An elliptic Painlevé equation from next-nearest-neighbor translation on the $E_8^{(1)}$ lattice

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Abstract

Elliptic Painlevé equations head the list of the differential and discrete Painlevé equations. The well known elliptic Painlevé equation is given by a nearest neighbor vector on the $E_8^{(1)}$ weight lattice.

In this poster, we show that the elliptic Painlevé equation found by Ramani, Carstea and Grammaticos in 2009 can be obtained from a half-translation of the affine Weyl group of type $E_8^{(1)}$. Moreover, we show that the generic version of the RCG equation is the elliptic Painlevé equation given by a nearest neighbor vector on the $E_8^{(1)}$ weight lattice.

Key Reference

N. Joshi and N. Nakazono, Elliptic Painlevé equations from next-nearest-neighbor translations on the $E_8^{(1)}$ lattice, *J. Phys. A*, (2017)

Introduction

[Painlevé equation] In the early 20th-century, in order to find a new class of special functions, Painlevé and Gambier classified all rational ordinary differential equations of second order of the form $y''=F(y',y,t)$, where $y=y(t)$ and $'=d/dt$, with the Painlevé property (solutions do not have movable branch points). As a result, they obtained six new equations (Painlevé equations).

[Discrete Painlevé equation] Discrete Painlevé equations are nonlinear ordinary difference equations of second order, which can be reduced to the Painlevé equations through appropriate limiting processes. Originally, discrete Painlevé equations appeared in the model of 2D quantum gravity and the theory of orthogonal polynomials [3]. In 1991, Grammaticos *et al.* introduced the singularity confinement criterion as the discrete version of the Painlevé property [2]. Since then, many kinds of discrete Painlevé systems were found. In 2001, Sakai showed the classification (definition) of discrete Painlevé equations by space of initial values [5]. Discrete Painlevé equations are characterized by their space of initial values constructed by the blow up of $\mathbb{P}^2$ at nine base points (i.e. points where the system is ill defined because it approaches $0/0$). They are classified into 19 types according to the configuration of the base points.

Discrete type	Type of surface
Elliptic	$A_0^{(1)}$
Multiplicative	$A_0^{(1)*}$, $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, …, $A_7^{(1)}$, $A_7^{(1)'}$
Additive	$A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$, $D_4^{(1)}$, …, $D_7^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$

[RCG equation] Ramani-Carstea-Grammaticos obtained the following ordinary difference equation (RCG equation) [4]:

$$\tilde{y}=\frac{(1-k^2\mathrm{sz}^4)\mathrm{c}_e\mathrm{d}_e\,xy-(\mathrm{c}_e^2-\mathrm{cz}^2)\mathrm{cz}\,\mathrm{dz}-(1-k^2\mathrm{s}_e^2\mathrm{sz}^2)\mathrm{cz}\,\mathrm{dz}\,x^2}{k^2(\mathrm{c}_e^2-\mathrm{cz}^2)\mathrm{cz}\,\mathrm{dz}\,x^2y-(1-k^2\mathrm{sz}^4)\mathrm{c}_e\mathrm{d}_e\,x+(1-k^2\mathrm{s}_e^2\mathrm{sz}^2)\mathrm{cz}\,\mathrm{dz}\,y},$$

$$\tilde{x}=\frac{(1-k^2\widehat{\mathrm{sz}}^4)\mathrm{c}_o\mathrm{d}_o\,\tilde{y}x-(\mathrm{c}_o^2-\widehat{\mathrm{cz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}-(1-k^2\mathrm{s}_o^2\widehat{\mathrm{sz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}\,\tilde{y}^2}{k^2(\mathrm{c}_o^2-\widehat{\mathrm{cz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}\,\tilde{y}^2x-(1-k^2\widehat{\mathrm{sz}}^4)\mathrm{c}_o\mathrm{d}_o\,\tilde{y}+(1-k^2\mathrm{s}_o^2\widehat{\mathrm{sz}}^2)\widehat{\mathrm{cz}}\,\widehat{\mathrm{dz}}\,x},$$

from the partial difference equation called Lattice Krichever-Novikov (KN) system. Here,

