Miquel dynamics for circle patterns

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Includes joint works with:

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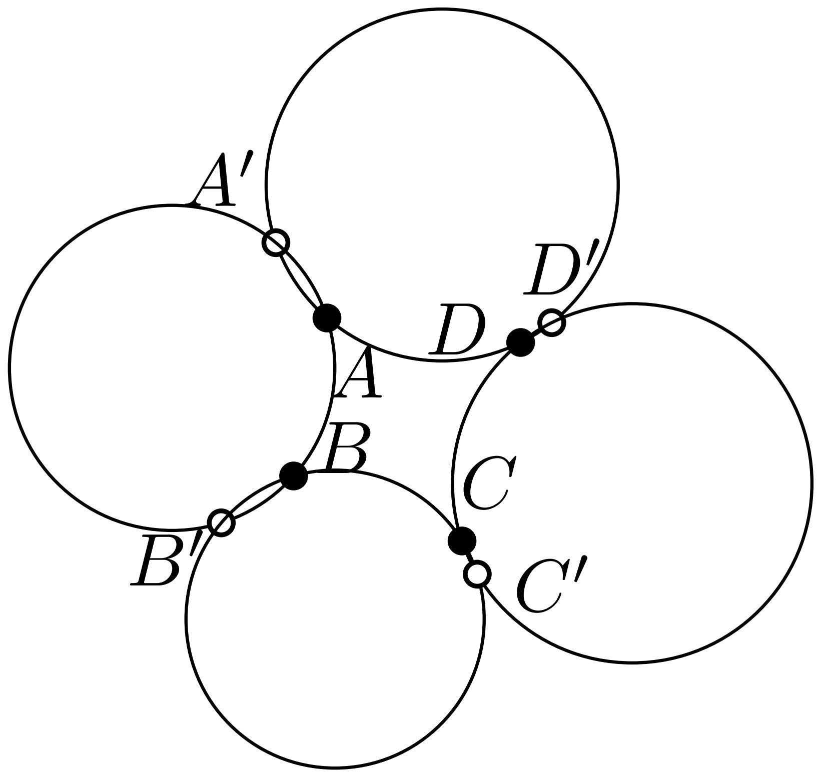
Fukuoka, November 15 2018

- Circle patterns form a well-studied class of objects in discrete differential geometry (discretization of conformal maps).

- Miquel dynamics: discrete integrable system on the space of circle patterns.

- Connection between circle patterns and the dimer model.

Miquel's theorem



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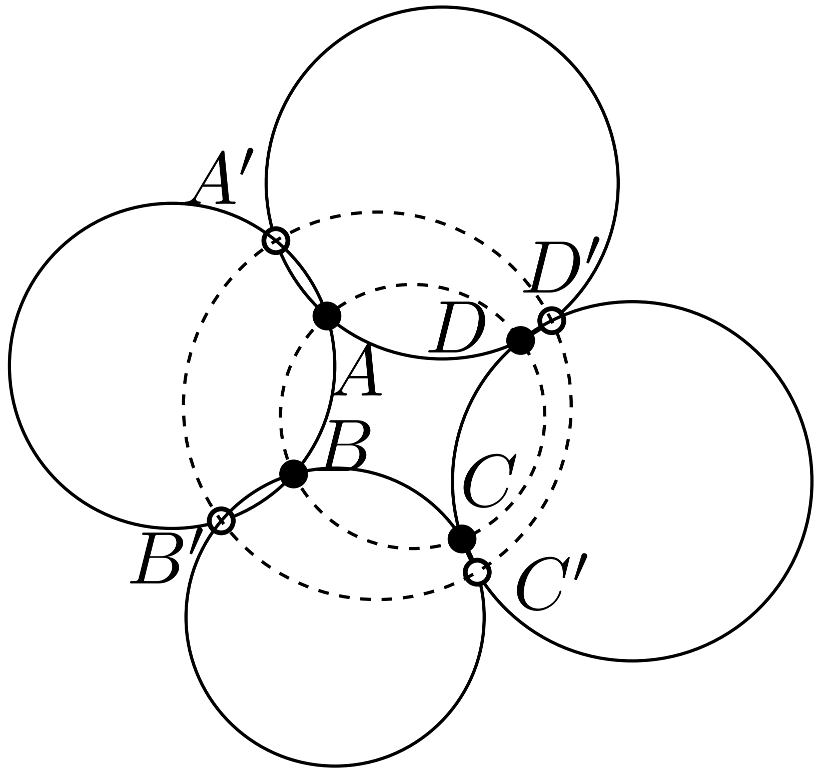
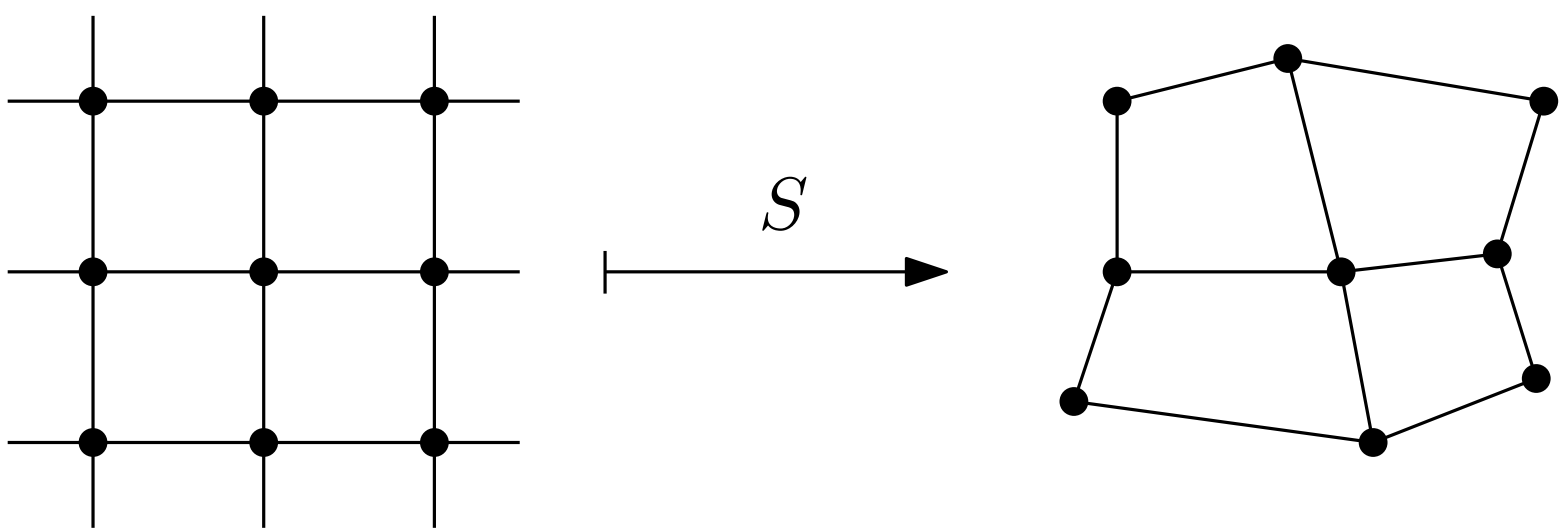


