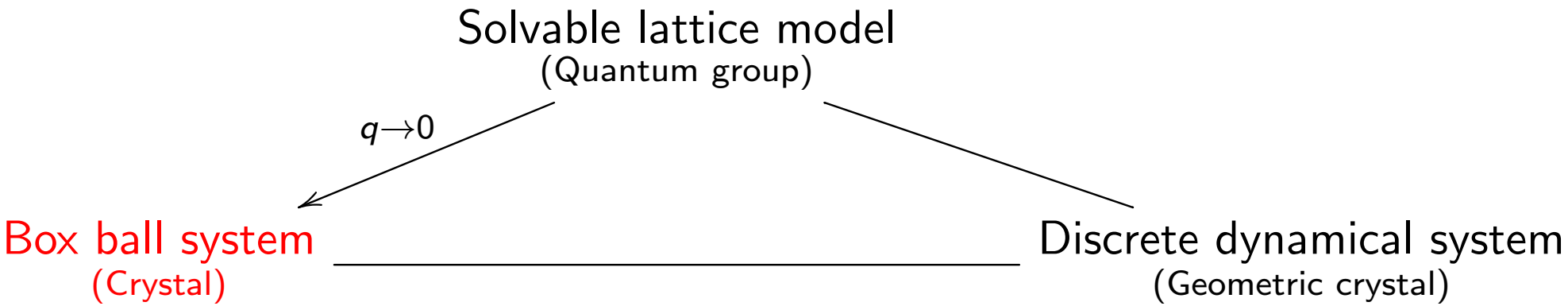
$E_6^{(2)}$:



$D_4^{(3)}$:



Box ball system



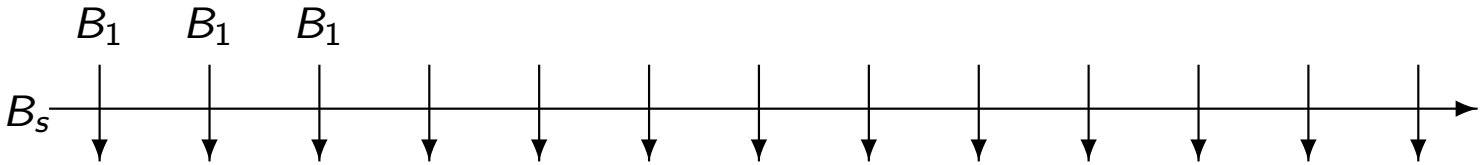
Takahashi-Satsuma's box ball system

The idea of commuting transfer matrices also works for Takahashi-Satsuma's box ball system.

Type $A_1^{(1)}$

$B_s = \{ \boxed{1}\boxed{1}\boxed{\cdots}\boxed{2} \}$: crystal of $W_s^{(1)}$

Time evolution operator $T_s : (B_1)^L \to (B_1)^L$



Combinatorial R-matrix $R : B_s \times B_1 \to B_1 \times B_s$ bijection!

$T_{s'} T_s = T_s T_{s'}$

Can construct a conserved quantity E_s under any $T_{s'}$.

Takahashi-Satsuma's box ball system (2)

$t = 0$: 2 2 2 1 1 2 2 1 1 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

$t = 1$: 1 1 1 2 2 1 1 2 2 1 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

$t = 2$: 1 1 1 1 1 2 2 1 1 2 1 1 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1

$t = 3$: 1 1 1 1 1 1 1 2 2 1 2 1 1 1 1 2 2 2 1 1 1 1 1 1 1 1 1 1 1

$t = 4$: 1 1 1 1 1 1 1 1 1 2 1 2 2 1 1 1 1 1 2 2 2 1 1 1 1 1 1 1 1

$t = 5$: 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 1 1 1 1 1 1 2 2 2 1 1 1 1 1

$t = 6$: 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 2 1 1 1 1 1 1 1 2 2 2 1 1

$t = 7$: 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 2 1 1 1 1 1 1 1 1 2 2

Nonlinear integrable system with solitons!

Kirillov-Reshetikhin crystal

Now we can replace B_s with other crystals to play a new game.

$B_s^{(r)}$: crystal of $W_s^{(r)}$, Kirillov-Reshetikhin crystal

Known to exist for all nonexceptional cases [Fourier-O-Schilling (2009)]

$R : B_s^{(r)} \times B_1^{(r)} \to B_1^{(r)} \times B_s^{(r)}$

There is a remarkable bijection that linearize this dynamics firstly constructed by Kerov-Kirillov-Reshetikhin in 1986.