$$\mathrm{sz}=\mathrm{sn}\,(z),\quad \widehat{\mathrm{sz}}=\mathrm{sn}\,(z+\gamma),\quad \mathrm{s}_e=\mathrm{sn}\,(\gamma_\mathrm{e}),\quad \mathrm{s}_o=\mathrm{sn}\,(\gamma_\mathrm{o}),\quad \mathrm{cz}=\mathrm{cn}\,(z),\quad \widehat{\mathrm{cz}}=\mathrm{cn}\,(z+\gamma),$$
$$\mathrm{c}_e=\mathrm{cn}\,(\gamma_\mathrm{e}),\quad \mathrm{c}_o=\mathrm{cn}\,(\gamma_\mathrm{o}),\quad \mathrm{dz}=\mathrm{dn}\,(z),\quad \widehat{\mathrm{dz}}=\mathrm{dn}\,(z+\gamma),\quad \mathrm{d}_e=\mathrm{dn}\,(\gamma_\mathrm{e}),\quad \mathrm{d}_o=\mathrm{dn}\,(\gamma_\mathrm{o}),$$
$$y=y(z),\quad x=x(z),\quad \gamma=\gamma_\mathrm{e}+\gamma_\mathrm{o},\quad \tilde{}:z\mapsto z+2\gamma,$$

sn, cn, dn are the Jacobian elliptic functions, and k is the modulus.

[Geometry of the RCG equation] In [1], the space of initial values of the RCG equation was investigated. The eight base points on $\mathbb{P}^1\times\mathbb{P}^1$ are given by

$$\begin{aligned}
p_1&:(x,y)=\big(\mathrm{cd}\,(\gamma_\mathrm{o}+2K+\mathrm{i}K'),\mathrm{cd}\,(z_0-\gamma_\mathrm{e}-\gamma_\mathrm{o}+2K+\mathrm{i}K')\big),\\
p_2&:(x,y)=\big(\mathrm{cd}\,(\gamma_\mathrm{o}+\mathrm{i}K'),\mathrm{cd}\,(z_0-\gamma_\mathrm{e}-\gamma_\mathrm{o}+\mathrm{i}K')\big),\\
p_3&:(x,y)=\big(\mathrm{cd}\,(\gamma_\mathrm{o}+2K),\mathrm{cd}\,(z_0-\gamma_\mathrm{e}-\gamma_\mathrm{o}+2K)\big),\\
p_4&:(x,y)=\big(\mathrm{cd}\,(\gamma_\mathrm{o}),\mathrm{cd}\,(z_0-\gamma_\mathrm{e}-\gamma_\mathrm{o})\big),\\
p_5&:(x,y)=\big(\mathrm{cd}\,(z_0+2K+\mathrm{i}K'),\mathrm{cd}\,(\gamma_\mathrm{e}+2K+\mathrm{i}K')\big),\\
p_6&:(x,y)=\big(\mathrm{cd}\,(z_0+\mathrm{i}K'),\mathrm{cd}\,(\gamma_\mathrm{e}+\mathrm{i}K')\big),\\
p_7&:(x,y)=\big(\mathrm{cd}\,(z_0+2K),\mathrm{cd}\,(\gamma_\mathrm{e}+2K)\big),\\
p_8&:(x,y)=\big(\mathrm{cd}\,(z_0),\mathrm{cd}\,(\gamma_\mathrm{e})\big),
\end{aligned}$$

where $\mathrm{cd}=\mathrm{cn}/\mathrm{dn}$ and $K=K(k)$, $K'=K'(k)$ are complete elliptic integrals. They lie on the following bi-degree $(2,2)$-curve:

$$x^2+y^2=\mathrm{sn}\,(z_0-\gamma_\mathrm{e})^2(1+k^2x^2y^2)+2\mathrm{cn}\,(z_0-\gamma_\mathrm{e})\,\mathrm{dn}\,(z_0-\gamma_\mathrm{e})\,xy.$$

Aim of this work

Although the geometry of the RCG equation has been clarified, its realization from the birational action of the affine Weyl group was missing since its base points are parametrized by the Jacobian elliptic function, and birational actions of the affine Weyl group on such setting were not explicitly known.

The present study fills this gap, that is, our main result provides the realization of the RCG equation as a half-translation of the affine Weyl group of type $E_8^{(1)}$. Moreover, we show that the generic version of the RCG equation is the elliptic Painlevé equation given by a nearest neighbor vector on the $E_8^{(1)}$ weight lattice.