Theorem (Miquel, 1838)**.** *In this setting,*
A, B, C, D *concyclic* $\Leftrightarrow$ A', B', C', D' *concyclic.*

Square grid circle patterns

Definition. *A map* $S : \mathbb{Z}^2 \to \mathbb{R}^2$ *is called a* square grid circle pattern *if any four vertices around a face of* $\mathbb{Z}^2$ *get mapped to four concyclic points.*



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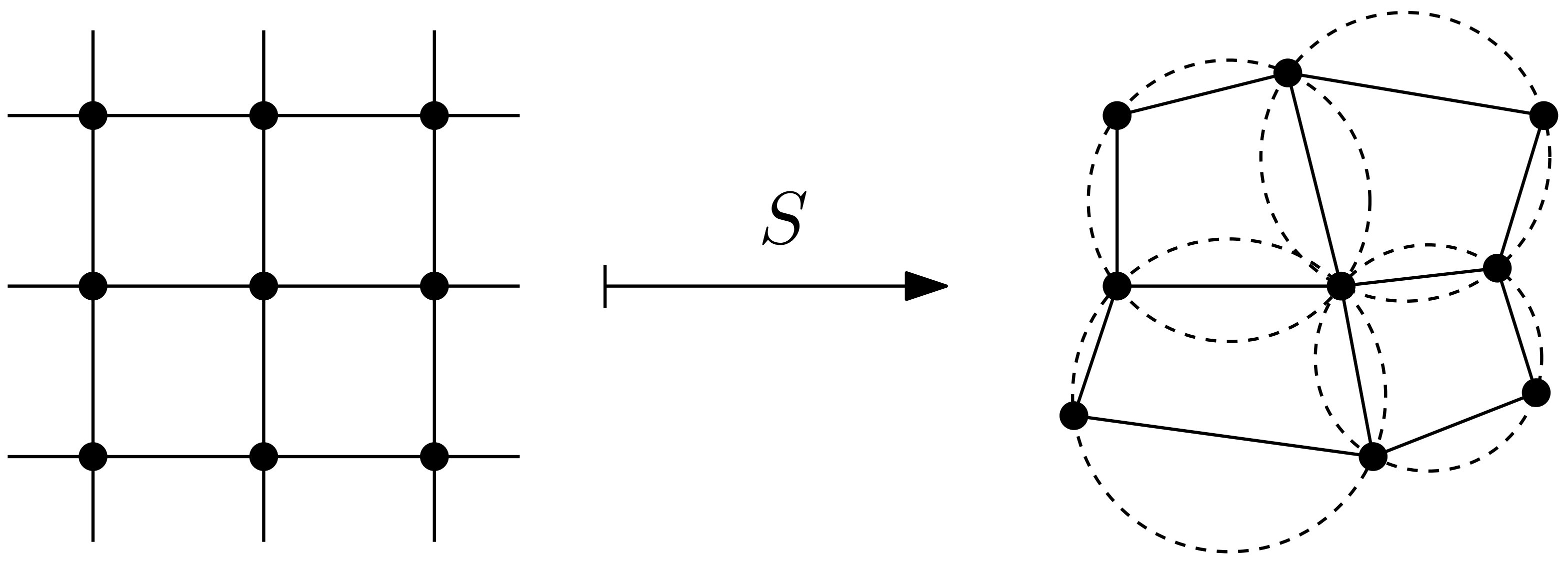
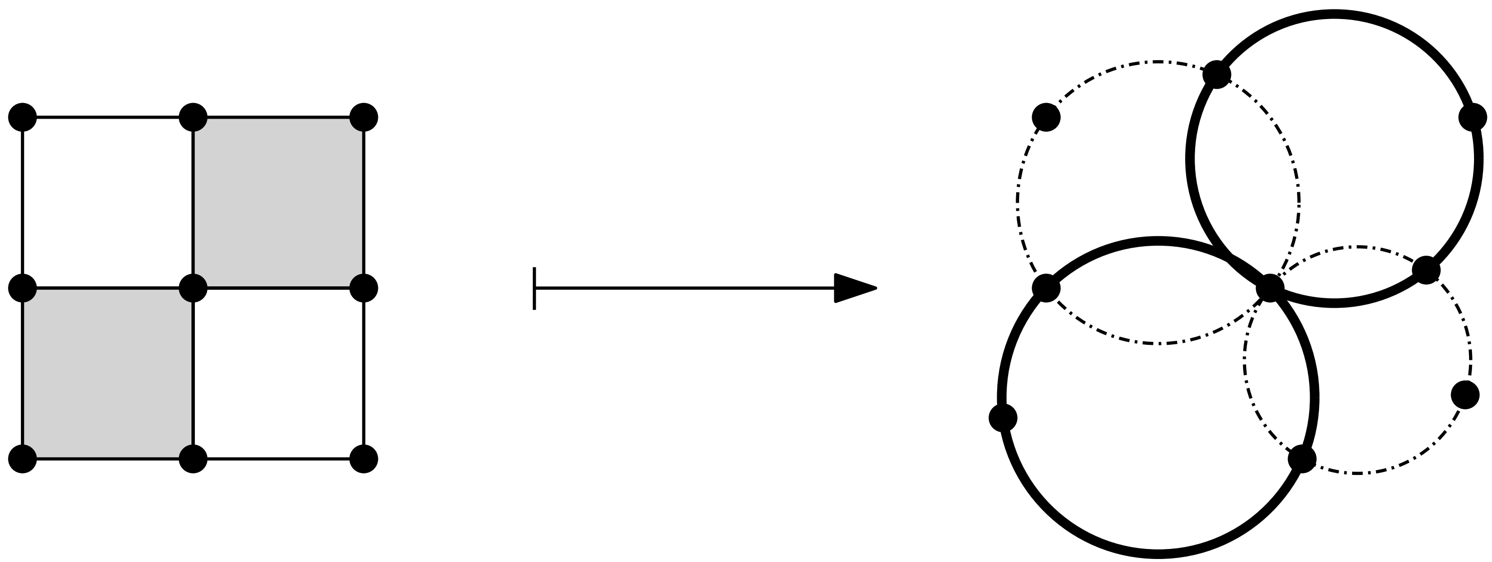


