Hidden solutions of discrete systems

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@monsoon0

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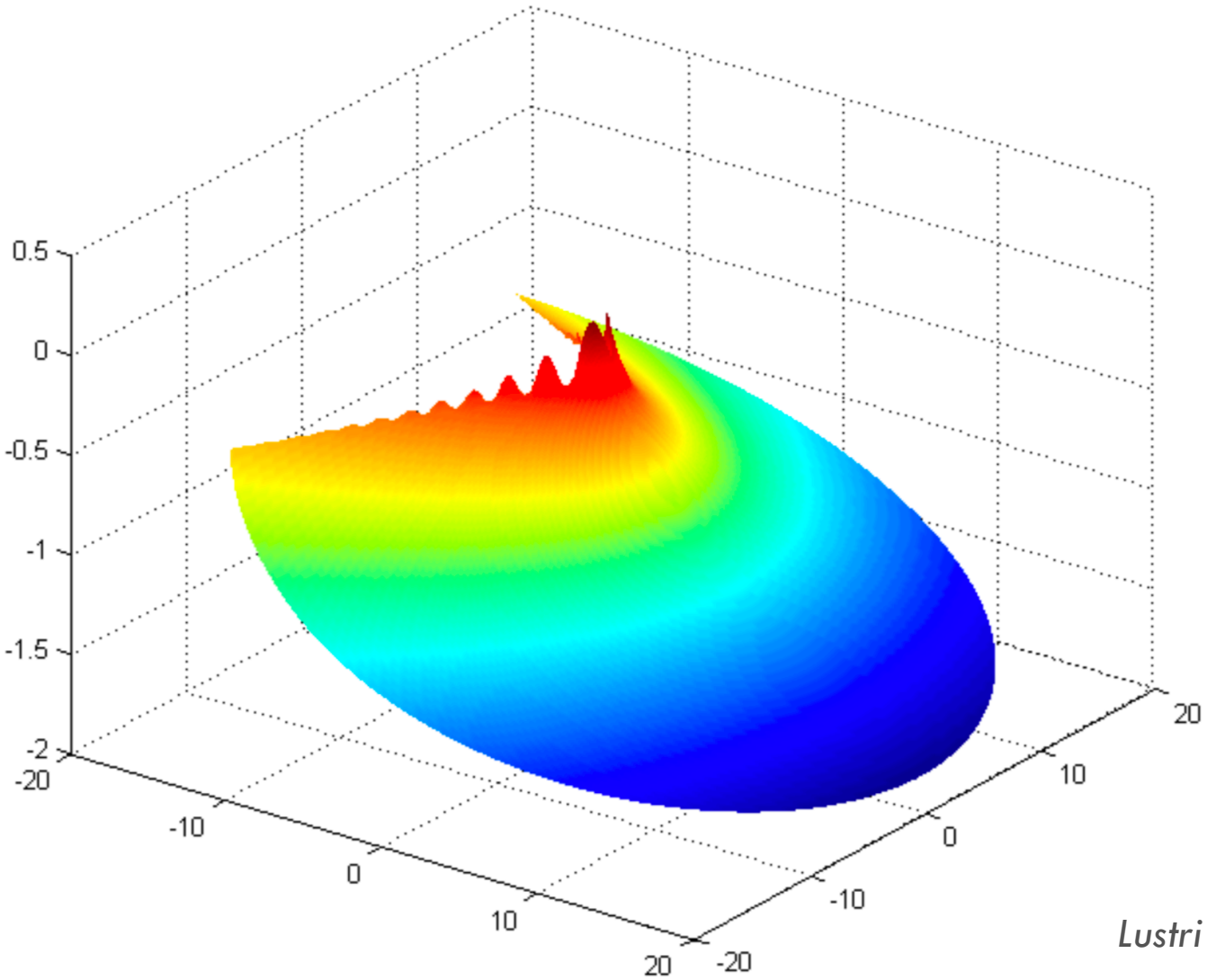

Lustri 2015

Real part of tritronquée solution of first Painlevé equation

$$y_{xx} = 6y^2 - x$$

Properties on $\mathbb{R}$

$$y \sim -\sqrt{\frac{x}{6}} \sum_{k=0}^{\infty} \frac{a_k}{x^{5k/2}}, \quad x \to \infty$$

- A one-parameter family of solution exists with this behaviour on the positive real line, real for $x \in \mathbb{R}$
- One unique solution is bounded, analytic and monotonically decaying for $x \in \mathbb{R}^+$