1 Generalized base points

Base points: $p_i:(x,y)=\big(\mathrm{cd}\,(c_i+\eta),\mathrm{cd}\,(\eta-c_i)\big),\quad i=1,\dots,8$

$(2,2)$-curve: $x^2+y^2=\mathrm{sn}\,(2\eta)^2(1+k^2x^2y^2)+2\mathrm{cn}\,(2\eta)\,\mathrm{dn}\,(2\eta)\,xy$

Specialization $\Downarrow$

$$\begin{array}{lll}
c_2=c_1+2K, & c_3=c_1+\mathrm{i}K', & c_4=c_1+2K+\mathrm{i}K'\\
c_6=c_5+2K, & c_7=c_5+\mathrm{i}K', & c_8=c_5+2K+\mathrm{i}K'\\
z_0=\eta+c_5+2K+\mathrm{i}K', & \gamma_\mathrm{e}=c_5-\eta+2K+\mathrm{i}K', & \gamma_\mathrm{o}=\eta+c_1+2K+\mathrm{i}K'
\end{array}$$

The base points and the $(2,2)$-curve for the RCG equation.

2 Birational action of the affine Weyl group

Using the geometric approach investigated by Sakai [5], we obtain the following birational action of $W\big(E_8^{(1)}\big)=\langle s_0,\dots,s_8\rangle$ on the coordinates (x,y) and parameters c_i, $i=1,\dots,8$, and η:

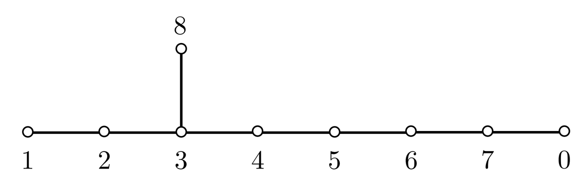
$$s_1(x)=y,\quad s_1(y)=x,$$
$$\left(\frac{s_2(y)-\mathrm{cd}\left(2\eta-\frac{c_1-c_2}{2}\right)}{s_2(y)-\mathrm{cd}\left(2\eta+\frac{c_1-c_2}{2}\right)}\right)\left(\frac{x-\mathrm{cd}\,(\eta+c_1)}{x-\mathrm{cd}\,(\eta+c_2)}\right)\left(\frac{y-\mathrm{cd}\,(\eta-c_2)}{y-\mathrm{cd}\,(\eta-c_1)}\right)$$
$$=\left(\frac{1-\frac{\mathrm{cd}(\eta-c_2)}{\mathrm{cd}(\eta)}}{1-\frac{\mathrm{cd}(\eta-c_1)}{\mathrm{cd}(\eta)}}\right)\left(\frac{1-\frac{\mathrm{cd}(\eta+c_1)}{\mathrm{cd}(\eta)}}{1-\frac{\mathrm{cd}(\eta+c_2)}{\mathrm{cd}(\eta)}}\right)\left(\frac{1-\frac{\mathrm{cd}\left(2\eta-\frac{c_1-c_2}{2}\right)}{\mathrm{cd}\left(\frac{c_1+c_2}{2}\right)}}{1-\frac{\mathrm{cd}\left(2\eta+\frac{c_1-c_2}{2}\right)}{\mathrm{cd}\left(\frac{c_1+c_2}{2}\right)}}\right),$$
$$s_0(c_7)=c_8,\quad s_0(c_8)=c_7,\quad s_1(\eta)=-\eta,\quad s_2(\eta)=\eta-\frac{2\eta+c_1+c_2}{4},$$
$$s_2(c_i)=c_i-\frac{3(2\eta+c_1+c_2)}{4},\quad i=1,2,\quad s_2(c_j)=c_j+\frac{2\eta+c_1+c_2}{4},\quad j\neq 1,2,$$
$$s_k(c_{k-1})=c_k,\quad s_k(c_k)=c_{k-1},\quad k=3,\dots,7,\quad s_8(c_1)=c_2,\quad s_8(c_2)=c_1.$$

Note that $\lambda=\sum_{i=1}^8 c_i$ is invariant under the action of $W\big(E_8^{(1)}\big)$. The following fundamental relations hold:

$$(s_is_j)^{l_{ij}}=1,$$

where

$$l_{ij}=\begin{cases}1, & i=j\\ 3, & i=j-1\quad (j=2,\dots,7),\quad \text{or if}\quad (i,j)=(3,8),(7,0)\\ 2, & \text{otherwise.}\end{cases}$$



