Deautonomization of cluster integrable systems

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arXiv:1711.02063 [math-ph]
arXiv:1804.10145 [math-ph]

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SIDE13

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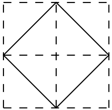
General agenda [BGM] (with example)

Newton polygon Δ



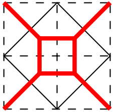
Cluster discrete flow →

Non-autonomous q-difference system

$$G(qZ)G(q^{-1}Z) = \left(\frac{G(Z)+Z}{G(Z)+1}\right)^2$$

Topological string partition function

= general solution (τ-function) of the equation



General agenda [BGM]

Given Newton polygon Δ:

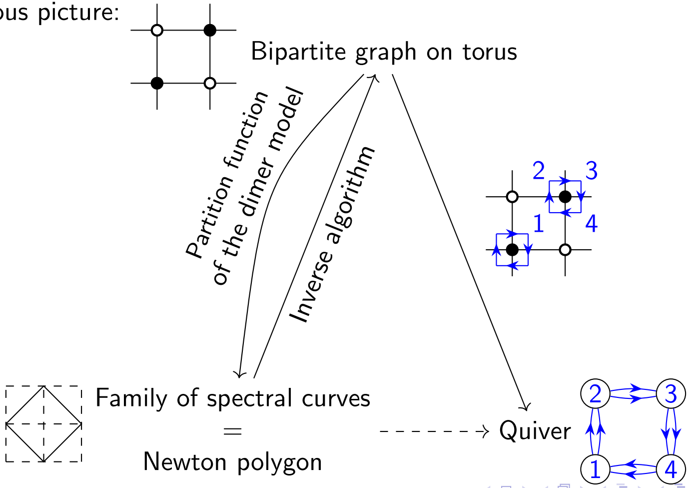
1. Construct q-difference system (non-linear, non-autonomous, but exactly solvable: like (q-)Painlevé equations):
   - Find quiver $\mathcal{Q} = \mathcal{Q}(\Delta)$ using Goncharov-Kenyon inverse procedure
   - Find mapping class group of $\mathcal{Q}$: $G_{\mathcal{Q}}$
   - Take lattice $G_{\Delta} \subset G_{\mathcal{Q}}$ and write down equations
2. Construct solution of the q-difference system:
   - Compute topological string partition function on corresponding CY_3. Possibly express it in terms of Nekrasov functions.
   - Take discrete Fourier transformation and guess appropriate shifts of variables to get several τ functions
   - In the $q = 1$ case: everything is solved by Θ-functions [Fock]
3. Quantize equation and solution

Anything is done only partly, for some concrete examples.

Goncharov-Kenyon integrable system (with example)

In the q=1 limit q-difference system turns into discrete symmetries of GK (cluster) integrable system. Switching on $q \neq 1$ is called *deautonomization*.

Autonomous picture:



Weights of the dimer configurations (=perfect matchings)

The weight of each dimer configuration D is a product of weights of the edges

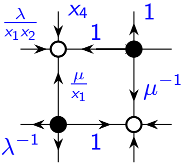
$$w(D) = \prod_{e \in D} w(e).$$

If we pick one dimer configuration D_0 then $D - D_0$ is cycle, $\partial(D - D_0) = 0$, for any D. Therefore, the weight $w(D_0)^{-1} w(D)$ is given by weights of elementary cycles — faces and A, B cycles on torus

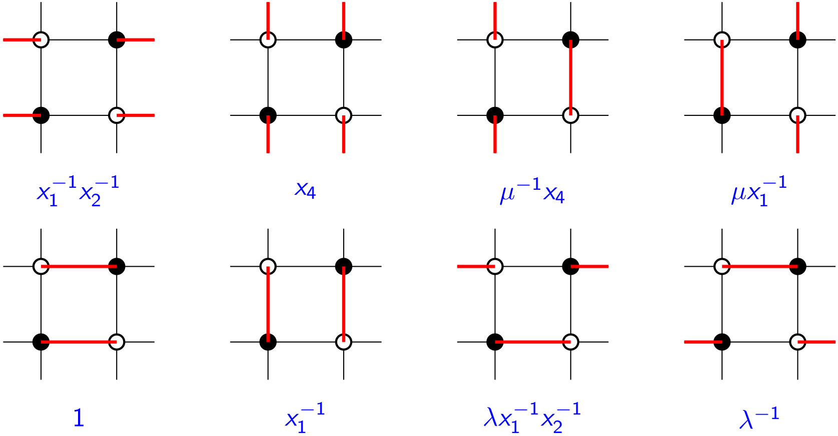
$$\prod_{e \in \partial Face_i} w(e) = x_i, \qquad \prod_{e \in A-cycle} w(e) = \lambda, \qquad \prod_{e \in B-cycle} w(e) = \mu$$