Joshi & Kitaev 2001

Coefficients of asymptotic series

$$a_k = c\left(\frac{25}{8\sqrt{6}}\right)^k ((k-1)!)\,(1+o(1))\,, \quad k \gg 1$$

- The asymptotic series diverges.
- It provides information about boundaries of validity.
- Late terms have the form

$$a_k \sim \frac{\alpha\Gamma(2k+\gamma)}{\chi^{2k+\gamma}}$$

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Singulant

In $\mathbb{C}$

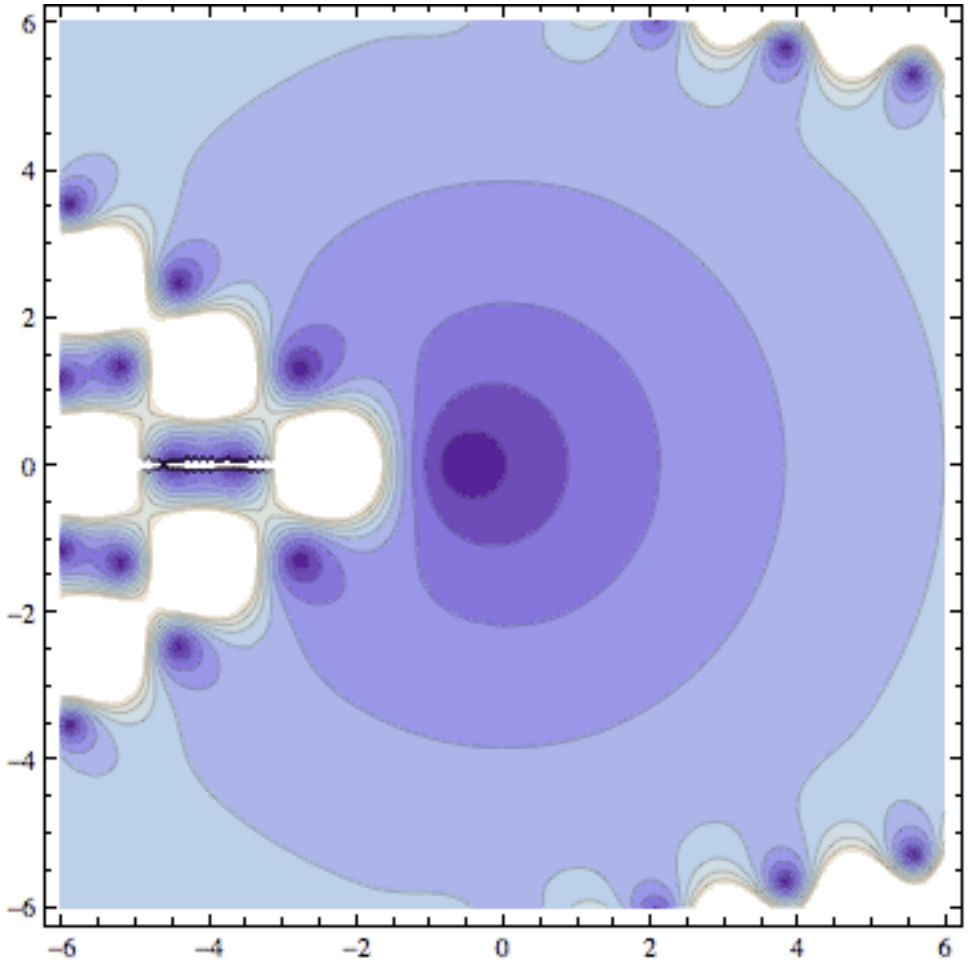


Image by H. Dullin

- The unique solution extends to $|\arg(x)| \leq 4\pi/5$
- But the behaviour is completely different for $x \in \mathbb{R}^-$

Do such solutions exist for discrete systems?

The answer is yes*.

* dPI *Joshi & Lustri 2015*
* dPII *Joshi, Lustri & Luu 2017*
* a-dPI *Joshi & Takei 2017*
* qPI *Joshi, Lustri & Luu 2018*

q-discrete first Painlevé equation

$$\overline{w}\,\underline{w} = \frac{1}{w} - \frac{1}{xw^2}$$

$$\overline{w} = w(qx),\ w = w(x),\ \ \underline{w} = w(x/q)$$

- For each $x \in \mathbb{C}$ the compactified & regularised initial value space is a foliated vector bundle with each fibre equivalent to $A_7^{(1)}$
- But the foliation fails in in the limit $x \to \infty$ when two base points merge.

Solutions?