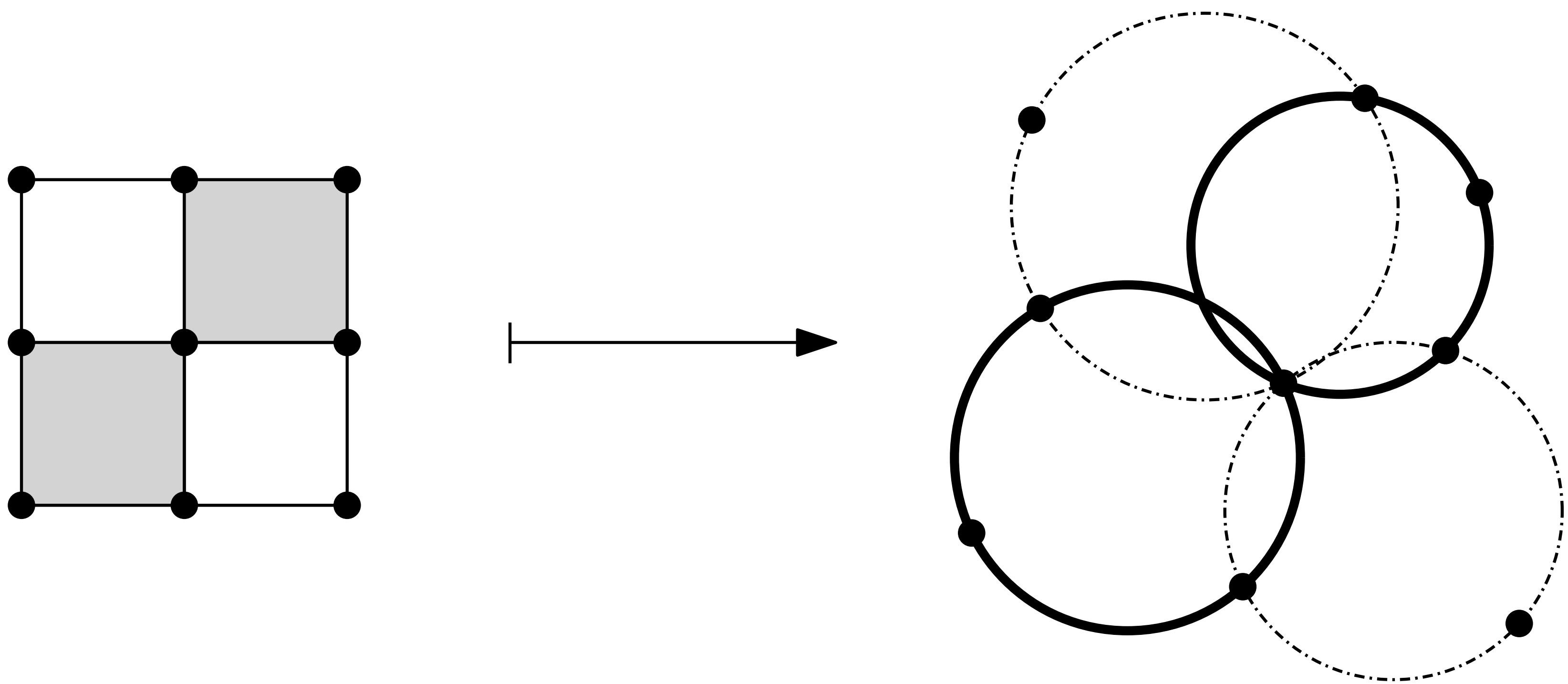
Miquel dynamics

- Checkerboard coloring of the faces of $\mathbb{Z}^2$: black and white circles.



- Define two maps from the set of square grid circle patterns to itself, black mutation μ_B and white mutation μ_W.