Moreover, by adding the transformations ι_i, $i=1,\dots,4$, $W\big(E_8^{(1)}\big)$ can be extended to $\widetilde{W}\big(E_8^{(1)}\big)=\langle\iota_1,\iota_2,\iota_3,\iota_4\rangle\rtimes W\big(E_8^{(1)}\big)$, where

$$\begin{aligned}
\iota_1&:(c_1,\dots,c_8,\eta,x,y)\mapsto\left(c_1-\frac{\mathrm{i}K'}{2},\dots,c_8-\frac{\mathrm{i}K'}{2},\eta-\frac{\mathrm{i}K'}{2},\frac{1}{kx},y\right),\\
\iota_2&:(c_1,\dots,c_8,\eta,x,y)\mapsto\left(c_1-\frac{\mathrm{i}K'}{2},\dots,c_8-\frac{\mathrm{i}K'}{2},\eta+\frac{\mathrm{i}K'}{2},x,\frac{1}{ky}\right),\\
\iota_3&:(c_1,\dots,c_8,\eta,x,y)\mapsto(c_1-K,\dots,c_8-K,\eta-K,-x,y),\\
\iota_4&:(c_1,\dots,c_8,\eta,x,y)\mapsto(c_1-K,\dots,c_8-K,\eta+K,x,-y).
\end{aligned}$$

The following relations hold:

$$(\iota_i\iota_j)^{m_{ij}}=1,\quad \iota_is_j=s_j\iota_i,\quad i=1,2,3,4,\ j\neq 1,2,\quad \iota_{\{1,2,3,4\}}s_1=s_1\iota_{\{2,1,4,3\}},$$
$$\iota_1s_2=s_2\iota_1\iota_2,\quad \iota_2s_2=s_2\iota_2,\quad \iota_3s_2=s_2\iota_3\iota_4,\quad \iota_4s_2=s_2\iota_4,$$

where

$$m_{ij}=\begin{cases}1, & i=j\\ 2, & \text{otherwise.}\end{cases}$$

Note that for convenience we use the following notations for the composition of the transformations:

$$s_{i_1\cdots i_m}=s_{i_1}\dots s_{i_m},\quad \iota_{i_1\cdots i_m}=\iota_{i_1}\dots\iota_{i_m}.$$

3 Derivations of the elliptic Painlevé equations

Let

$$\phi=s_{56453483706756452348321564534837067564523483217067348304 68}\,\iota_{4321}.$$

The action of ϕ on the parameter space is not translational, but when the parameters take the following special values:

$$c_2=c_1+2K,\quad c_3=c_1+\mathrm{i}K',\quad c_4=c_1+2K+\mathrm{i}K',$$
$$c_6=c_5+2K,\quad c_7=c_5+\mathrm{i}K',\quad c_8=c_5+2K+\mathrm{i}K',$$

the action of ϕ becomes the translational motion in the parameter subspace as the following:

$$\phi:(\gamma_\mathrm{e},\gamma_\mathrm{o},z)\mapsto(\gamma_\mathrm{e},\gamma_\mathrm{o},z+2(\gamma_\mathrm{e}+\gamma_\mathrm{o})),$$

where

$$z=\eta+c_5+2K+\mathrm{i}K',\quad \gamma_\mathrm{e}=c_5-\eta+2K+\mathrm{i}K',\quad \gamma_\mathrm{o}=\eta+c_1+2K+\mathrm{i}K'.$$

Then, the action of ϕ on the coordinates $\phi:(x,y)\mapsto(\tilde{x},\tilde{y})$ gives the RCG equation. Moreover, the action of $\phi^2:(c_i,\eta,x,y)\mapsto(\overline{c_i},\eta+\lambda-4K-2\mathrm{i}K',\overline{x},\overline{y})$ gives the generic version of the RCG equation, which corresponds to a nearest neighbor vector on the $E_8^{(1)}$ weight lattice.

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Acknowledgements

This research was supported by an Australian Laureate Fellowship # FL120100094 and grant # DP160101728 from the Australian Research Council and JSPS KAKENHI Grant Number JP17J00092.