Symmetries of the Eikonal Equation

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Objectives

We consider the Eikonal Equation in dimension n:

$$\sum_{i=1..n} p_i^2 = 1$$

where $p_i = \frac{\partial z}{\partial x^i}$.
We want to find:

1. infinitesimal algebra of Lie symmetries of the Eikonal equation.
2. Give an explicit basis for the Lie symmetries in dimension n.
3. Give one-dimensional inequivalent subalgebras.
4. Construct solutions.

Introduction

1. we consider the problem of classifying the Lie symmetries of the Eikonal equation with an arbitrary number of independent variables.
2. The Eikonal equation arises in the study of geometric optics and electro-magnetism and is discussed in [E].
3. it is too complicated to be able to find a closed form general solution in terms of an arbitrary function.
4. Since it is hard to find solutions, the approach of finding Lie symmetries to map solutions to solutions becomes particularly attractive.

Derivation of the PDE system

$$\sum_{i=1..n} p_i^2 = 1 \tag{1}$$

where $p_i = \frac{\partial z}{\partial x^i}$. We shall also find it convenient to use

$$p_i p_i = 1 \tag{2}$$

where the summation convention is in force *without* necessarily having to have the sum run over a subscript and superscript.
Define:

$$D_i = \frac{\partial}{\partial x^i} \tag{3}$$

and

$$X = a^i(x^j, z)\frac{\partial}{\partial x^i} + c(x^j, z)\frac{\partial}{\partial z}. \tag{4}$$

The first prolongation of X is given by

$$\tilde{X} = X + (D_i c - p_j(D_i a^j))\frac{\partial}{\partial p_i}. \tag{5}$$

Now we apply $\tilde{X}$ to 1 so as to make X a symmetry and we obtain the following system:

$$\frac{\partial c}{\partial x^i} - \frac{\partial a^i}{\partial z} = 0 \tag{6}$$

$$\frac{\partial a^i}{\partial x^j} + \frac{\partial a^j}{\partial x^i} = 0 \quad (i \neq j). \tag{7}$$

$$\frac{\partial a^i}{\partial x^i} = \frac{\partial c}{\partial z} \tag{8}$$

Solving the PDE system

$$\frac{\partial}{\partial x^i}, \frac{\partial}{\partial z}, x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, z\frac{\partial}{\partial x^i} + x^i\frac{\partial}{\partial z}, x^i\frac{\partial}{\partial x^i} + z\frac{\partial}{\partial z}, \\ (z^2 - x^k x^k)\frac{\partial}{\partial x^i} + 2x^i\Delta, (x^k x^k - z^2)\frac{\partial}{\partial z} + 2z\Delta \tag{9}$$

where $\Delta = x^i\frac{\partial}{\partial x^i}$ is the dilation or Liouville vector field. In 9 the summation convention applies to the repeated indices i and k and $1 \leq i \leq n$ except for the R_{ij} where $1 \leq i < j \leq n$.
The basis of symmetries given in 9 provides just about the simplest solution to the PDE consisting of 9. However, it is not very convenient in terms of identifying the algebra itself. As such we introduce the following new basis:

$$R_i^j = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j} \tag{10}$$

$$A^i = \frac{1}{\sqrt{8}}((z^2 - x^k x^k + 2)\frac{\partial}{\partial x^i} + 2x^i\Delta), \tag{11}$$

$$\Delta = x^i\frac{\partial}{\partial x^i} + z\frac{\partial}{\partial z}, \tag{12}$$

$$F = \frac{1}{\sqrt{8}}((x^k x^k - z^2 - 2)\frac{\partial}{\partial z} + 2z\Delta), \tag{13}$$

$$E^i = \frac{1}{\sqrt{8}}((z^2 - x^k x^k - 2)\frac{\partial}{\partial x^i} + 2x^i\Delta), \tag{14}$$

$$B^i = z\frac{\partial}{\partial x^i} + x^i\frac{\partial}{\partial z}, \tag{15}$$

$$C = \frac{1}{\sqrt{8}}((x^k x^k - z^2 + 2)\frac{\partial}{\partial z} + 2z\Delta). \tag{16}$$