- Black mutation μ_B: each vertex gets moved to the other intersection point of the two white circles it belongs to. All the vertices move simultaneously.



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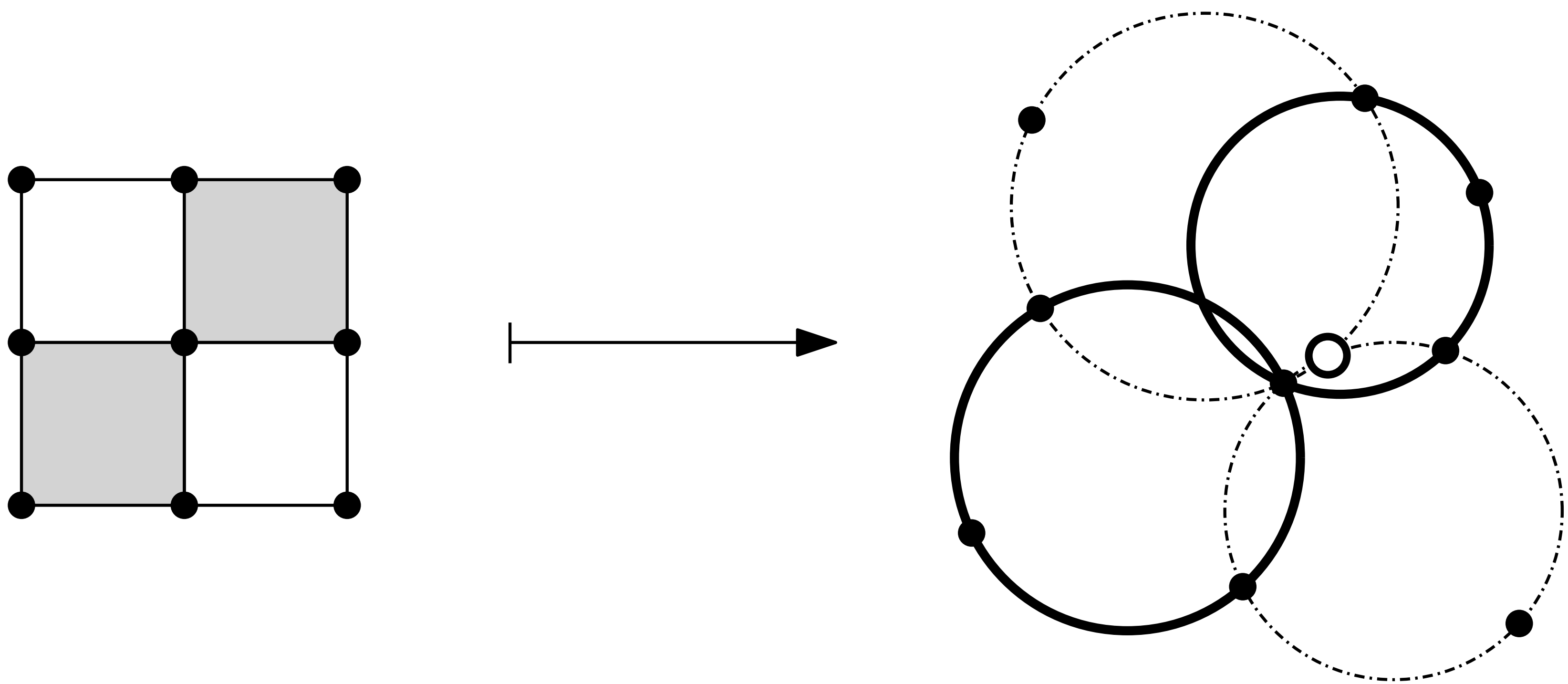