Note that λ, μ depend on concrete choice of A, B cycles.

We have $\prod_i x_i = \prod_{e \in \partial \mathbb{T}^2} W(e) = 1$, since $\partial \mathbb{T}^2 = 0$.



Dimer partition function



Partition function $\mathcal{Z}(\lambda,\mu) = W(D_0)^{-1}\sum W(D)$, where

$$\mathcal{Z}(\lambda,\mu) = x_1^{-1}x_2^{-1}\lambda + \lambda^{-1} + \mu x_1^{-1} + \mu^{-1}x_4 + H,$$

where

$$H = 1 + x_1^{-1} + x_1^{-1}x_2^{-1} + x_4$$

Cluster structure

Poisson bracket is

$$\{x_1, x_2\} = 2x_1x_2, \{x_2, x_3\} = 2x_2x_3, \{x_3, x_4\} = 2x_3x_4, \{x_4, x_1\} = 2x_4x_1$$

Using the $(\mathbb{C}^\times)^3$ action $\mathcal{Z}(\lambda, \mu) \mapsto t_{\mathcal{Z}} \cdot \mathcal{Z}(t_\lambda \cdot \lambda, t_\mu \cdot \mu)$ one can get

$$\mathcal{Z}(\lambda, \mu) = \lambda + z\lambda^{-1} + \mu + \mu^{-1} + H = 0$$

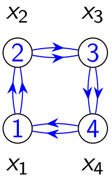
After rescaling coefficients of $\mathcal{Z}$ become Casimirs and Hamiltonians:

$$1 = x_1x_2x_3x_4, \quad z = x_1x_3, \quad H = \sqrt{x_1x_2} + \frac{1}{\sqrt{x_1x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$$

It's relativistic Toda lattice.

Mapping class group $G_{\mathcal{Q}}$

We have to find all combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver. This is purely combinatorial problem. Example:



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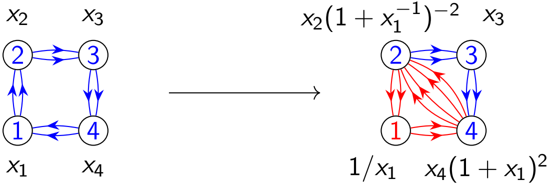

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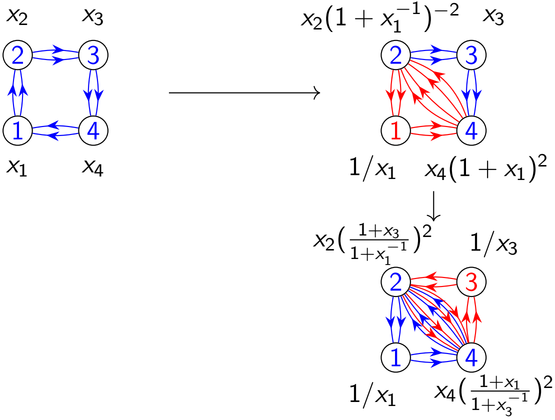

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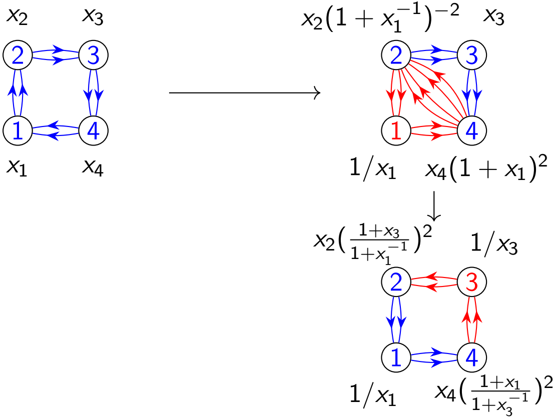

