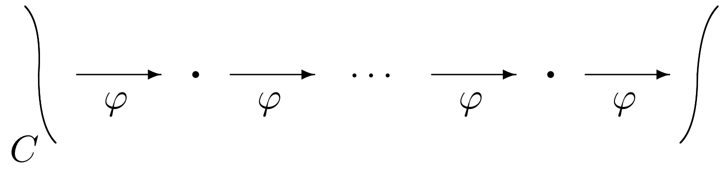
Dynamical degrees and singularity patterns (2)

Setting: $f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, autonomous and confining, i.e. all singularities have the singularity confinement property:



Def. *The dynamical degree* of the mapping f: $\boxed{\lambda_* = \lim_{n\to+\infty} \left(\deg f^{(n)}\right)^{1/n}}$ $\quad(\lambda_* \geq 1)$

Fact: If $f \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ is confining, then f is conjugate (by a birational transformation) to an *automorphism* on a rational surface X (= its "space of initial conditions").

$$\begin{array}{ccc} X & \xrightarrow[\sim]{\varphi} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \overset{f}{\dashrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

$\varphi : X \to X$ induces an invertible linear action on $\mathrm{Pic}\, X$, the Picard group of X,

$\varphi_* : \mathrm{Pic}\, X \to \mathrm{Pic}\, X$, the largest eigenvalue of which is the dynamical degree of f.

[**T. Takenawa** *J. Phys. A* **34 (2001) 10533–10545.**]

Thm. [**J. Diller and C. Favre** *Amer. J. Math.* **123 (2001) 1135–1169.**]

Let $\Phi \in GL_\rho(\mathbb{Z})$ represent $\varphi_* : \mathrm{Pic}\, X \to \mathrm{Pic}\, X$ for a confining birational mapping on $\mathbb{P}^1 \times \mathbb{P}^1$. The Jordan normal form of the matrix Φ can only take one of the following three forms:

(a) $\begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_\rho \end{pmatrix}$

ν_j: roots of unity
$\deg f^{(n)}$: bounded growth

(b) $\begin{pmatrix} 1 & 1 & 0 & & & \\ 0 & 1 & 1 & & & \\ 0 & 0 & 1 & & & \\ & & & \nu_1 & & \\ & & & & \ddots & \\ & & & & & \nu_{\rho-3} \end{pmatrix}$

ν_j: roots of unity
$\deg f^{(n)} \sim n^2$

(c) $\begin{pmatrix} \lambda & & & & \\ & 1/\lambda & & & \\ & & \nu_1 & & \\ & & & \ddots & \\ & & & & \nu_{\rho-2} \end{pmatrix}$

$\lambda > 1\,,\ |\nu_j| = 1$
$\deg f^{(n)} \sim \lambda^n$

Fact: There exists a $(\mathbb{Z})$ basis for $\mathrm{Pic}\, X$ s.t.: $\varphi_* \sim \begin{pmatrix} U & * \\ 0 & A \end{pmatrix}$, $U \in \mathrm{M}_{s\times s}(\mathbb{Z})$, unitary, $A \in \mathrm{GL}_{\rho-s}(\mathbb{Z})$

The sub-matrix A represents the action $\boxed{\overline{\varphi}_* : \mathrm{Pic}\, X/\mathrm{P}_X \to \mathrm{Pic}\, X/\mathrm{P}_X, \quad \overline{\varphi}_*(F) = \varphi_*(F) \bmod \mathrm{P}_X,}$

where $\boxed{\mathrm{P}_X := \left\{ F \in \mathrm{Pic}\, X \,\middle|\, {}^\exists m \in \mathbb{Z} : \ \varphi_*^m F = F \right\}.}$

From now on we only consider the cases (b) and (c), i.e. the case $\mathrm{P}_X \neq X$.