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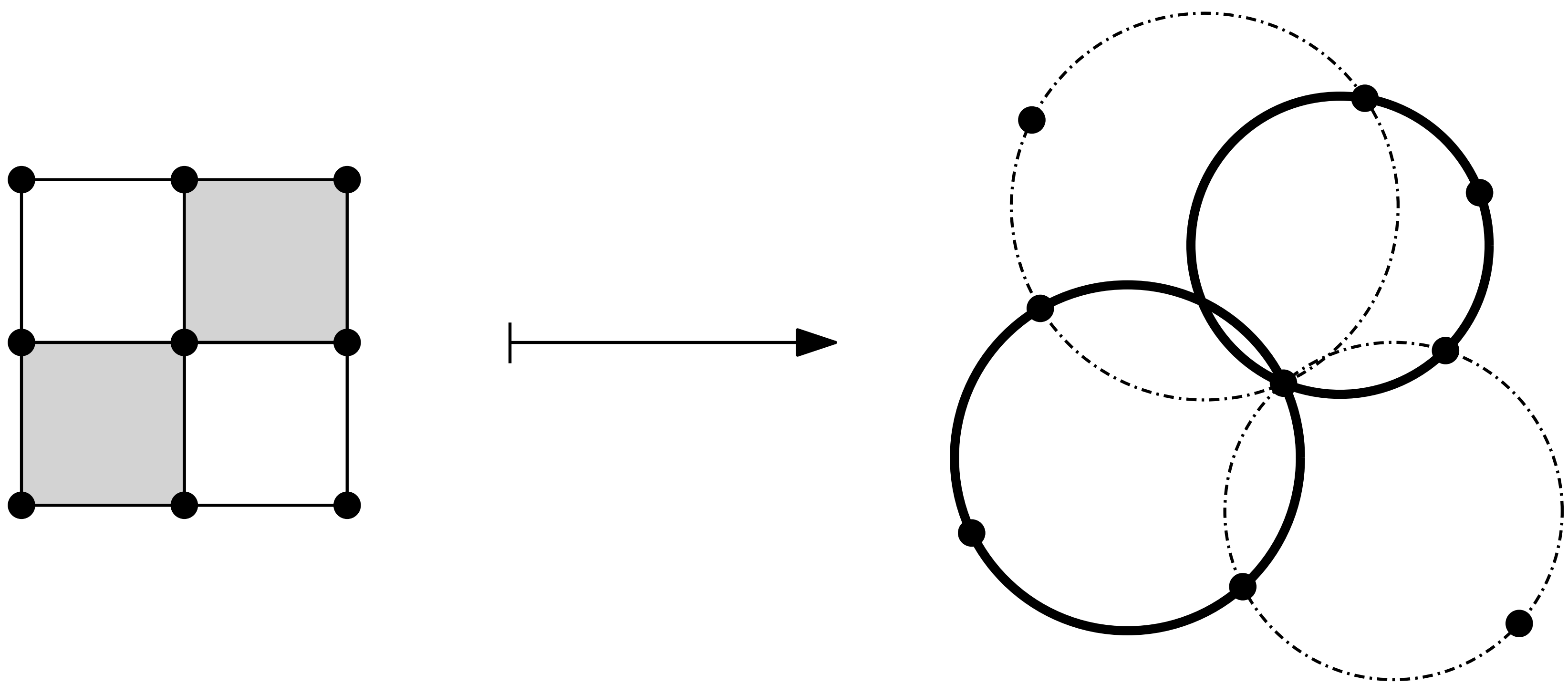

