Berlin Mathematical School

TU berlin

Mats Vermeeren
TU Berlin

DGD SFB TRR 109 Discretization in Geometry and Dynamics

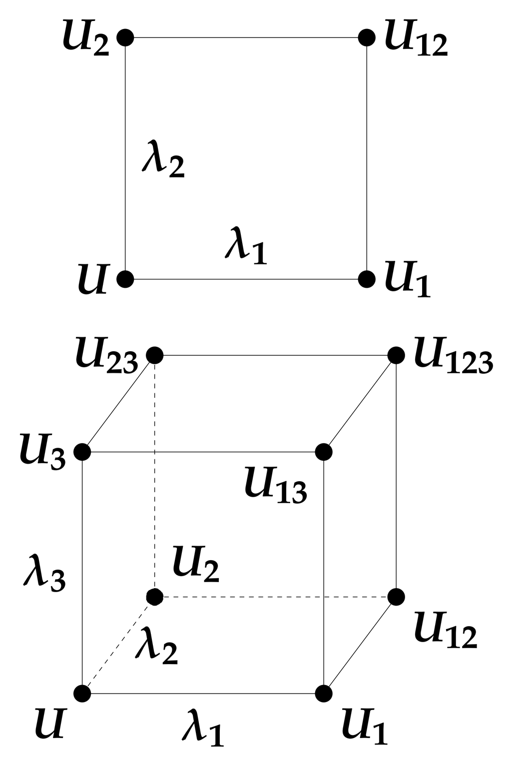
Continuum limits of integrable lattice equations
and their variational structure

Multidimensionally consistent lattice equations

Quad equation: $\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$, invariant under symmetries of the square, affine in each of the entries U, U_1, U_2, U_{12}.

Subscripts of U denote lattice shifts, λ_1 and λ_2 are lattice parameters.

Integrable if consistent around the cube: initial data U, U_1, U_2, U_3 lead to a well-defined value of U_{123}. Then it can be imposed on a higher-dimensional lattice $\mathbb{Z}^N$.



Integrable hierarchies of PDEs

Hierarchy of commuting PDEs:

$$u_{t_2} = f_2(u, u_x, u_{xx}, \ldots),$$
$$u_{t_3} = f_3(u, u_x, u_{xx}, \ldots), \ \ldots$$

A typical example is the potential KdV hierarchy

$$u_{t_3} = 3u_x^2 + u_{xxx},$$
$$u_{t_5} = 10u_x^3 + 5u_{xx}^2 + 10u_x u_{xxx} + u_{xxxxx}, \ \ldots$$

By commutativity, we can consider a single field $u(x, t_2, \ldots, t_N)$ on higher-dimensional multi-time $\mathbb{R}^N$.

Lagrangian structure: discrete

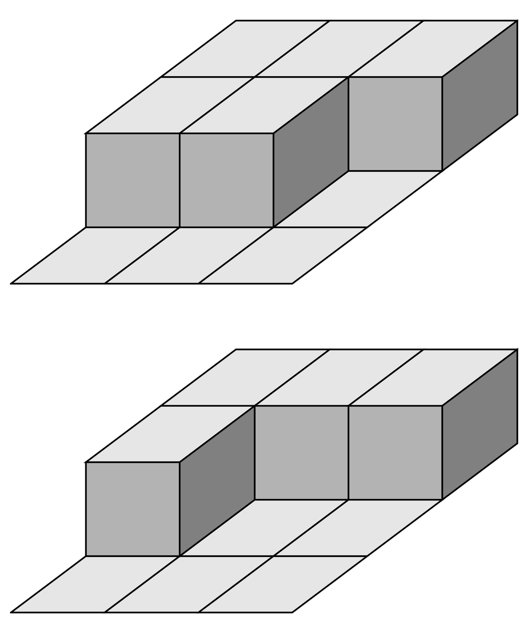
A field $U : \mathbb{Z}^N \to \mathbb{R}$ is a solution to the pluri-Lagrangian (or Lagrangian multiform) problem for

$$L(\square_{ij}) = L(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j)$$

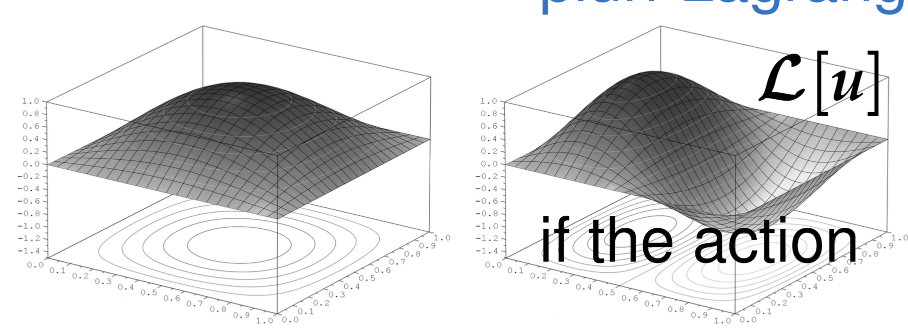
if the action

$$\sum_{\square \in \Gamma} L(\square)$$

is critical on all discrete surfaces Γ in $\mathbb{Z}^N$.