Corol. If $\mu_{\varphi_*}(t)$ and $\mu_{\overline{\varphi}_*}(t)$ are the minimal polynomials for φ_* and $\overline{\varphi}_*$, then:

(b) $\mu_{\varphi_*}(t) = (t-1)^3 \times \big($a product of cyclotomic polynomials in t, all different and different from $(t-1)\big)$

$\boxed{\mu_{\overline{\varphi}_*}(t) = (t-1)^2}$

(c) $\mu_{\varphi_*}(t) = \mu_\lambda(t) \times \big($a product of cyclotomic polynomials, all different$\big)$

$\boxed{\mu_{\overline{\varphi}_*}(t) = \mu_\lambda(t) \ : \ \text{minimal polynomial for } \lambda}$

Prop. 1 Suppose we have $\psi(t) \in \mathbb{Q}[t] \backslash \{0\}$ and $F \in \mathrm{Pic}\, X/\mathrm{P}_X, F \neq 0$, such that $\psi(\varphi_*)\, F = 0 \bmod \mathrm{P}_X$.

i) If $\psi(t)$ does not have a root greater (or less) than 1, then the mapping is integrable. It has dynamical degree $\lambda_* = 1$ and quadratic degree growth.

ii) If $\psi(1) \neq 0$, then we are in the case (c) and the mapping is nonintegrable $(\lambda_* > 1)$.

Proof: For any endomorphism f of a finite dimensional vector space V over a field K, if there exists a polynomial $\psi(t) \in K[t] \setminus \{0\}$ such that $\psi(f)v = 0$ for some non-zero $v \in V$, then $\psi(t)$ and the minimal polynomial for f share a common factor over $K[t]$.

Hence, if $\psi(t) \in \mathbb{Q}[t] \setminus \{0\}$ and $F \in \mathrm{Pic}\, X/\mathrm{P}_X, F \neq 0 \mod \mathrm{P}_X$, such that $\psi(\varphi_*)\, F = 0 \mod \mathrm{P}_X$, then $\psi(t)$ and $\mu_{\overline{\varphi}_*}(t)$ must have a common factor. Since we excluded the case (a), it then follows that if $\psi(t)$ does not have a root greater (or less) than 1, we are not in the case (c) and therefore must be in the case (b).
Conversely, if $\psi(1) \neq 0$ we must be in the case (c) since $\mu_{\overline{\varphi}_*}(t) = (t-1)^2$ in the case (b).

Def. the k-th cyclotomic polynomial $\boxed{\phi_k(t) := \prod_{\text{all } \iota_j} (t - \iota_j),}$ (ι_j : primitive k-th root of unity)

Prop. 2 Let $\varphi : X \to X$ be an automorphism on the space of initial conditions X for a mapping with unbounded degree growth. Let $C_1, \ldots, C_m$ be the divisor classes of curves that appear in the non-cyclic patterns for φ on X, let $a_1, \ldots, a_m > 0$ and consider $F = \sum_{j=1}^{m} a_j C_j \in \mathrm{Pic}\, X/\mathrm{P}_X$.

Then, $$\boxed{(\varphi_*^{m_0} - 1)\left(\prod_{j=1}^{\ell} \phi_{k_j}(\varphi_*^{m_j})\right) F \notin P_X,}$$ for $\ell \geq 0$, $k_j \geq 2$ and $m_j \geq 1$ $(j = 0, 1, \ldots, \ell)$.

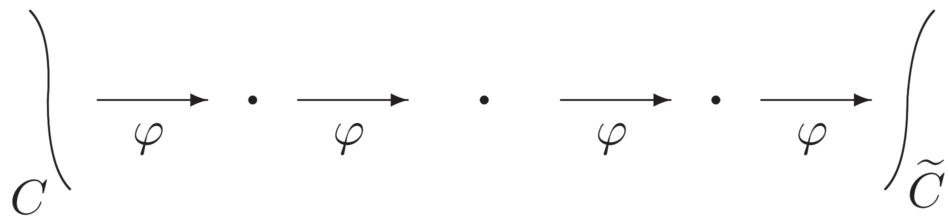
In particular, F itself and $(\varphi_* - 1)F \notin P_X$. *Proof: cf.* **[arXiv:1810.02693]**

Lemma If $C \subset X$ is a curve that appears in an open pattern on X (the equivalent of a non-cyclic singularity pattern for the mapping, after blow-up), then its corresponding divisor class does not lie in the cyclic part of the Picard group, i.e.: $[C] \notin \mathrm{P}_X$.
On the other hand, if $C \subset X$ is a curve that appears in a cyclic pattern or if it is a curve that arises in the blow-up process but is not part of any open pattern on X, then $[C] \in \mathrm{P}_X$.