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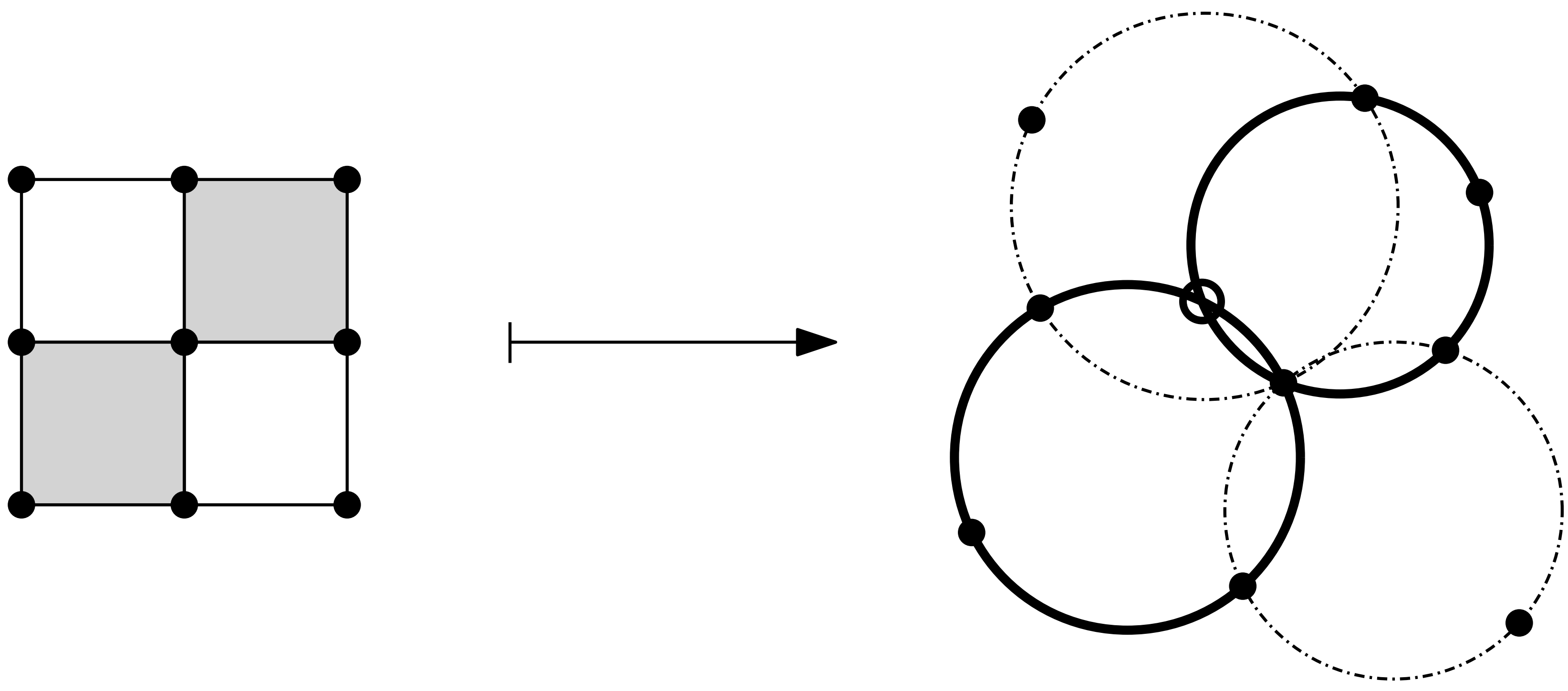