The Symmetry Lie Algebra

The non-zero brackets of the symmetry Lie algebra are given by:

$$\begin{aligned}
&[R^{ij}, R^{km}] = \delta_i^k R^{jm} + \delta_m^j R^{ik} + \delta_m^i R^{jk} + \delta_k^j R^{mi}\\
&[R^{ij}, A^k] = \delta_k^i A^j - \delta_k^j A^i,\\
&[R^{ij}, E^k] = \delta_k^i E^j - \delta_k^j E^i,\\
&[R^{ij}, B^k] = \delta_k^i B^j - \delta_k^j B^i,\\
&[A^i, A^j] = R^{ij}, [\Delta, A^i] = E^i,\\
&[A^i, E^j] = \delta_j^i \Delta,\ [A^i, F] = B^i,\\
&[A^i, B^j] = -\delta_j^i F,\ [B^i, B^j] = -R^{ij},\\
&[\Delta, F] = C,\ [\Delta, E^i] = A^i,\\
&[C, F] = \Delta,\ [B^i, F] = A^i,\ [E^i, E^j] = -R^{ij},\\
&[C, E^i] = B^i, [B^i, E^j] = \delta_i^j C,\ [B^i, C] = E^i,\ [\Delta, C] = F.
\end{aligned}$$

The Lie algebra $o(p, q)$

$$g = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \tag{17}$$

then $A \in O(p, q)$ if and only if

$$A^t g A = g \tag{18}$$

and a matrix R is in $o(p, q)$ if and only if

$$gR + (gR)^t = 0. \tag{19}$$

It follows that the Lie algebra $o(p, q)$ is the set of matrices of the form

$$\begin{bmatrix} a & b \\ b^t & c \end{bmatrix} \tag{20}$$

where a is $p \times p$, $a^t = -a$, the matrix c is $q \times q$ and $c^t = -c$ and b is $p \times q$ arbitrary.

Canonical basis for $o(p, q)$

F_{ab} is the $(p+q) \times (p+q)$ matrix whose only non-zero entries are 1 in the (a, b)-th and -1 in the (b, a)-th positions, respectively, and $1 \leq a < b \leq p$; H_{ij} is the $(p+q) \times (p+q)$ matrix whose only non-zero entries are 1 in the $(p+i, p+j)$-th and -1 in the $(p+j, p+i)$-th positions, respectively, and $1 \leq i < j \leq q$; G_{aj} is the $(p+q) \times (p+q)$ matrix whose only non-zero entries are 1 in the $(a, p+j)$-th and 1 in the $(p+j, a)$-th positions, respectively, and $1 \leq a \leq p$ and $p+1 \leq j \leq p+q$

None-Zero Brackets for $o(p, q)$

brackets are non-zero:

$$[F_{ab}, F_{cd}] = \delta_{ac}F_{db} + \delta_{bc}F_{ad} + \delta_{ad}F_{bc} + \delta_{bd}F_{ca} \tag{21}$$

$$[H_{ij}, H_{kl}] = \delta_{ki}H_{lj} + \delta_{jk}H_{il} + \delta_{jl}H_{jk} + \delta_{lj}H_{ki} \tag{22}$$

$$[F_{ab}, G_{cj}] = \delta_{bc}G_{aj} \tag{23}$$

$$[G_{ai}, H_{jk}] = \delta_{ij}G_{ak} \tag{24}$$

$$[G_{ai}, G_{bj}] = \delta_{ij}F_{ab} + \delta_{ab}H_{ij}. \tag{25}$$

None-Zero Brackets for $o(3, 2)$

In the case of $o(3, 2)$ we obtain the following non-zero brackets:

$$\begin{aligned}
&[e_1, e_2] = -e_3, [e_1, e_3] = e_2, [e_1, e_4] = -e_6, [e_1, e_5] = -e_7,\\
&[e_1, e_6] = e_4, [e_1, e_7] = e_5, [e_2, e_3] = -e_1, [e_2, e_4] = -e_8,\\
&[e_2, e_9] = e_5, [e_3, e_6] = -e_8, [e_3, e_7] = -e_9, [e_3, e_8] = e_6,\\
&[e_3, e_9] = e_7, [e_4, e_5] = e_{10}, [e_4, e_6] = e_1, [e_4, e_8] = e_2,\\
&[e_4, e_{10}] = e_5, [e_4, e_8] = e_2, [e_4, e_{10}] = e_5, [e_5, e_7] = e_1,\\
&[e_5, e_9] = e_2, [e_5, e_{10}] = -e_4, [e_6, e_7] = e_{10}, [e_6, e_8] = e_3,\\
&[e_6, e_{10}] = e_7, [e_7, e_9] = e_3, [e_7, e_{10}] = -e_6, [e_8, e_9] = e_{10},\\
&[e_8, e_{10}] = e_9, [e_9, e_{10}] = -e_8,
\end{aligned} \tag{26}$$

where, in accordance with previous definitions, $e_1 = F_{12}, e_2 = F_{13}, e_3 = F_{23}, e_4 = G_{14}, e_5 = G_{15}, e_6 = G_{24}, e_7 = G_{25}, e_8 = G_{34}, e_9 = G_{35}, e_{10} = H_{12}$.

Identifying the algebra in dimension n

In order to identify the algebra in general, where the brackets are given by 17, it is convenient not to adhere strictly to the canonical ordering that gives the brackets in 21-25. Instead we shall use the following $(n+3) \times (n+3)$ matrix:

$$\begin{bmatrix} R & A & E & B \\ -A^t & 0 & \Delta & F \\ E^t & \Delta & 0 & C \\ B^t & F & -C & 0 \end{bmatrix} \tag{27}$$

where R is $n \times n$ and $A, B, E \in \mathbb{R}^n$ and $C, \Delta, F \in \mathbb{R}$. Each of A, B, E, C, Δ, F are arbitrary and the only restriction on R is that it should be skew-symmetric. The idea is that the space R^{ij} of vector fields in 10 is made to correspond bi-uniquely to the submatrix R giving the standard representation of $o(n)$. Likewise, the spaces of vector fields A^i, B^i, E^i correspond to the partial rows and columns in 27 and the single vector fields Δ, C, F to the pair of entries of the same name in 27. As such, it is straightforward, although tedious, to check that one obtains precisely the same brackets from the matrix algebra in 27 as one does for the corresponding vector fields in 17. The subspace spanned by R and A gives the required subalgebra that is isomorphic to (n) and similarly the subspace spanned by C gives the required subalgebra that is isomorphic to $o(2)$. The conclusion is that the symmetry algebra of 1 is isomorphic to $o(n+1, 2)$.

One-dimensional subalgebras of $o(3, 2)$

one-dim	Eigenvalues	Comment
$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & -b & 0 \end{bmatrix}$	$\pm ia, \pm ib, 0\,(a > 0, b > 0)$	semi-simple
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \end{bmatrix}$	$\pm a, \pm b, 0\,(a^2 + b^2 \neq 0)$	semi-simple
$\begin{bmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\pm a, 0, 0, 0$	nilp. part of index 3
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & a & 0 \\ 0 & -b & 0 & 0 & a \\ 0 & a & 0 & 0 & b \\ 0 & 0 & a & -b & 0 \end{bmatrix}$	$\pm a \pm ib, 0\,(b > 0)$	semi-simple
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha-1 & 1 & 0 \\ 0 & -\alpha+1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \alpha+1 \\ 0 & 0 & 1 & -\alpha-1 & 0 \end{bmatrix}$	$\pm i\alpha, \pm i\alpha, 0$	nilpotent part index 2
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & a \\ 0 & 1 & 0 & a & 1 \\ 0 & 1 & a & 0 & 1 \\ 0 & a & 1 & -1 & 0 \end{bmatrix}$	$\pm a, \pm a, 0$	nilpotent part index 2
$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & b & -b & 0 \end{bmatrix}$	$\pm ia, 0, 0, 0,\ (a > 0)$	nilpotent part index 3
$\begin{bmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\pm ia, \pm b, 0$	semi-simple
$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -2 & -1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$	$0, 0, 0, 0, 0$	nilpotent of index 5

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