- Nishioka (2010) proved that qPI has no solution expressible in terms of solutions of first-order difference equations.
- So it has no rational solutions or q-hypergeometric-type solutions.
- The only explicit information about solutions we have is their asymptotic behaviours.

Asymptotic behaviours

- qPI has an unstable (*quicksilver**) solution

$$w \sim 1/x, \ |x| \to \infty$$

- Its asymptotic expansion is divergent, hiding exponentially small terms with free coefficients.

**Joshi 2015*

How does it behave in $\mathbb{C}$?

Rescaling

$$x = x_0 q^n, s = \epsilon\, n, q = 1 + \epsilon, w(x) = W(s)$$

$$W(s+\epsilon)W(s)^2W(s-\epsilon) = W(s) - \frac{1}{x_0(1+\epsilon)^{s/\epsilon}}$$

To leading order as $\epsilon \to 0$ we have

$$W(s) \sim W_0(s)$$

$$W_0^4 = W_0 - e^{-s}$$

Asymptotic expansion

$$W(s) \sim \sum_{r=0}^{\infty} \epsilon^{2r} W_r(s) + \sum_{r=1}^{\infty} \epsilon^{2r+1} V_r(s), \epsilon \to 0$$

For simplicity, we focus on one of the 4 behaviours given by

$$W_0^4 = W_0 - e^{-s}$$

J., Luu and Lustri, arXiv:1807.00450 [math-ph] 2018

Late order terms

$$\begin{cases} W_r(s) & \sim \dfrac{F(s)\Gamma(2r+\gamma_1)}{\xi(s)^{2r+\gamma_1}}, \\ V_r(s) & \sim \dfrac{G(s)\Gamma(2r+\gamma_2)}{\mu(s)^{2r+\gamma_2}} \end{cases} \qquad r \gg 1$$

It can be shown that the even coefficients dominate the odd ones in the limit.

The Singulant

Matching terms of order $W_r(s)$ gives

$$\cosh(-\xi_s) = \frac{1-2W_0^3}{2W_0^3}$$

which implies

$$\xi^{\pm}(s) = \pm \int_{s_0}^{s} \cosh^{-1}\big(\sigma(t)\big)\,dt$$

with

$$\sigma(s) = \frac{1-2W_0^2}{2W_0^3}$$

The Prefactor

Matching terms of order $W_{r-1/2}(s)$ gives

$$F(s) = \frac{\Upsilon W_0(s)}{\left(\sigma(s)^2 - 1\right)^{1/4}}$$

and matching near singularity gives $\gamma_1 = -1/2$.

Optimal truncation

For each s, there is a smallest term in the series, occurring at

$$N \sim \frac{1}{2}\left(|\xi|/\epsilon + \kappa\right), \;\; \kappa \in [0, 1)$$