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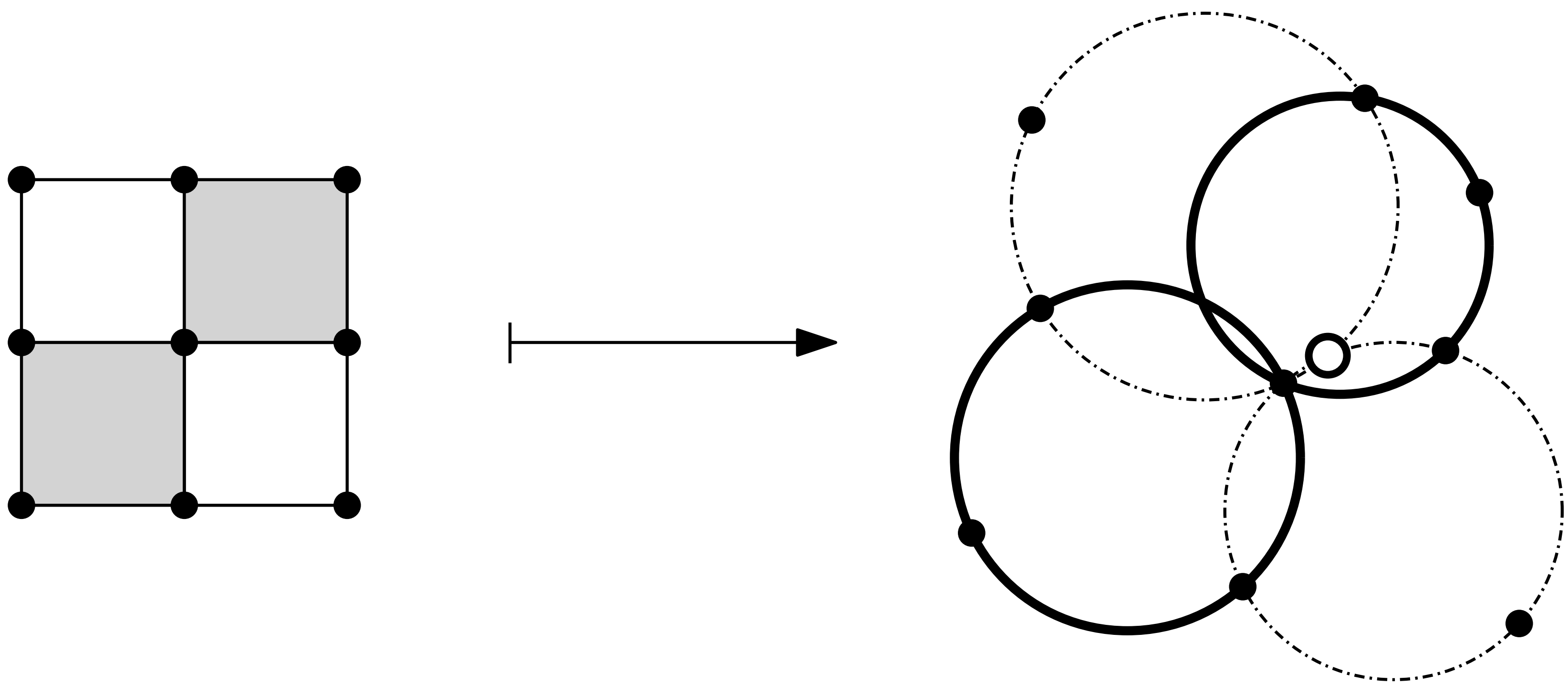

- Why does μ_B produce a square grid circle pattern ?

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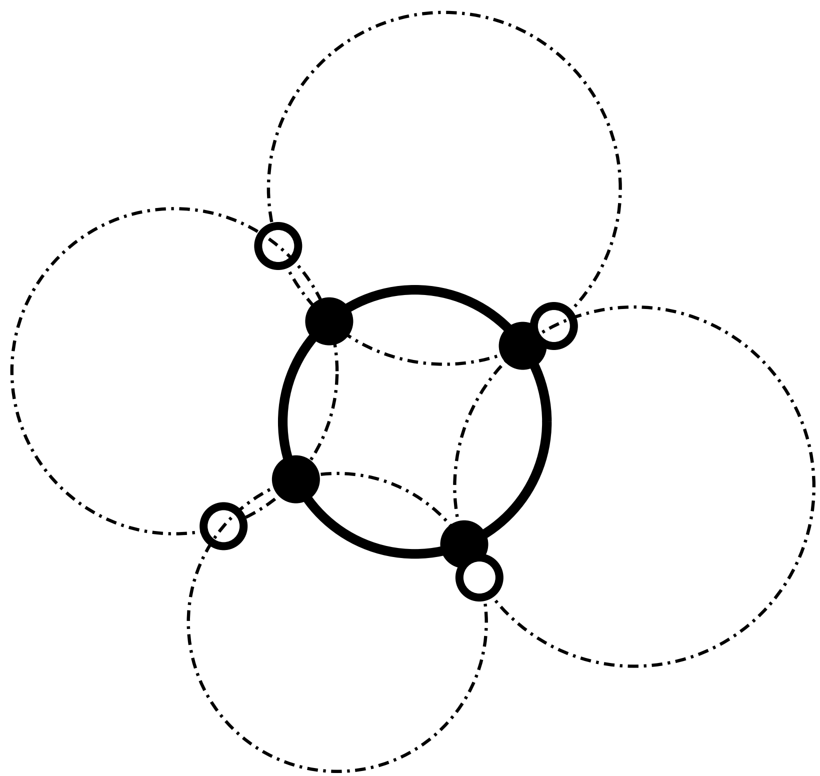