Extended to all nonexceptional types [O-Schilling-Scrimshaw (2018)].

Type $A_3^{(1)}, r = 1$

$t = 0$: 1 1 1 1 2 2 2 2 1 1 3 3 2 1 1 1 4 2 1 3 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

$t = 1$: 1 1 1 1 1 1 1 1 2 2 2 2 1 3 3 2 1 1 4 2 3 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

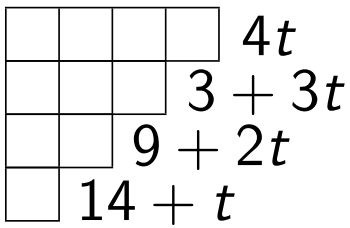
$t = 2$: 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 1 3 3 2 1 2 4 3 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

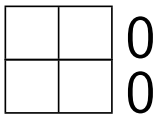
$t = 3$: 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 1 3 1 3 2 1 4 3 2 2 1 1 1 1 1 1 1 1 1 1 1

$t = 4$: 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 3 2 1 3 2 1 1 1 4 3 2 2 1 1 1 1 1 1 1

$t = 5$: 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 3 2 1 3 2 2 1 1 1 1 4 3 2 2 1 1 1 1

$t = 6$: 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 1 3 2 1 1 3 2 2 1 1 1 1 1 4 3 2 2





Type $D_4^{(1)}, r = 2$

$t = 0$:

$$\begin{array}{cccccccccccccccccccccccc} 2 & 2 & & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & \overline{2} & \phi & \overline{3} & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

$t = 1$:

$$\begin{array}{ccccccccccccccccccccccccc} 1 & 1 & 2 & 1 & 1 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & \overline{1} & 2 & 4 & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

$t = 2$:

$$\begin{array}{ccccccccccccccccccccccccc} 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \overline{1} & 2 & 2 & 4 & 3 & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

$t = 3$:

$$\begin{array}{ccccccccccccccccccccccccc} 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & \overline{1} & 2 & 2 & 2 & 4 & 3 & 2 & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

$t = 4$:

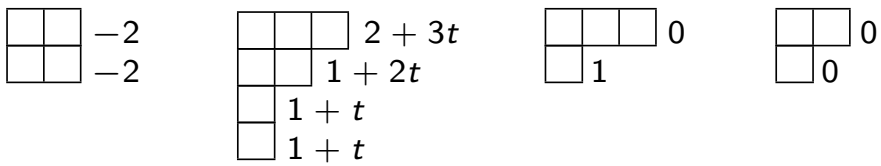
$$\begin{array}{ccccccccccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & \overline{1} & 2 & 2 & 2 & 2 & 4 & 3 & 2 & 2 & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{array}$$

$t = 5$:

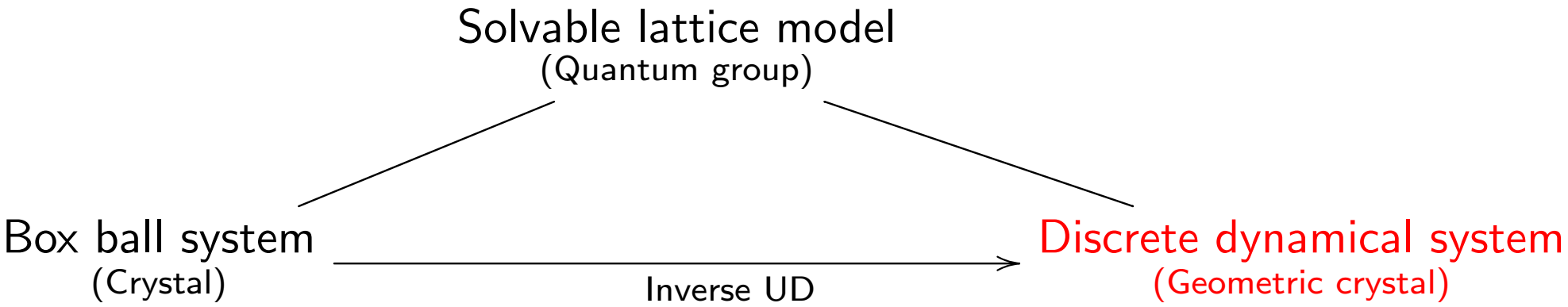
$$\begin{array}{ccccccccccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & \overline{1} & 2 & 2 & 2 & 2 & 2 & 4 & 3 & 2 & 2 & 2 & \overline{3} & \overline{3} & 3 & 2 & 2 & 2 & 2 & 2 \end{array}$$