We truncate at this term and write

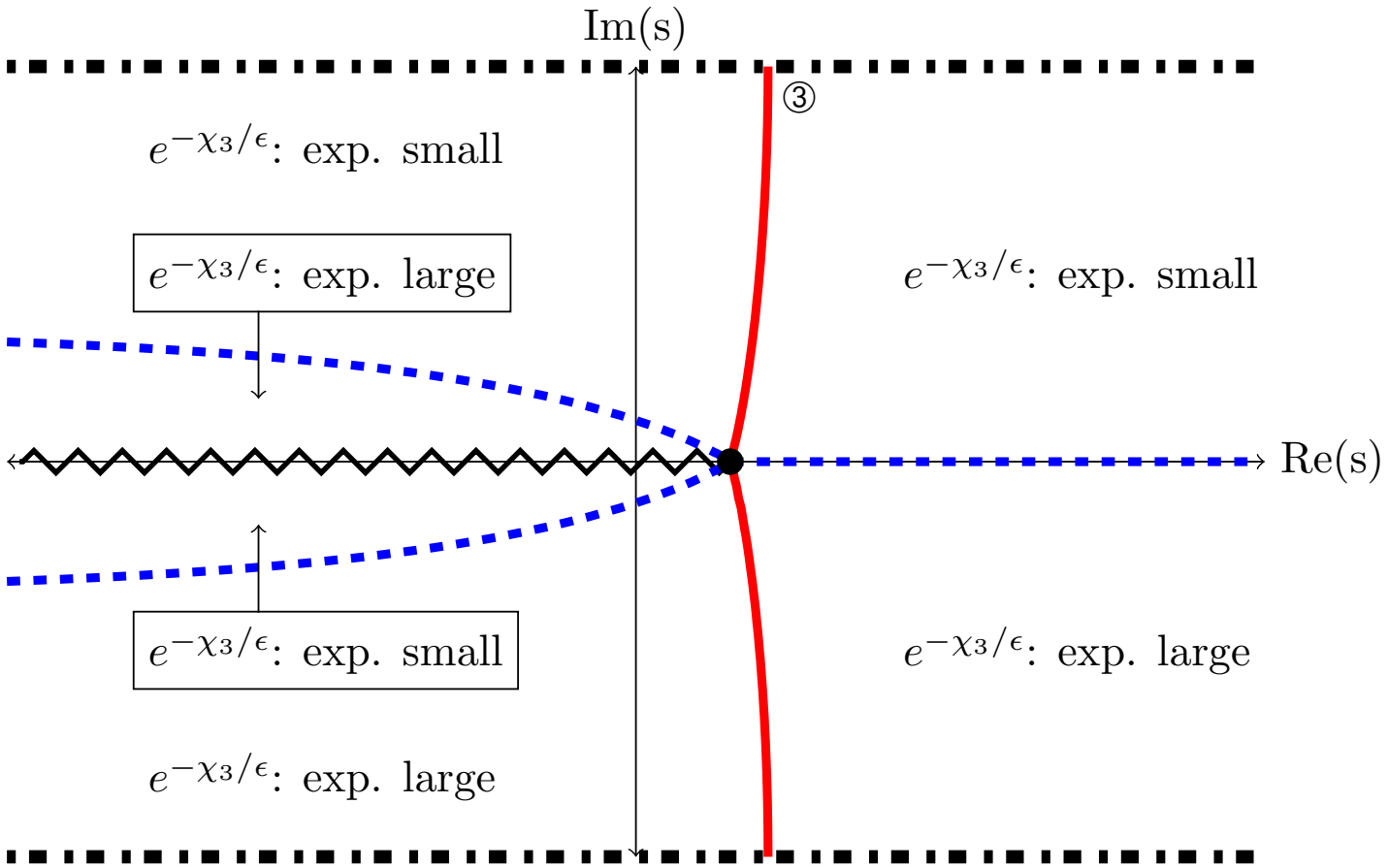
$$W(s) = \sum_{r=0}^{2N-1} \epsilon^r W_r(s) + R_N(s)$$

and matching near singularity gives .

