Integrability in Super Chern-Simons (=ABJM) Matrix Model

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ABJM Matrix Model (originally from Localication, here as Definition)

[Aharony-Bergman-Jafferis-Maldacena 2008, Hosomichi-Lee-Lee-Lee-Park 2008, Aharony-Bergman-Jafferis 2008, Kapustin-Willett-Yaakov 2009]

$$\left\langle s_\lambda \right\rangle_k (N_1 \mid N_2) = \frac{i^{-\frac{1}{2}\left(N_1^2-N_2^2\right)}}{N_1!N_2!}\int \prod_{m=1}^{N_1}\frac{d\mu_m}{2\pi}\prod_{n=1}^{N_2}\frac{d\nu_n}{2\pi}$$

$$\frac{\prod_{m<m'}^{N_1}\left(2\sinh\frac{\mu_m-\mu_{m'}}{2}\right)^2\prod_{n<n'}^{N_2}\left(2\sinh\frac{\nu_n-\nu_{n'}}{2}\right)^2}{\prod_{m=1}^{N_1}\prod_{n=1}^{N_2}\left(2\cosh\frac{\mu_m-\nu_n}{2}\right)^2}$$

$$\exp\frac{ik}{4\pi}\left(\sum_{m=1}^{N_1}\mu_m^2-\sum_{n=1}^{N_2}\nu_n^2\right)$$

Super Schur

$$s_\lambda\left(e^\mu \mid e^\nu\right)$$

$$\prod_{m<m'}^{N_1}\left(2\sinh\frac{\mu_m-\mu_{m'}}{2}\right)^2$$

Trigonometric Deformation

$\mu \to \sinh \mu$

$$\frac{\prod_{m<m'}^{N_1}\left(\mu_m-\mu_{m'}\right)^2\prod_{n<n'}^{N_2}\left(\nu_n-\nu_{n'}\right)^2}{\prod_{m=1}^{N_1}\prod_{n=1}^{N_2}\left(\mu_m+\nu_n\right)^2}$$

Supergroup Deformation

$U(N_1) \to U(N_1|N_2)$

$$\exp\frac{-k}{4\pi}\sum_{m=1}^{N_1}\mu_m^2$$

Gaussian

$$s_\lambda\left(e^\mu\right) = s_\lambda\left(e^{\mu_1},\ldots,e^{\mu_{N_1}}\right)$$

Schur Poly.

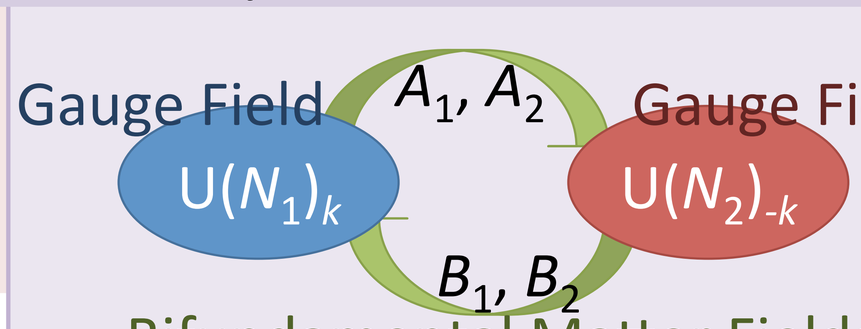
$$\prod_{m<m'}^{N_1}\left(\mu_m-\mu_{m'}\right)^2$$

Grand Canonical ABJM Matrix Model

[Marino-Putrov 2012]

$$\left\langle s_\lambda \right\rangle_{k,M}^{GC}(z) = \sum_{N=\max(0,-M)}^{\infty} z^N \left\langle s_\lambda \right\rangle_k (N \mid N+M)$$

ABJM Theory = $\mathcal{N}$=6 Chern-Simons Theory



ABJM is Integrable !!

$\min(N_1,N_2)$ x M2 + $|N_2-N_1|$ x fractional M2
& Wilson Loop on C^4 / Z_k

$$\left\langle\left\langle s_\lambda \right\rangle\right\rangle_{k,M}^{O} = \left\langle s_\lambda \right\rangle_{k,M}^{GC}(z) / \left\langle 1 \right\rangle_{k,M}^{GC}(z)$$ (Open Normalization)

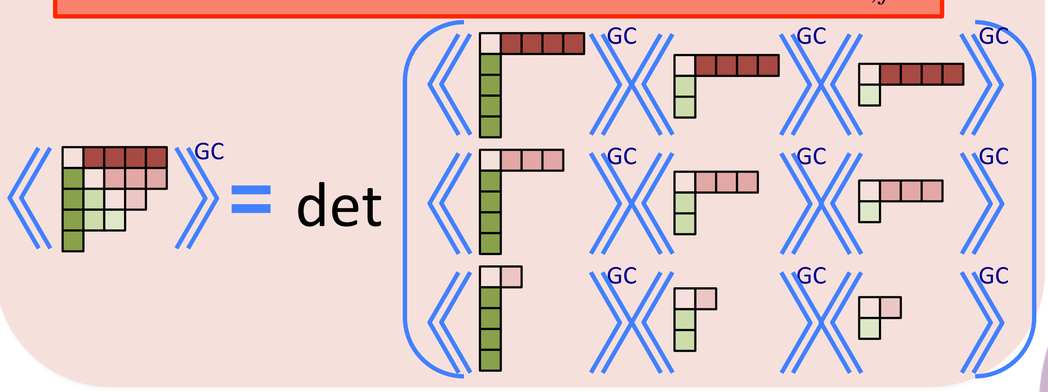
Lowest Component [Kubo-M 2018]

$$\left\langle s_\lambda \right\rangle_k (R, M+R) = \frac{1}{k^{R+\frac{M}{2}}}\frac{\prod_{a,a'\in A}2\sin\frac{\pi(a-a')}{k}\prod_{l,l'\in L}2\sin\frac{\pi(l-l')}{k}}{\prod_{a\in A}\prod_{l\in L}2\cos\frac{\pi(a+l)}{k}}$$