Lagrangian structure: continuous



A field $u : \mathbb{R}^N \to \mathbb{R}$ is a solution to the pluri-Lagrangian problem for the 2-form

$$\mathcal{L}[u] = \sum_{i,j} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

if the action

$$\int_\Gamma \mathcal{L}[u]$$

is critical on all smooth surfaces Γ in $\mathbb{R}^N$.

Embedding the lattice into continuous multi-time: Miwa shifts

Correspondence between discrete and continuous fields:

$$\begin{cases} U = U(\mathbf{n}) & = u(t_1, t_2, \ldots, t_N) \\ U_i = U(\mathbf{n} + \mathfrak{e}_i) & = u\left(t_1 - 2\lambda_i, t_2 + \lambda_i^2, \ldots, t_N + \frac{2(-1)^N}{N} \lambda_i^N\right) \end{cases}$$

[Miwa. On Hirota's difference equations. Proc. Japan Acad. A. 1982]

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$.
Discrete U is a sampling of the continuous u.

We write the quad equation in terms of u, expand in λ_1,λ_2, and equate all coefficients of the power series to 0.
$\hookrightarrow$ Whole hierarchy from single difference equation.

In the leading order, we only see t_1-derivatives of u, but we want to obtain PDEs. $\longrightarrow$ Leading order cancellation required.

Continuum limit of a Lagrangian

Using Miwa shifts, turn discrete $L(\square_{12})$ into power series $\mathcal{L}_{\text{disc}}([u], \lambda_1, \lambda_2)$. The action for $\mathcal{L}_{\text{disc}}$ is still a sum.

With the Euler-Maclaurin formula we turn the action sum into an integral, with integrand

$$\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i! j!} \partial_1^i \partial_2^j \mathcal{L}_{\text{disc}}([u], \lambda_1, \lambda_2),$$

where the differential operators are $\partial_k = \sum_j (-1)^{j+1} \frac{2\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$.

Then

$$L(\square_{12}) = \int_{\blacksquare_{12}} \mathcal{L}_{\text{Miwa}}([u(\mathbf{t})], \lambda_1, \lambda_2) \, \eta_1 \wedge \eta_2,$$

where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

Pluri-Lagrangian 2-form for the continuum limit hierarchy:

$$\mathcal{L} = \sum_{i<j} \mathcal{L}_{\text{Miwa}}([u], \lambda_i, \lambda_j) \, \eta_i \wedge \eta_j = \sum_{i<j} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

where $\mathcal{L}_{\text{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i \lambda_2^j}{i \ j} \mathcal{L}_{ij}[u]$.

Example

Lattice potential KdV equation:

$$\left(\lambda_1^{-1} + \lambda_2^{-1} + U_{1,2} - U\right)\left(\lambda_2^{-1} - \lambda_1^{-1} + U_2 - U_1\right) = \lambda_2^{-2} - \lambda_1^{-2} \xrightarrow{\text{Miwa shifts}} \sum_{i,j \geq 0} f_{i,j}[u] \, \lambda_1^i \lambda_2^j = 0 \xrightarrow{f_{i,j}=0} \begin{cases} u_2 = 0, \\ u_3 = 3u_1^2 + u_{111} \\ u_4 = 0, \\ u_5 = 10u_1^3 + 5u_{11}^2 + 10u_1 u_{111} + u_{11111}, \\ \ldots \end{cases}$$

where $u_i = u_{t_i}$.

Lagrangian:

$$L = \frac{1}{2}\left(U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}\right)\left(U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}\right) + \left(\lambda_i^{-2} - \lambda_j^{-2}\right) \log\left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}}\right).$$

$$\xrightarrow[\text{EM formula}]{\text{Miwa shifts}} \begin{cases} \mathcal{L}_{1,3} = -2u_1^3 - u_1 u_{111} + u_1 u_3 \\ \mathcal{L}_{1,5} = -5u_1^4 + 10u_1 u_{11}^2 - u_{111}^2 + u_1 u_5 \\ \mathcal{L}_{3,5} = 6u_1^5 - 15u_1^2 u_{11}^2 + 20u_1^3 u_{111} - 10u_1^3 u_3 + 7u_{11}^2 u_{111} + 6u_1 u_{111}^2 - 12u_1 u_{11} u_{1111} \\ \quad + 3u_1^2 u_{11111} + 20u_1 u_{11} u_{13} - 5u_{11}^2 u_3 - 10u_1 u_{111} u_3 + 3u_1^2 u_5 - u_{1111}^2 \\ \quad + u_{111} u_{11111} - 2u_{111} u_{113} + u_1 u_{115} + 2u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + u_{111} u_5 \end{cases}$$

V. Continuum limits of pluri-Lagrangian systems. arXiv:1706.06830 • V. Continuum Limits of Variational Sytems. PhD Thesis, 2018.
V. A variational perspective on continuum limits of ABS and lattice GD equations. arXiv:1811.01855 • http://page.math.tu-berlin.de/~vermeer/