$$R_N(s) \sim \mathcal{S}\phi(s)F(s)e^{-\xi/\epsilon}$$

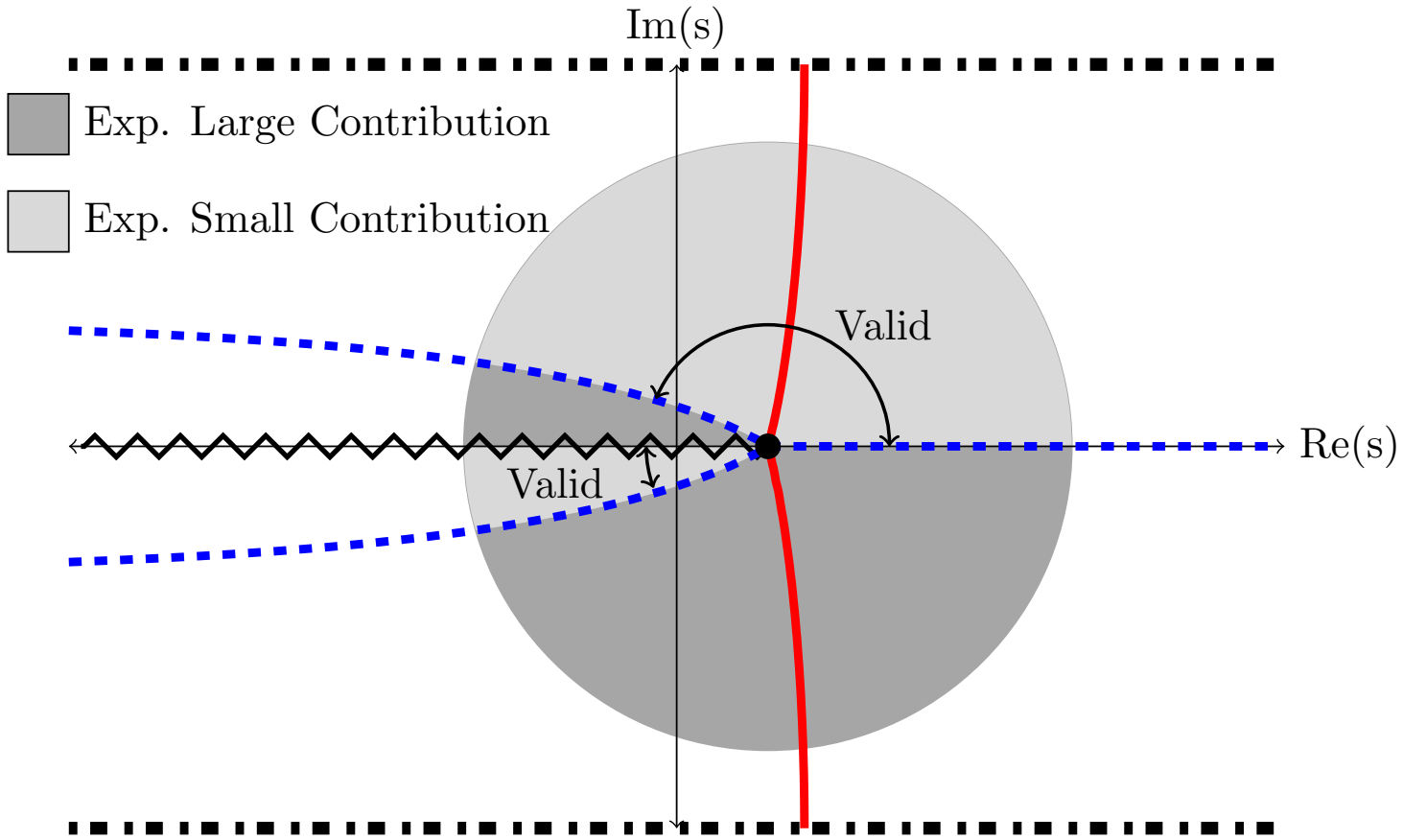
Stokes multiplier

$$\mathcal{D}_0 = \{s \in \mathbb{C} \mid \Im(s) \in (-\pi, \pi]\}, \quad F(s) = \chi_3(s)$$