Giambelli Identity

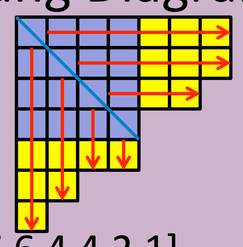
[Hatsuda-Honda-M-Okuyama 2013, Matsuno-M 2016]

$$\left\langle\left\langle s_{(a_1,\ldots,a_R\mid l_1,\ldots,l_R)}\right\rangle\right\rangle_{k,M}^{O} = \det\left(\left\langle\left\langle s_{(a_i\mid l_j)}\right\rangle\right\rangle_{k,M}^{O}\right)_{1\le i,j\le R}$$



$$\left\langle\left\langle s_Y \right\rangle\right\rangle_{k,M}^{C} = \left\langle\left\langle s_\lambda \right\rangle\right\rangle_{k,M}^{GC}(z) / \left\langle s_\lambda \right\rangle_k (R \mid M+R)$$ (Closed Norm.)

Notations (Young Diagram)

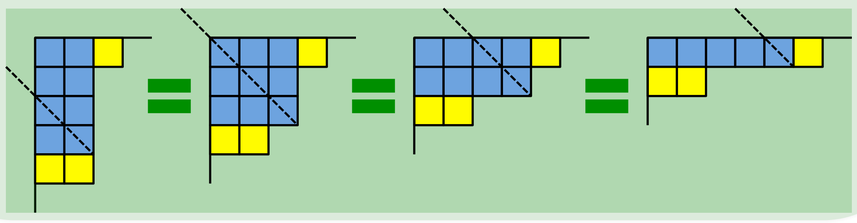


Y = [7,7,6,4,4,2,1]
= (6,5,3,0|6,4,2,1)
= ([4,4,4,4]+[3,3,2])
∪[4,2,1]

Open-Closed Duality

[Hatsuda-Okuyama 2016, Kiyoshige-M 2016]

$$\left\langle\left\langle s_{\left(\left[(M+R)^R\right]+Y\right)\cup Y'}\right\rangle\right\rangle_{k,M}^{C} = \left\langle\left\langle s_{\left(\left[(M+R+1)^{R-1}\right]+Y\right)\cup Y'}\right\rangle\right\rangle_{k,M+2}^{C}$$

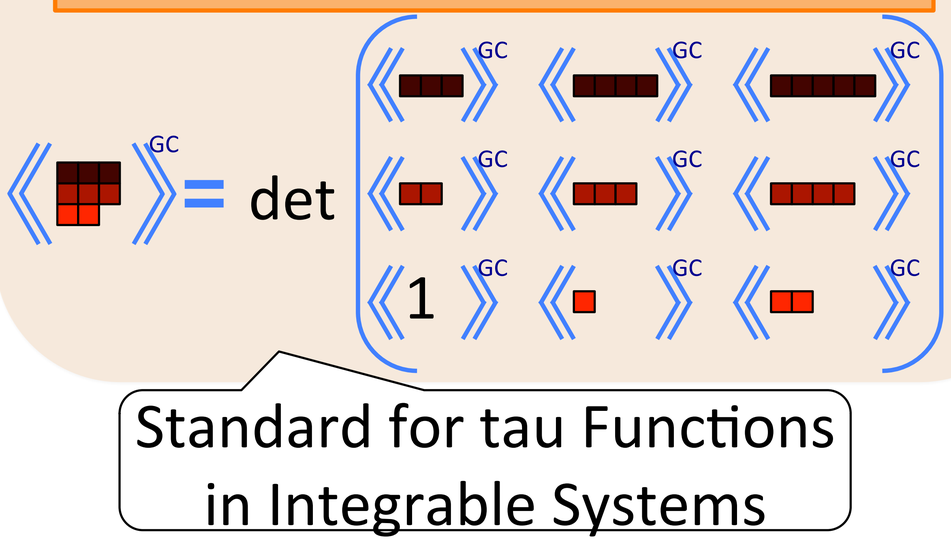


Quantum Jacobi-Trudi Id.

(9^{th} Variation of Schur Polynomial)

[Matsumoto-M 2013, Furukawa-M 2017]

$$\left\langle\left\langle s_{[\lambda_1,\ldots,\lambda_L]}\right\rangle\right\rangle_{k,M}^{O} = \det\left(\left\langle\left\langle s_{[\lambda_i-i+j]}\right\rangle\right\rangle_{k,M+j-1}^{O}\right)_{1\le i,j\le L}$$



Standard for tau Functions in Integrable Systems

Many Interesting (but unproven) Relations

- Two-Point Function [Kubo-M 2017]
- Painleve Bilinear Equation [Bonelli-Grassi-Tanzini 2017]
- Wronskian Relation [Grassi-Hatsuda-Marino 2014]
- Exact Large N Expansion via Topological Strings [Marino-Putrov 2009, ... , Hatsuda-Marino-M-Okuyama 2013]