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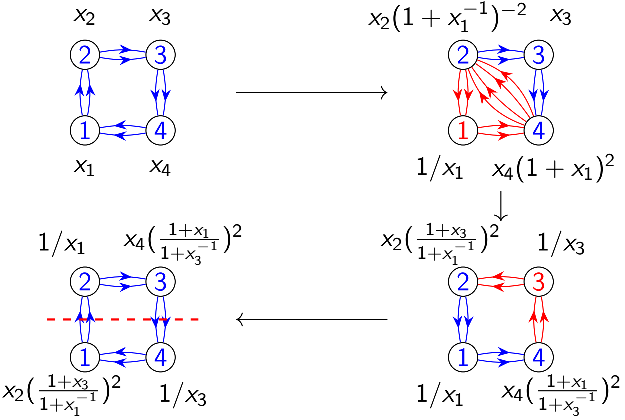


Discrete flow

Take a map coming from quiver automorphism.
Forget about $x_1x_2x_3x_4 = 1 \implies$ no integrable system. $x_1x_2x_3x_4 = q$

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 (\frac{1+x_3}{1+x_1^{-1}})^2, x_1^{-1}, x_4 (\frac{1+x_1}{1+x_3^{-1}})^2, x_3^{-1} \right)$$

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 (\frac{x_1+z}{x_1+1})^2, x_1^{-1}, qz, q \right)$$

Casimir z becomes "time", so introduce $x_i = x_i(z)$, $T : x_i(z) \mapsto x_i(qz)$.

$$x_1(qz)x_1(q^{-1}z) = \left(\frac{x_1(z)+z}{x_1(z)+1} \right)^2$$

This is q-Painlevé III_3 equation, or $P(A_7^{(1)'})$.

Only for $q = 1$ flow T preserves $H = \sqrt{x_1x_2} + \frac{1}{\sqrt{x_1x_2}} + \sqrt{\frac{x_1}{x_2}} + Z\sqrt{\frac{x_2}{x_1}}$.

Quantization (one of the main advantages)

- In addition to non-autonomous parameter q one may add quantum deformation p:

$$\hat{x}_i \hat{x}_j = p^{-2\epsilon_{ij}} \hat{x}_j \hat{x}_i$$

- There are quantum mutations

$$\mu_j: \quad \hat{x}_j \mapsto \hat{x}_j^{-1}, \quad \hat{x}_i^{1/|\epsilon_{ij}|} \mapsto \hat{x}_i^{1/|\epsilon_{ij}|} \left(1 + p\hat{x}_j^{\operatorname{sgn} \epsilon_{ij}}\right)^{\operatorname{sgn} \epsilon_{ij}}, \; i \neq j$$

- All groups G_Q are the same.
- And so there are quantum deformations of all systems. For example, quantum q-Painlevé III_3:

$$\begin{cases} \hat{x}_1(q^{-1}z)^{1/2}\, \hat{x}_1(qz)^{1/2} = \dfrac{\hat{x}_1(z) + pz}{\hat{x}_1(z) + p}, \\ \hat{x}_1(z)\hat{x}_1(q^{-1}z) = p^4 \hat{x}_1(q^{-1}z)\hat{x}_1(qz). \end{cases}$$

Different approaches to quantization were also considered long before by K. Hasegawa, G. Kuroki, H. Nagoya, Y. Yamada.