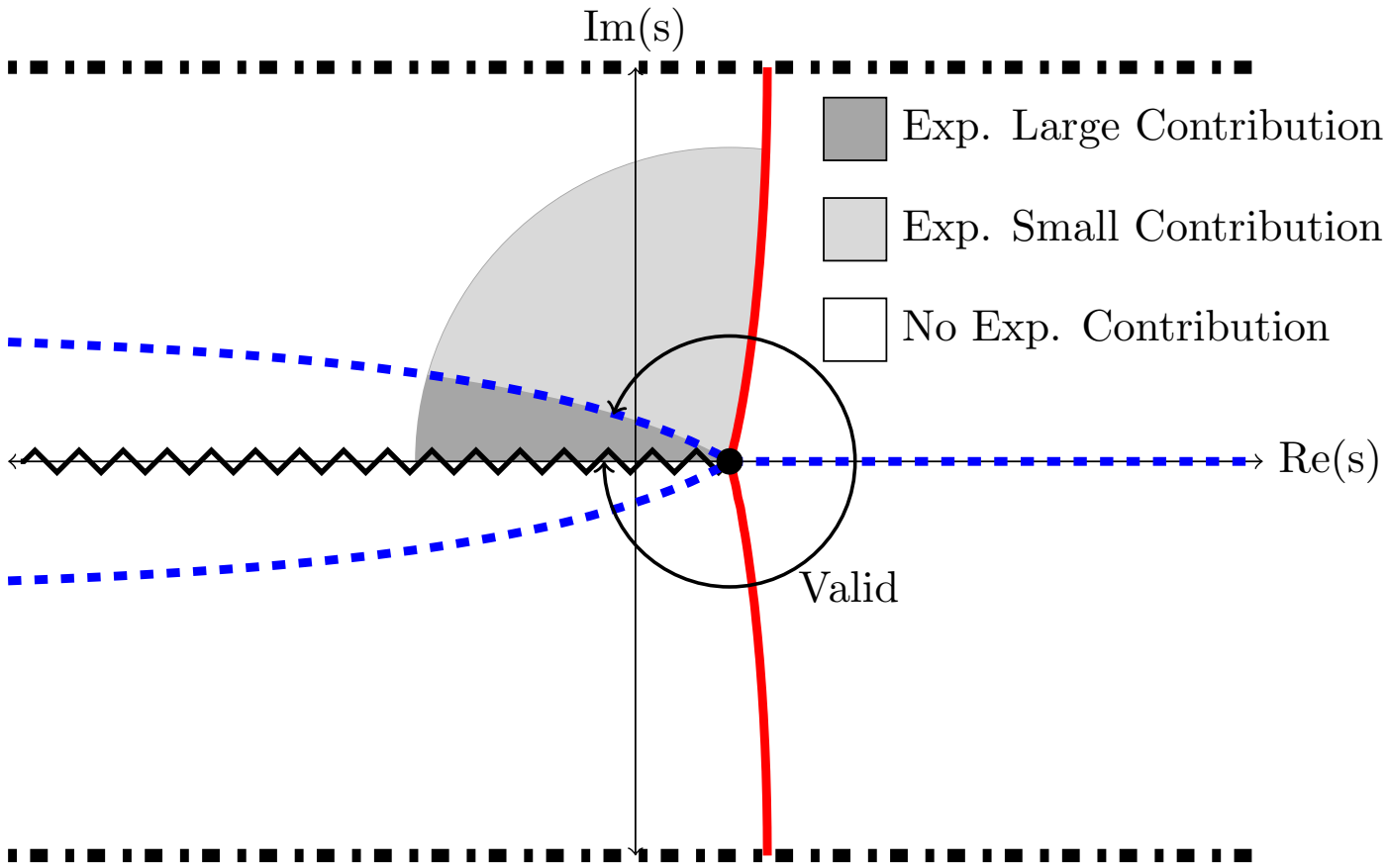
Stokes curve anti-Stokes curve

Region of validity



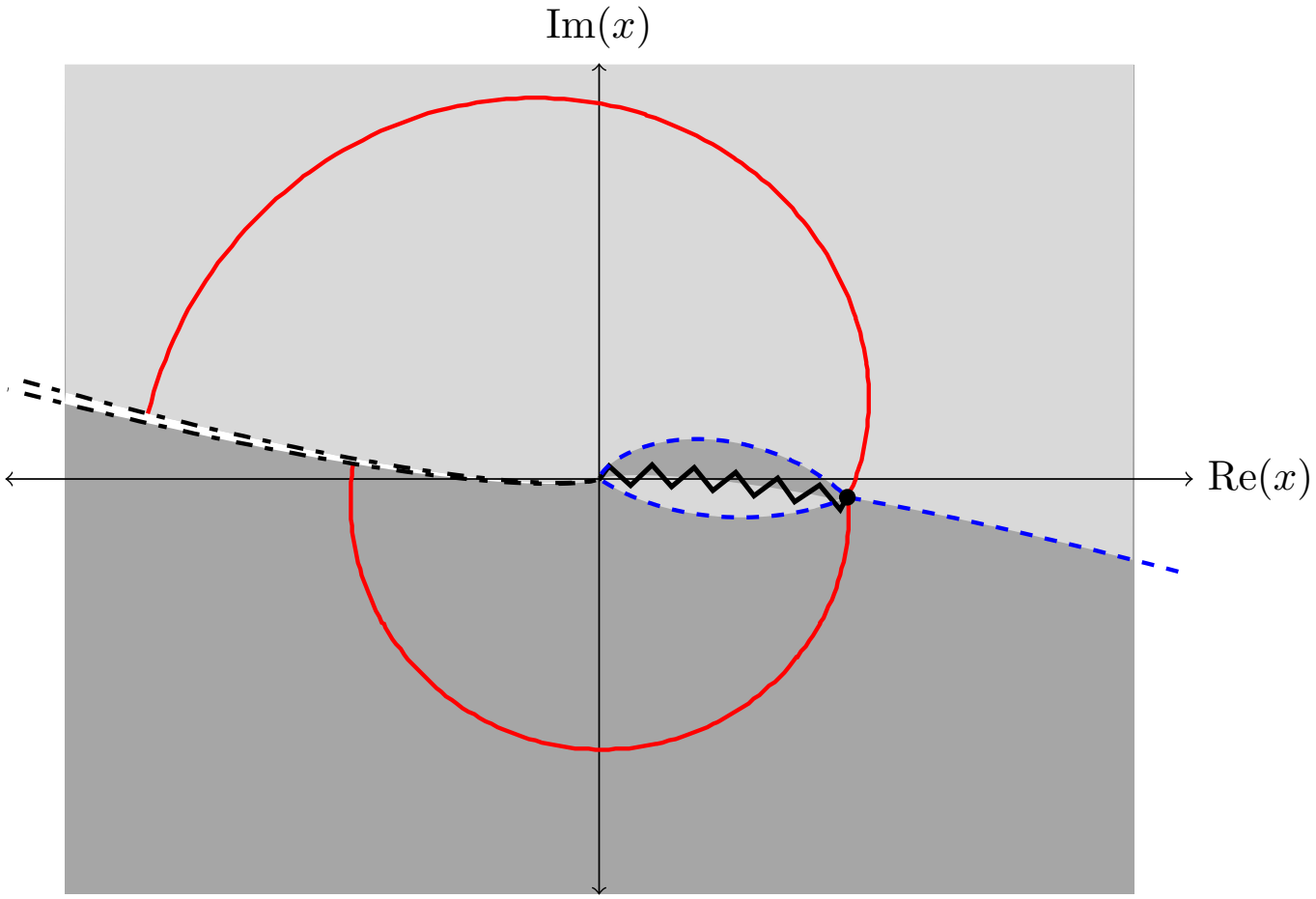
For a generic value of S.

