Positiveness of the soliton solution of the discrete BKP equation

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1 Introduction and Summary

Purpose

Our purpose is ultradiscretization of the discrete BKP equation and to obtain the soliton solution of it.

Difficulty

- The ultra discretization of the discrete BKP equation becomes implicite if we assume the parameters arrising in the discrete BKP equation are real.
- The sign of the value of the soliton solution may change.

Conclusion

We simplify the coefficients of the discrete BKP equation to avoid the impliciteness and obtain the soliton solution of it. The soliton solution can be expressed in two different forms, and using these expressions, we can obtain the condition for the value of the soliton solution to be always positive.

2 discrete BKP equation

$$\begin{aligned}&Z_1 f(k_1+1,k_2,k_3)f(k_1,k_2+1,k_3+1)\\&+Z_2 f(k_1,k_2+1,k_3)f(k_1,k_2+1,k_3+1)\\&+Z_3 f(k_1,k_2,k_3+1)f(k_1+1,k_2+1,k_3)\\&-Z_4 f(k_1,k_2,k_3)f(k_1+1,k_2+1,k_3+1)=0 \qquad (1)\end{aligned}$$

- Relation to the usual form of the discrete BKP equation is as follows.

$$\begin{aligned}Z_1&=(z_1+z_2)(z_2-z_3)(z_3+z_1),\\Z_2&=(z_1+z_2)(z_2+z_3)(z_3-z_1),\\Z_3&=(z_1-z_2)(z_2+z_3)(z_3+z_1),\\-Z_4&=(z_1-z_2)(z_2-z_3)(z_3-z_1)\end{aligned}$$

  Two of the coefficients become positive and other two become negative, under the condition that all of the parameters z_i are real.

- We can obtain the explicite max plus equation assuming the coefficients Z_i are positive, and taking the ultradiscrete limit.

3 Pfaffian expression

The elements of the Pfaffian

$$< d_0\ i >= \sum_{l_i=1}^{2} c_i^{(l_i)}\phi\left(t_i^{(l_i)}\right),$$

$$< i\ j >= \sum_{l_i=1}^{2}\sum_{l_j=1}^{2} c_i^{(l_i)}c_j^{(l_j)} cf\left(t_i^{(l_i)},\ t_j^{(l_j)}\right)\phi\left(t_i^{(l_i)}\right)\phi\left(t_j^{(l_j)}\right)$$

Here,

$t_i^{(1)},\ t_i^{(2)}$: free parameters corresponding to the wave number of the i-th soliton.

$$\phi(t)=\prod_{i=1}^{3} A_i(t)^{k_i},$$

$$A_1(t)=\frac{Z_2Z_4+Z_3t}{Z_1-t},\ A_2(t)=\frac{Z_1Z_4-Z_3t}{Z_2+t},\ A_3(t)=t,$$

$$cf(t,\ s)=\frac{t-s}{tsZ_3+Z_1Z_2Z_4}$$

: coefficient that represent the collision of solitons.

↓

2N soliton

$$f =< 1\ 2\ \cdots\ 2N > \qquad (2)$$

2N+1 soliton

$$f =< d_0\ 1\ 2\ \cdots\ 2N+1 > \qquad (3)$$

Using Pfaffian identities, we can show that this function actually satisfies the discrete BKP equation (1).

4 Expanded form

Using the definition of the Pfaffian and the identity of the coefficient $cf(t,\ s)$,

$$\begin{aligned}&Z_1Z_2Z_3Z_4 cf(t_1,\ t_2)cf(t_1,\ t_3)cf(t_2,\ t_3)\\&\quad=-cf(t_1,\ t_2)+cf(t_1,\ t_3)-cf(t_2,\ t_3)\end{aligned}$$

, we can derive the expanded form of the soliton solution from the Pfaffian expression (2) and (3).

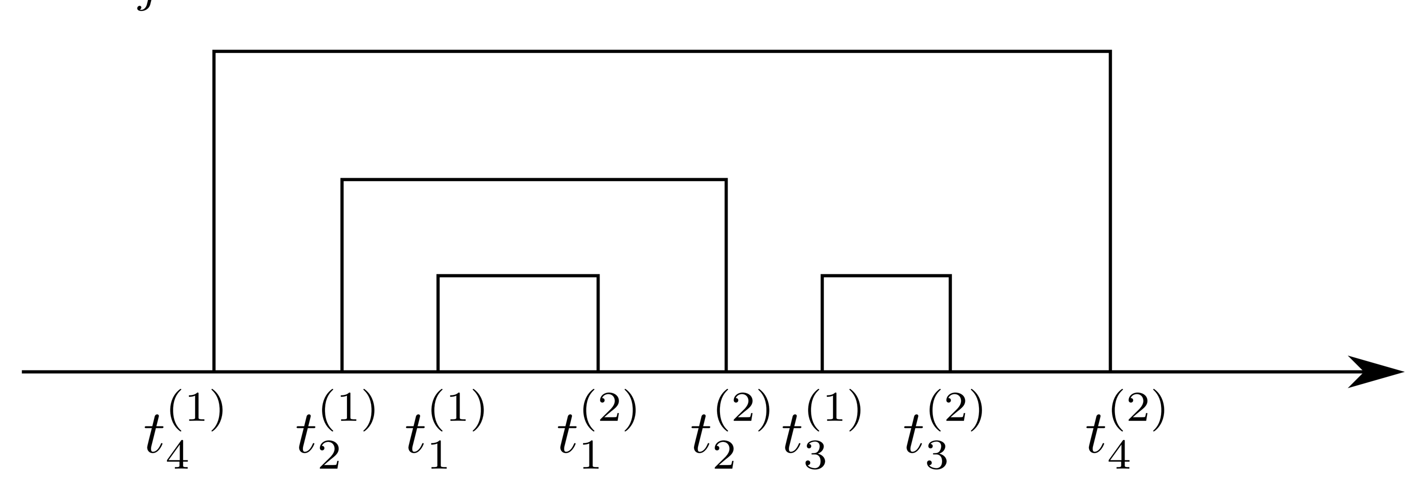
$$(1\ 2\ \cdots N)=\sum_{k_l=1}^{2}\cdots\sum_{k_N=1}^{2}\prod_{1\le i<j\le N} cf(t_i^{(k_i)},\ t_j^{(k_j)}) \times\prod_{i=1}^{N} c_i^{(k_i)}\phi(t_i^{(k_i)}) \qquad (4)$$

5 The condition of the positiveness

Since $\prod cf(t_i,t_j)$ in the expression (4) is a product of the coefficients $cf(t,\ s)$ which are antisymmetric with respect to t and s, we can easily decide the sign of each terms. We can show that there are combinations of $c_i^{(k_i)}$ for which all terms in the expression (4) are positive, if the parameters $t_i^{(k_i)}$ satisfy the following conditions.

(1) Both $t_j^{(1)}$ and $t_j^{(2)}$ take values between $t_i^{(1)},\ t_i^{(2)}$.

(2) Neither of $t_i^{(1)}, t_i^{(2)}$ does not take value between $t_j^{(1)}$ and $t_j^{(2)}$.



Reference

[1] N. Shinzawa, Japan J. Indust. Appl. Math.(Vol. 35, Issue 2, 915-937)