Generic solution [BShch]/[BGM] (quantum q-GIL formula)

$$\hat{x}_1(z) = p z^{1/2} \hat{\mathcal{T}}_1(z)^2 \hat{\mathcal{T}}_3(z)^{-2}$$

$$\hat{\mathcal{T}}_1(z) = \hat{a} \sum_{n\in\mathbb{Z}} \hat{s}^n Z^{2,0}(\hat{u} q^{2n}; q_1 q_2^{-1}, q_2^2 | z)$$

$$\hat{\mathcal{T}}_3(z) = i\hat{a} \sum_{n\in\frac{1}{2}+\mathbb{Z}} \hat{s}^n Z^{2,0}(\hat{u} q^{2n}; q_1 q_2^{-1}, q_2^2 | z)$$

Where

$$q_2 = q^{1/2}, \quad q_1 = q_2^{-1} p^2, \quad \hat{u}\hat{s} = p^4 \hat{s}\hat{u}$$

and also

$$q_2^2 \hat{a} = p^{-2} \hat{a} q_2^2 = \hat{a} q_1^{-1} q_2, \quad q_1 q_2^{-1} \hat{a} = p^2 \hat{a} q_1 q_2^{-1} = \hat{a} q_1^2$$

So here we have *operator* Fourier transformation.

$Z^{2,0}(\hat{u} q^{2n}; q_1 q_2^{-1}, q_2^2 | z)$ is a topological string partition function (with prefactor), or 5D Nekrasov function for $SU(2)$ pure gauge theory.

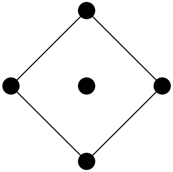
Proofs for the classical case: M. Bershtein, A. Shchechkin (also conjectured this); M. Jimbo, H. Nagoya, H. Sakai ($P(A_3^{(1)})$) + Y. Matsuhira, H. Nagoya (limit)

Numerology of cluster integrable systems

- (# of variables) $=$ 2*Area(Δ).
- Dimension of the phase space $=$ 2*(# of internal points).
- Number of Casimirs (without q) $=$ (# of boundary points) - 3.
- (# of discrete flows) $=$ number of Casimirs (without q)

Simplest cases: 1) one discrete flow, 2) one Hamiltonian.

Directions of the generalization



4 boundary points, internal points on one line →

Non-autonomous discrete Hirota equations

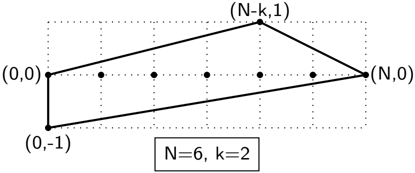
One internal point ↓

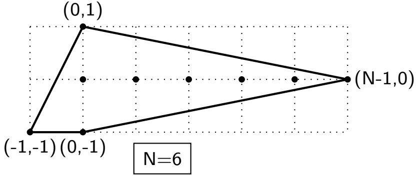
q-difference Painlevé equations

? General Newton polygons

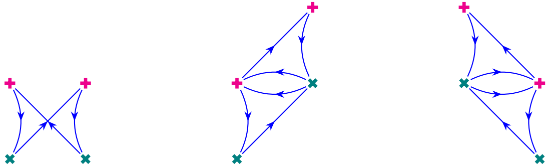
4 boundary points, hyperelliptic curves (Toda family)

Classification: $Y^{N,k}$ polygons with $0 \leq k \leq N$ (left picture) and $L^{1,2N-1,2}$ polygons (right picture):





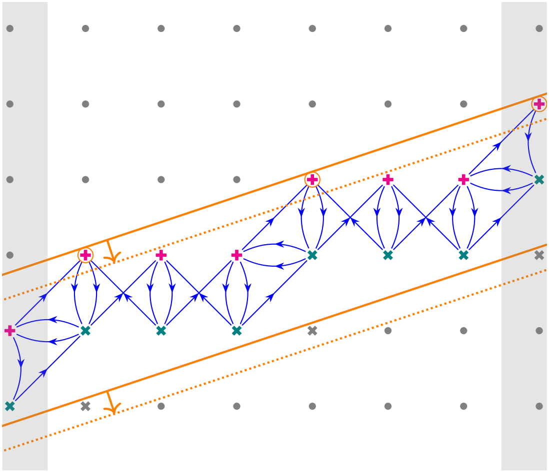
Quivers for $Y^{N,k}$ theories can be glued from blocks of three types 0, 1, -1, respectively. $N = N_1 + N_0 + N_{-1}$, $k = N_1 - N_{-1}$.



Similar non-cyclic quivers appeared in Di Francesco's papers.