Extended region of validity



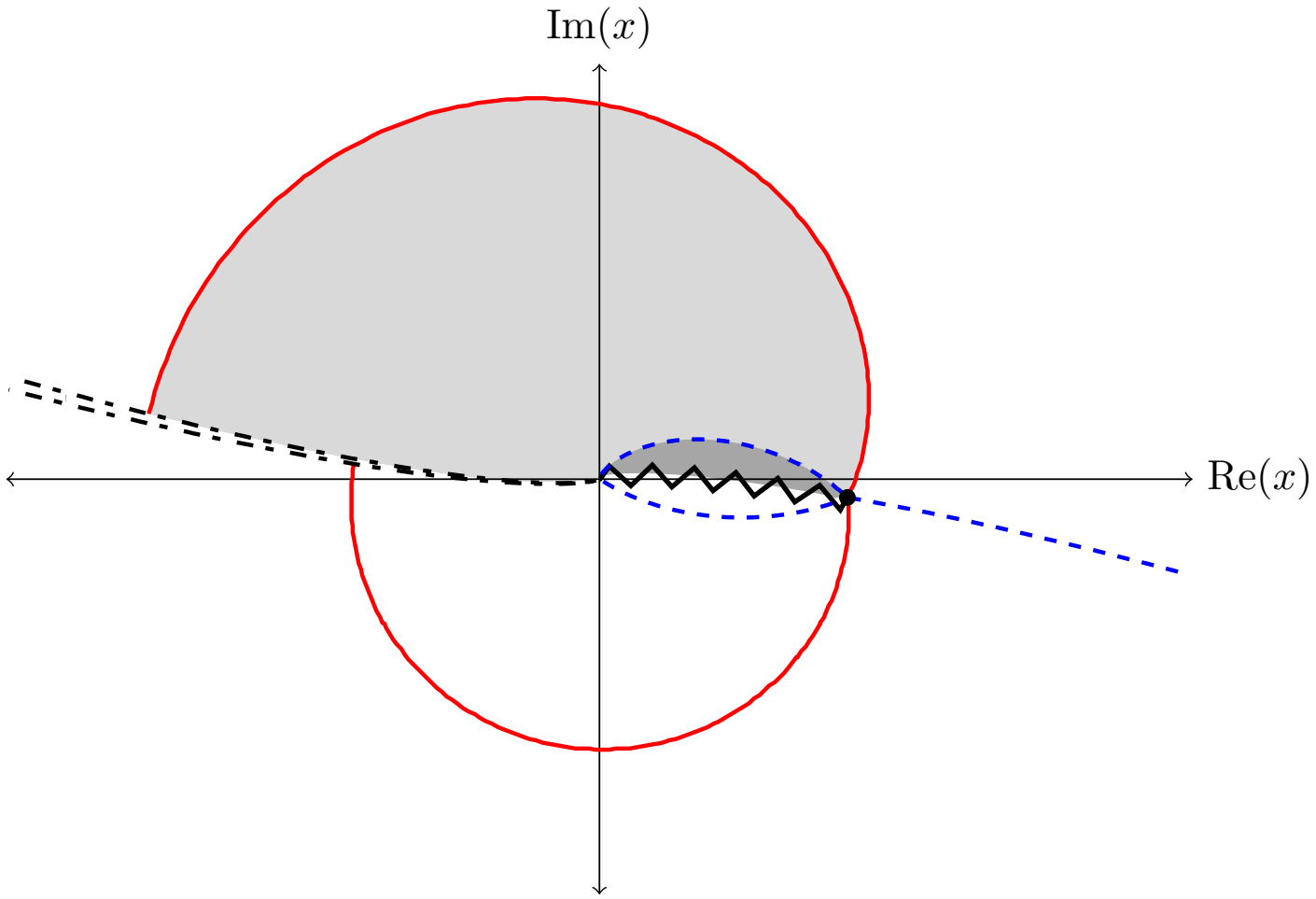
For a special value of $S=0$.