$t = 6$:

$$\begin{array}{ccccccccccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & \overline{1} & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 3 & 2 & 2 & 2 & 2 & \overline{3} & \overline{3} & 3 & 2 & 2 \end{array}$$



Discrete dynamical system

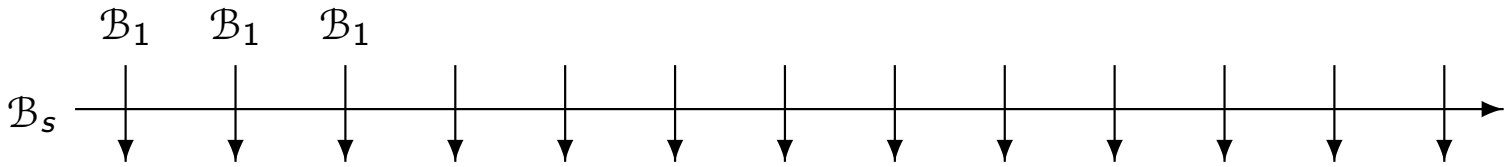


Discrete mKdV equation

Consider an inverse ultra-discretization of B_s for type $A_1^{(1)}$.

$\mathcal{B}_s = \{x = (x_1, x_2) \mid x_1 x_2 = s\} \subset (\mathbb{C}^\times)^2$

Time evolution operator $T_s : (\mathcal{B}_1)^L \to (\mathcal{B}_1)^L$



Geometric R-matrix $R : \mathcal{B}_s \times \mathcal{B}_1 \to \mathcal{B}_1 \times \mathcal{B}_s; (x, y) \mapsto (y', x')$ given by

$$x_1' = x_1 \frac{s y_1 + x_2}{s y_2 + x_1}, \quad y_1' = y_1 \frac{s y_2 + x_1}{s y_1 + x_2}.$$

Setting $M(x) = \begin{pmatrix} x_1^{-1} & -\zeta \\ -1 & x_2^{-1} \end{pmatrix}^{-1}$, we have $M(x)M(y) = M(y')M(x')$.

$$M(x)M(y^{(1)})M(y^{(2)}) \cdots M(y^{(L)}) = M(y'^{(1)})M(y'^{(2)}) \cdots M(y'^{(L)})M(x')$$

If $x = x'$, $\det(M(y^{(1)})M(y^{(2)}) \cdots M(y^{(L)}) - tE)$ is a conserved quantity.

In[1]:=
```
iterate[gamma_, xi_, eta_] := {xi * (eta + gamma / xi) / (1 / eta + gamma * xi),
    eta * (1 / eta + gamma * xi) / (eta + gamma / xi)};

te[gamma_, st_] := Module[{xx = 1, out = {}, aux, i},
  For[i = 1, i ≤ Length[st], i++,
   aux = iterate[gamma, xx, st[[i]]];
   out = Append[out, aux[[2]]];
   xx = aux[[1]]];
  (*Print[xx];*)out]

M[mu_, eta_, z_] := 1 / (mu + Sqrt[z]) * {{mu * eta, z}, {1, mu / eta}}

gamma = 0.8; mu = 0.6; L = 30; ini = {2, 3, 2, 4, 1, 2, 5, 3, 2}; T = 10;
st = Join[ini, Table[1, {L - Length[ini]}]]
For[t = 1, t ≤ T, t++,
 st = te[gamma, st];
 Print[st];
 mat = {{1, 0}, {0, 1}};
 For[i = 1, i ≤ Length[st], i++,
  mat = mat.M[mu, st[[i]], 1]];
 Print[N[CharacteristicPolynomial[mat, t]]]]
```