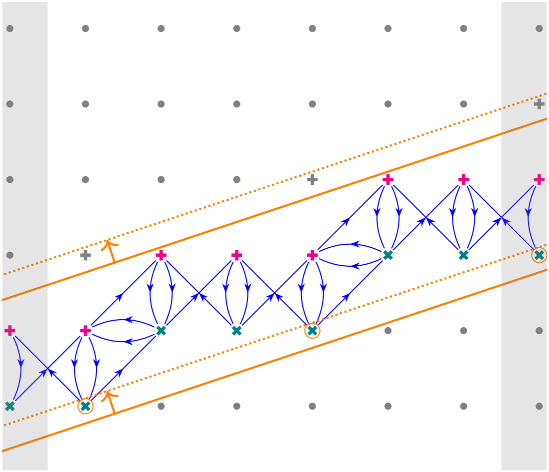
Graphs also computed by S. Franco, A. Hanany, K. Kennaway, D. Vegh, B. Wecht

Action of the automorphism group



N=6, k=2

Action of the automorphism group



N=6, k=2

Equations [BGM?]

Mutable "+"-variables labelled by the points of integer lattice: $x_{(n,m)}$. They satisfy periodicity condition and Y-system in order to be compatible with mutations:

$$\frac{x_{(n,m+1)}x_{(n,m-1)}}{x_{(n,m)}^2} = \frac{(1+x_{(n+1,m)})(1+x_{(n-1,m)})}{(1+x_{(n,m)})^2}, \quad x_{(n,m)} = x_{(n+N,m+k)}$$

One can move from Y-system to T-system (from X-clusters to A-clusters):

$$x_{(n,m)} = z_0^{1/N} q^{(kn-Nm+N)/N^2} \frac{\tau_{(n-1,m-1)}\tau_{(n+1,m-1)}}{\tau_{(n,m-1)}^2}, \quad \tau_{(n,m)} = \tau_{(n+N,m+k)}$$

$$\tau_{(n,m+1)}\tau_{(n,m-1)} = \tau_{(n,m)}^2 + z_0^{1/N} q^{(kn-Nm)/N^2} \tau_{(n+1,m)}\tau_{(n-1,m)}$$

And after some change of labeling:

$$\tau_j(qz)\,\tau_j(q^{-1}z) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

Solution [BGM]

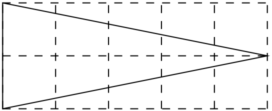
$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

Fourier transformation of partition function of 5D $SU(N)$ pure gauge theory with Chern-Simons term at level k:

$$\tau_j(z) = \sum_{\vec{\Lambda}\in Q_{N-1}+\omega_j} \prod_i (s_i^{\Lambda_i}) \cdot Z^{N,k}(\{u_i q^{\Lambda_i}\}; q, q^{-1}|z), \qquad j \in \mathbb{Z}/N\mathbb{Z}\,.$$

where Q_{N-1} is $SL(N)$ root lattice, and ω_i are $SL(N)$ fundamental weights ($\omega_0 = 0$).

Compare with K. Takasaki's talk: $u_i \approx q^{\frac{N+1-2i}{2N}}$, $k = N$



Nekrasov functions

$$Z^{N,k}(\vec{u}; q_1, q_2|z) = Z_{\mathrm{cl}}^{N,k}(\vec{u}; q_1, q_2|z) \cdot Z_{1-\mathrm{loop}}^{N}(\vec{u}; q_1, q_2) \cdot Z_{\mathrm{inst}}^{N,k}(\vec{u}; q_1, q_2|z)\,,$$

where

$$Z_{\mathrm{cl}}^{N,k}(\vec{u}; q_1, q_2|z) = \exp\left(\log z \frac{\sum (\log u_i)^2}{-2\log q_1 \log q_2} + k\frac{\sum (\log u_i)^3}{-6\log q_1 \log q_2}\right),$$

$$Z_{1-\mathrm{loop}}^{N}(\vec{u}; q_1, q_2) = \prod_{1\le i\ne j\le N} (u_i/u_j; q_1, q_2)_\infty\,,$$