Region of validity in *x*



For a generic value of *S*

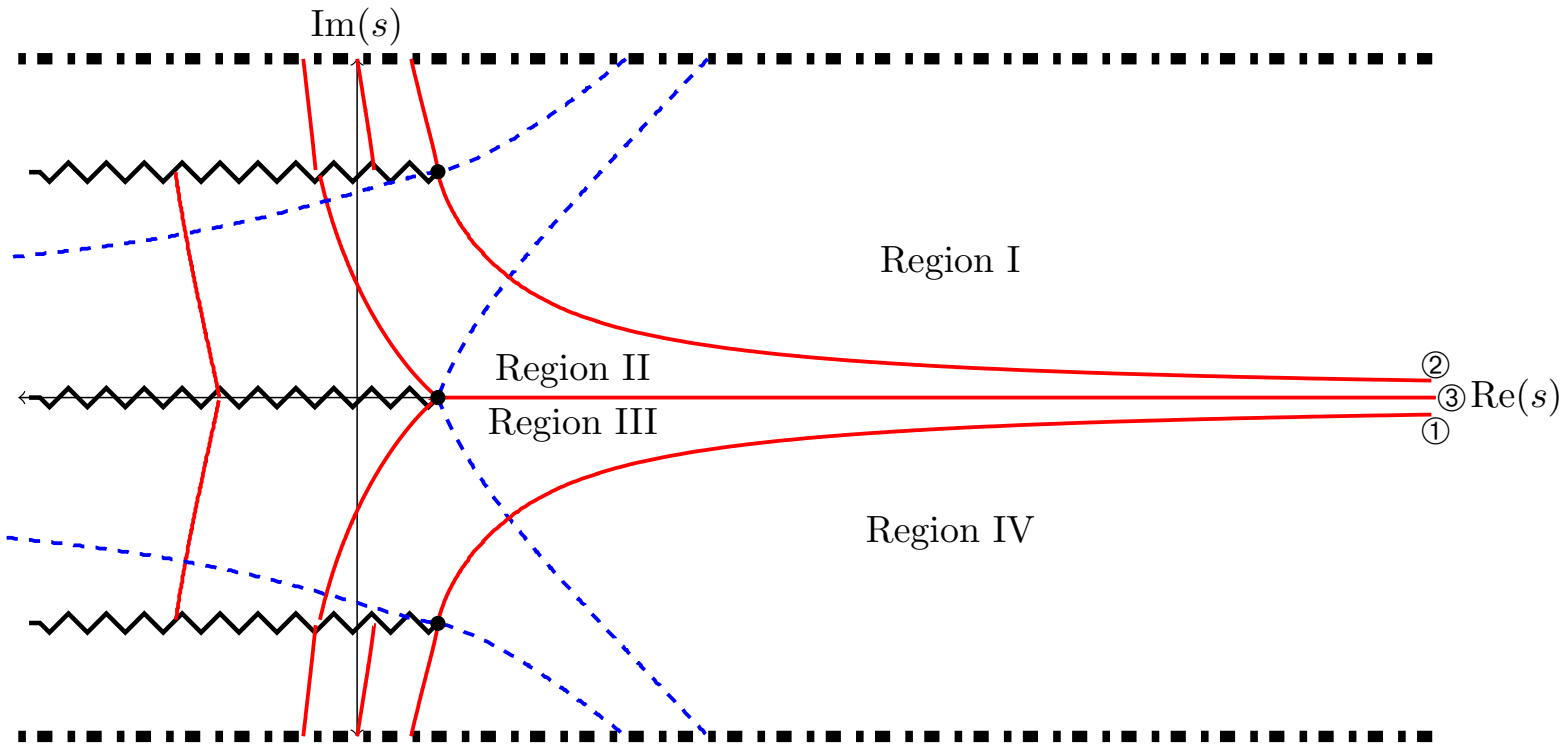
Extended region of validity in x



For a special value of $S=0$.

Type B solutions

- There also exists another type of solution, which matches with the quicksilver solution.
- Its Stokes structure (indicated below) is more complicated.



Summary

- Asymptotic behaviours suggest there exist "hidden" solutions of discrete systems.
- Their asymptotic expansions are divergent.
- Estimates of late-order terms show Stokes phenomena in $\mathbb{C}$.
- The results suggest the existence of unique solutions with interesting properties.
- These are analogous to *tritronquée* solutions of the Painlevé equations.