Rem. That the divisor class of a curve in an open pattern on X cannot lie in P_X follows from the fact that it is not part of any cyclic pattern and from the fact such a curve must have negative self-intersection, which implies that it cannot be expressed as a non-trivial sum of effective classes in $\mathrm{Pic}\, X$.

Example 1: $\boxed{x_{n+1} + x_{n-1} = x_n + \dfrac{1}{x_n^k}}$ $(k \in 2\mathbb{Z}_{>0})$ has two singularity patterns, a *cyclic* one $(\, x_0\, \infty\, \infty\,)$,

and an *open* (confined) one $\boxed{(\, 0\, \infty^k\, \infty^k\, 0\,)}$:



on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{pmatrix} u \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ \infty^k \end{pmatrix} \mapsto \begin{pmatrix} \infty^k \\ \infty^k \end{pmatrix} \mapsto \begin{pmatrix} \infty^k \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ u \end{pmatrix}$$

after b-up : $\boxed{\{y=0\} \to C_1 \to C_2 \to C_3 \to \{x=0\}}$ (= open pattern on X)

($\{y=0\}$ and $\{x=0\}$ represent the strict transforms of the corresponding lines on $\mathbb{P}^1 \times \mathbb{P}^1$)

$$\left(\longrightarrow \quad 1 \quad - k\,\lambda \quad - k\,\lambda^2 \quad + \lambda^3 \quad !? \right)$$

Let $H_y \in \operatorname{Pic} X$ be the divisor class for the total transform of the line $y = 0$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Using the above Lemma, and neglecting any contributions from the cyclic part P_X of $\operatorname{Pic} X$, we can express H_y in terms of the curves that appear in the open pattern as:

$$\begin{aligned} H_y &= [y=0] + [C_3] \mod \mathrm{P}_X &\equiv \big(\varphi_*^{-1} + \varphi_*^2\big)\,[C_1] \mod \mathrm{P}_X \\ &= k[C_1] + k[C_2] \mod \mathrm{P}_X &\equiv k\big(1 + \varphi_*\big)\,[C_1] \mod \mathrm{P}_X, \end{aligned}$$

which implies that $$\boxed{\big(\varphi_*^3 - k\varphi_*^2 - k\varphi_* + 1\big)\,[C_1] = 0 \mod \mathrm{P}_X.}$$

$$t^3 - kt^2 - kt + 1 = (t+1)\big(t^2 - (k+1)t + 1\big) \xrightarrow{\text{Prop. 2}} \boxed{\big(\varphi_*^2 - (k+1)\varphi_* + 1\big)\,F = 0 \mod \mathrm{P}_X}$$

for $F := \big(\varphi_* + 1\big)\,[C_1] \neq 0 \mod \mathrm{P}_X$

Hence, from Prop. 1 one concludes that since $\psi(t) = (t^2 - (k+1)t + 1)$ has no root $t = 1$, this mapping must have a dynamical degree that is greater than 1 and is therefore non-integrable.

Moreover, as $\psi(t)$ is in fact irreducible, we have $\mu_{\overline{\varphi}_*}(t) = \psi(t)$ and therefore that the dynamical degree

$$\boxed{\lambda_* = \frac{k + 1 + \sqrt{(k+1)^2 - 4}}{2}}$$

Example 2: $\boxed{x_{n+1}x_{n-1} = \dfrac{x_n^4 - 1}{x_n^4 + 1}}$ has one cyclic singularity pattern $\big(x\,\infty\,x'\,0\big)$ and 8 open ones

$$\boxed{(\pm 1 \quad 0 \quad \mp 1), \qquad (\pm i \quad 0 \quad \pm i), \qquad (\pm r \quad \infty \quad \mp ir), \qquad (\pm ir \quad \infty \quad \mp r)} \quad (r = \exp i\pi/4)$$