Out[29]=
```
{2, 3, 2, 4, 1, 2, 5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```

{0.928571, 1.82986, 2.98662, 1.97471, 4.14039, 1.05925, 1.81454,
 4.93005, 3.09235, 2.16416, 1.08364, 1.00896, 1.00099, 1.00011,
 1.00001, 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

$-9.93056\,la + la^2$

{1.00826, 0.870175, 1.65521, 2.95892, 1.94598, 4.26451, 1.13211,
 1.64781, 4.8397, 3.1726, 2.31902, 1.17809, 1.02732, 1.00398, 1.00055,
 1.00007, 1.00001, 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

$-9.93056\,la + la^2$

{0.999086, 1.02384, 0.825285, 1.48136, 2.91414, 1.91368, 4.36999,
 1.21961, 1.4999, 4.7287, 3.2422, 2.4625, 1.28143, 1.05527, 1.00994, 1.00164,
 1.00025, 1.00004, 1.00001, 1., 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

$8.82011 \times 10^{-15} - 9.93056\,la + la^2$

{1.0001, 0.996485, 1.04547, 0.794017, 1.31385, 2.84951, 1.87754, 4.45501,
 1.32276, 1.37046, 4.59712, 3.3023, 2.59365, 1.39133, 1.09274, 1.01974,
 1.00379, 1.00067, 1.00011, 1.00002, 1., 1., 1., 1., 1., 1., 1., 1., 1., 1.}

$8.82011 \times 10^{-15} - 9.93056\,la + la^2$

{0.999989, 1.00049, 0.991629, 1.07155, 0.776276, 1.15781, 2.76267, 1.83708,
 4.5183, 1.44247, 1.25882, 4.44547, 3.35371, 2.71239, 1.50526, 1.13935, 1.03415,
 1.00747, 1.00149, 1.00027, 1.00005, 1.00001, 1., 1., 1., 1., 1., 1., 1., 1.}

$-9.93056\,la + la^2$

{1., 0.999934, 1.00142, 0.984202, 1.10012, 0.771905, 1.01729, 2.65222, 1.79167,
 4.55906, 1.57946, 1.16411, 4.27488, 3.39698, 2.81924, 1.62072, 1.19443,
 1.05377, 1.01318, 1.00291, 1.00059, 1.00011, 1.00002, 1., 1., 1., 1., 1., 1., 1.}

$8.82011 \times 10^{-15} - 9.93056\,la + la^2$

{1., 1.00001, 0.999778, 1.00317, 0.974169, 1.12889, 0.780788, 0.894882, 2.51834,
 1.74046, 4.57686, 1.73409, 1.08534, 4.08713, 3.43238, 2.91505, 1.73544, 1.25702,
 1.07903, 1.02144, 1.00521, 1.00116, 1.00024, 1.00005, 1.00001, 1., 1., 1., 1., 1.}

$8.82011 \times 10^{-15} - 9.93056\,la + la^2$

{1., 0.999999, 1.00003, 0.99943, 1.00603, 0.961783, 1.15532, 0.802907, 0.791644, 2.36341,
 1.68244, 4.57145, 1.90626, 1.02153, 3.88472, 3.45995, 3.00085, 1.84757, 1.3259,
 1.11014, 1.03274, 1.00869, 1.00209, 1.00047, 1.0001, 1.00002, 1., 1., 1., 1.}

$-9.93056\,la + la^2$

{1., 1., 0.999995, 1.0001, 0.998774, 1.01025, 0.947559, 1.17677, 0.838332, 0.707345,
 2.19228, 1.61653, 4.54263, 2.0953, 0.971736, 3.67085, 3.47953, 3.07763, 1.95569,
 1.3997, 1.14711, 1.04754, 1.01367, 1.00356, 1.00085, 1.00019, 1.00004, 1.00001, 1., 1.}

$8.82011 \times 10^{-15} - 9.93056\,la + la^2$

{1., 1., 1., 0.999985, 1.00023, 0.997668, 1.01605, 0.932234, 1.19072, 0.887145, 0.640835,
 2.012, 1.54176, 4.49005, 2.2998, 0.935121, 3.44924, 3.49078, 3.14634, 2.05888,
 1.47695, 1.18974, 1.06621, 1.02051, 1.00573, 1.00146, 1.00035, 1.00008, 1.00002, 1.}

$-9.93056\,la + la^2$

Other types available

Birational map R and the matrix M are available for

- Type $A_n^{(1)}, r = 1$; [Yamada (2001)], [Kajiwara-Noumi-Yamada (2002)]
- Type $D_n^{(1)}, r = 1$; [Kuniba-O-Takagi-Yamada (2003)]
- All nonexceptional types, $r = 1$; [Kashiwara-Nakashima-O (2010)]
- Type $A_n^{(1)}, r$: arbitrary; [Frieden (2017)]

Thank you!