$$Z_{\mathrm{inst}}^{N,k}(\vec{u}; q_1, q_2|z) = \sum_{\vec{\lambda}} \frac{z^{|\vec{\lambda}|}\prod_{i=1}^N (\mathsf{T}_{\lambda^{(i)}}(u_i; q_1, q_2))^k}{\prod_{i,j=1}^N \mathsf{N}_{\lambda^{(i)},\lambda^{(j)}}(u_i/u_j; q_1, q_2)}\,,$$

$$\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)}), \ \ |\vec{\lambda}| = \sum |\lambda^{(i)}|, \ \ |\lambda| = \sum \lambda_j,$$

$$\mathsf{N}_{\lambda,\mu}(u, q_1, q_2) = \prod_{s\in\lambda}(1 - u q_2^{-a_\mu(s)-1} q_1^{\ell_\lambda(s)}) \cdot \prod_{s\in\mu}(1 - u q_2^{a_\lambda(s)} q_1^{-\ell_\mu(s)-1})\,,$$

$$\mathsf{T}_\lambda(u; q_1, q_2) = u^{-|\lambda|} q_1^{|\lambda'|-\frac{1}{2}(\|\lambda'\|)} q_2^{\frac{1}{2}(|\lambda|-\|\lambda\|)} = \prod_{(i,j)\in\lambda} u^{-1} q_1^{1-i} q_2^{1-j},$$

$$\|\lambda\| = \sum \lambda_j^2\,.$$

Differential limit (5D $\to$ 4D)

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right)=\tau_j(z)^2+z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$

We take $q=\exp R$, $z=R^{2N}\mathrm{z}$ and send $R\to 0$:

$$(\partial_{\log \mathrm{z}})^2\log\tau_j=\mathrm{z}^{1/N}\frac{\tau_{j+1}\tau_{j-1}}{\tau_j^2}\,,\quad j\in\mathbb{Z}/N\mathbb{Z}$$

So we see no dependence on k. In the different variables

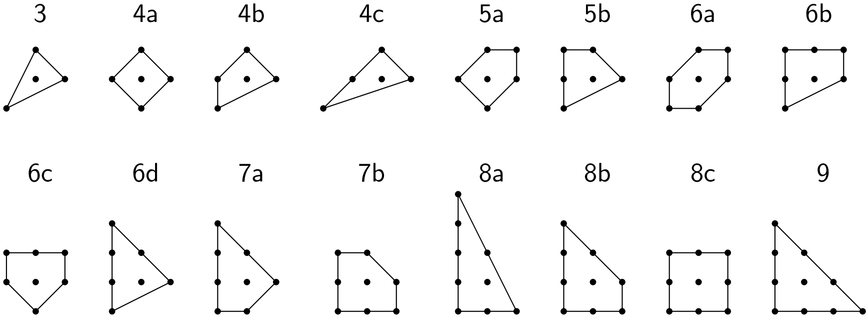
$$\phi_j=\log\tau_j-\log\tau_{j-1},\quad r=2N\mathrm{z}^{\frac{1}{2N}}$$

We have

$$\frac{d^2\phi_n}{dr^2}+\frac{1}{r}\frac{d\phi_n}{dr}=e^{\phi_{n+1}-\phi_n}-e^{\phi_n-\phi_{n-1}}$$

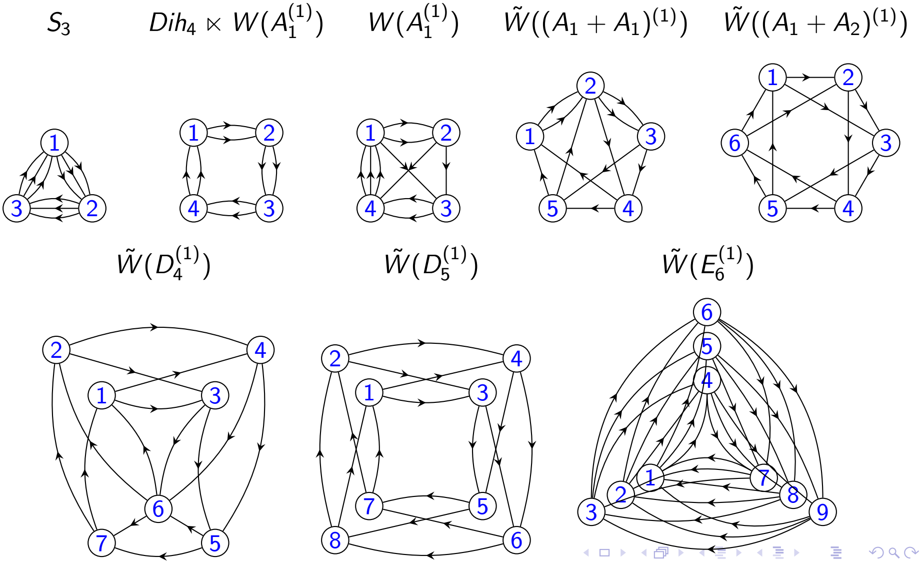
This is radial Toda equation, for $N=2$ — radial sinh-Gordon equation (PIII_3).