After b-up one obtains 16 special curves $A_1^+, \cdots, D_2^-$ (2 for each pattern), on a rational surface X, that form the open patterns:

$$\boxed{\begin{array}{ll} \{y=\pm 1\} \to A_1^\pm \to A_2^\pm \to \{x = \mp 1\}, & \{y = \pm i\} \to B_1^\pm \to B_2^\pm \to \{x = \pm i\}, \\ \{y = \pm r\} \to C_1^\pm \to C_2^\pm \to \{x = \mp ir\}, & \{y = \pm ir\} \to D_1^\pm \to D_2^\pm \to \{x = \mp r\} \end{array}}$$

The (divisor classes of) $\{y = 0\}$ and $\{y = \infty\}$ are part of P_X and we find that (modulo P_X)

$$\begin{aligned} H_y &= [y=1] + [A_2^-] = [y=-1] + [A_2^+] = [y=i] + [B_2^+] = [y=-i] + [B_2^-] \\ &= [y=r] + [D_2^-] = [y=-r] + [D_2^+] = [y=ir] + [C_2^-] = [y=-ir] + [C_2^+] \\ &= [A_1^+] + [A_1^-] + [B_1^+] + [B_1^-] = [C_1^+] + [C_1^-] + [D_1^+] + [D_1^-], \end{aligned}$$

from which we can derive:

$$\left(\varphi_* + \varphi_*^{-1}\right) F = 4F \mod \mathrm{P}_X$$

$$\Leftrightarrow \qquad \boxed{\left(\varphi_*^2 - 4\varphi_* + 1\right) F = 0 \mod \mathrm{P}_X,}$$

where $F := [A_1^+] + [A_1^-] + [B_1^+] + [B_1^-] + [C_1^+] + [C_1^-] + [D_1^+] + [D_1^-] \neq 0 \mod \mathrm{P}_X$ (due to Prop. 2).

As $(t^2 - 4t + 1)$ does not contain a factor $(t-1)$, Prop. 1 tells us that this mapping is <u>nonintegrable</u>.

Moreover, since this polynomial is irreducible, $\mu_{\overline{\varphi}_*}(t) = (t^2 - 4t + 1)$ and hence we find $\boxed{\lambda_* = 2 + \sqrt{3}.}$

Example 3: $\boxed{x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^k}}$ $(k \in \mathbb{Z}_{>1}, \text{odd})$ has a cyclic $\big(x_0 \infty \infty\big)$, as well as an *unconfined* singularity pattern $\big(0\, \infty^k\, \infty^k\, 0\, \infty^k\, \infty^k\, 0\, \infty^k\, \infty^k\, 0 \cdots\big)$.

Writing Z_n for the # of "spontaneous" occurrences of the value 0 in the iteration at step n, we have for the # of pre-images of the values 0 and ∞ (neglecting those that arise from the cyclic pattern):

$$Z_n + Z_{n-3} + Z_{n-6} + Z_{n-9} + \cdots = k(Z_{n-1} + Z_{n-2}) + k(Z_{n-4} + Z_{n-5}) + \cdots$$

(where we set $Z_{n<0} = 0$).

Postulating a behaviour $Z_n \sim \lambda^n$ for some $\lambda > 1$ we can re-sum the series in the limit $n \to +\infty$:

$$\sum_{k=0}^{+\infty} \lambda^{-3k} \equiv \frac{\lambda^3}{\lambda^3 - 1} = \left(\frac{1}{\lambda} + \frac{1}{\lambda^2}\right) \frac{k\lambda^3}{\lambda^3 - 1} \equiv k \sum_{k=0}^{+\infty} \left(\lambda^{-1-3k} + \lambda^{-2-3k}\right)$$

$$\Leftrightarrow \qquad \lambda^2 - k\lambda - k = 0$$

$$\longrightarrow \qquad \lambda_* = \frac{k + \sqrt{k(k+4)}}{2}$$

: How can this be justified geometrically ?

[**RIAM proceedings, University of Kyushu, 29AO-S7 (2018) Article No.01. http://hdl.handle.net/2324/1956707**]