Newton polygons with one internal point

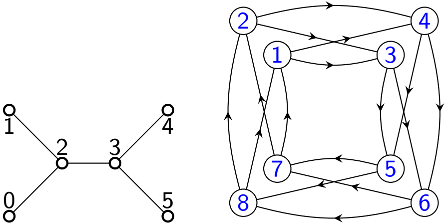


Same area $\Leftrightarrow$ same quivers, except for 4_a and 4_b. 3 is trivial.

Quivers and their mapping class groups [BGM]

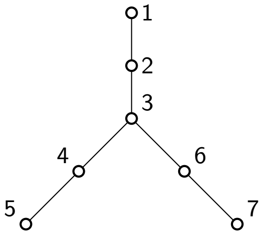


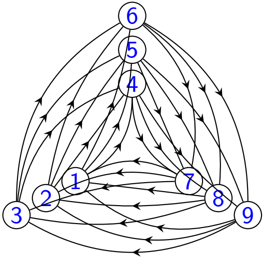
$8_{a,b,c}$ case — $q - PVI$ [BGM]



$s_0 = (1,2),$
$s_1 = (5,6),$
$s_2 = (1,5) \circ \mu_5 \circ \mu_1,$
$s_3 = (3,7) \circ \mu_3 \circ \mu_7,$
$s_4 = (3,4),$
$s_5 = (7,8),$
$\pi = (1,7,5,3)(2,8,6,4),$
$\sigma = (1,7)(2,8)(3,5)(4,6) \circ \varsigma,$
here ς — inversion of all arrows

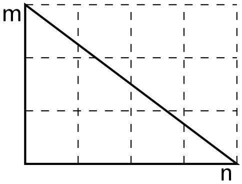
9 case [BGM]





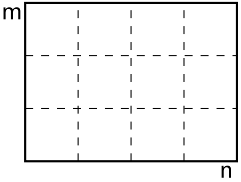
$s_1 = (2,3)$,
$s_2 = (1,2)$,
$s_4 = (4,5)$,
$s_5 = (5,6)$,
$s_6 = (7,8)$,
$s_0 = (8,9)$,
$s_3 = (4,7) \circ \mu_1 \circ \mu_4 \circ \mu_7 \circ \mu_1$,
$\pi = (1,4,7)(2,5,8)(3,6,9)$,
$\sigma = (1,7)(2,8)(3,9) \circ \varsigma$

More complicated examples

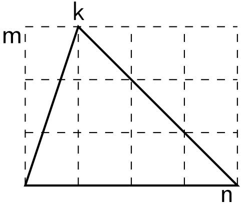


K. Kajiwara, M. Noumi, Y. Yamada: action of the $W(A_{n-1}^{(1)}) \times W(A_{m-1}^{(1)})$ Weyl group

(?) $m = 2$, $n = 2k$ case: N. Okubo, T. Suzuki (?)



(work in progress) M. Semenyakin, A. Marshakov: deautonomization of classical XXZ spin chain (with small $GL(m)$ orbits) $\stackrel{?}{=}$ q-Schlesinger. Solutions are given by Fourier transform of quiver Nekrasov partition functions with matter. $W(A_{n-1}^{(1)})^2 \times W(A_{m-1}^{(1)})^2$



(partly work in progress) R. Inoue, T. Ishibashi, T. Lam, H. Oya, P. Pylyavskyy: quiver $Q_m(A_{n-1}^{(1)})_k$

Thank you for your attention!