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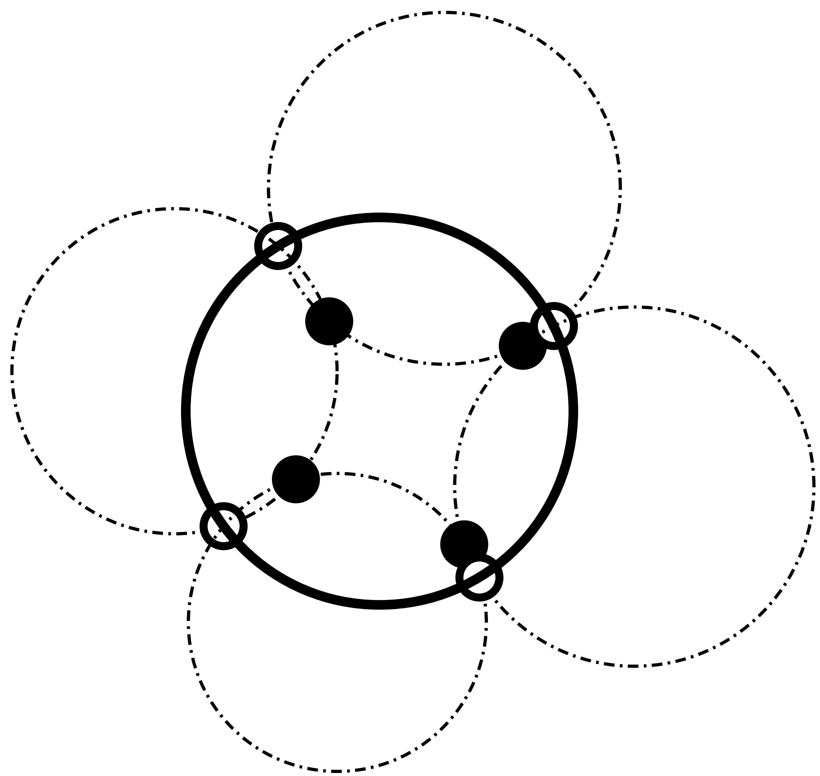


Miquel's theorem !

- Why does μ_B produce a square grid circle pattern ?

- The maps μ_B and μ_W are involutions.

- Miquel dynamics: discrete-time dynamics obtained by alternating between μ_B and μ_W.

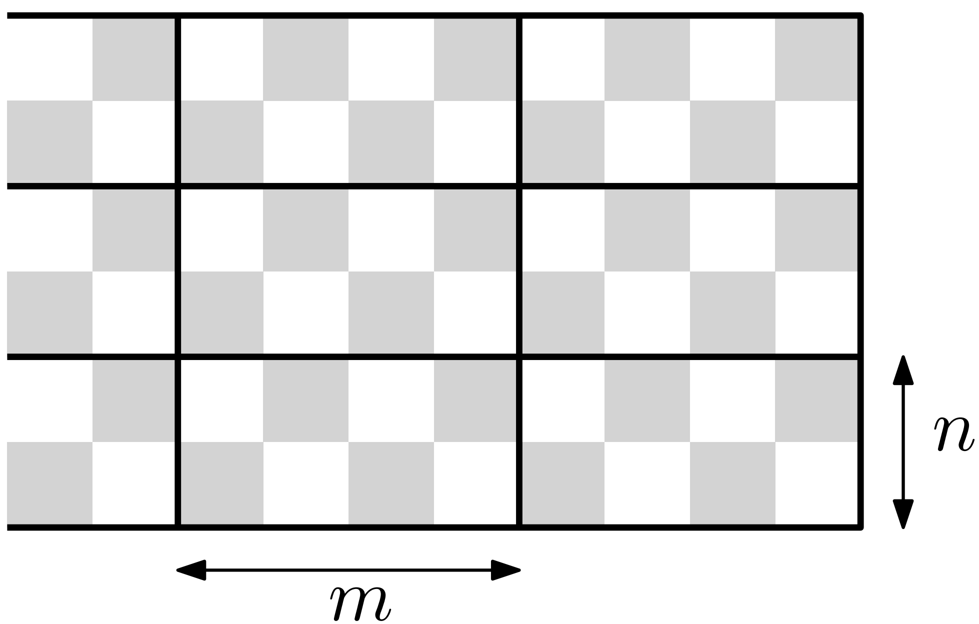
- Invented by Richard Kenyon.

- Resembles (but a priori unrelated to) the dynamics on circle configurations in three dimensions considered by Bazhanov-Mangazeev-Sergeev.

Biperiodic square grid circle patterns

- A circle pattern S is spatially biperiodic if there exist m, n integers and $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that for all $(x, y) \in \mathbb{R}^2$,

$$S(x+m, y) = S(x, y) + \vec{u}$$
$$S(x, y+n) = S(x, y) + \vec{v}$$



$m = 4$

$n = 2$

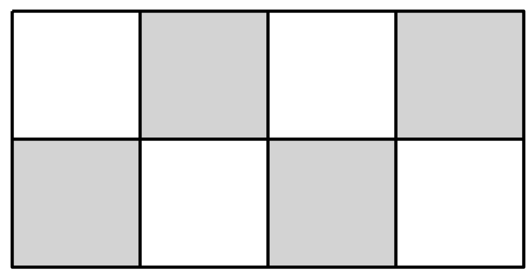
- A biperiodic circle pattern is mapped by Miquel dynamics to another biperiodic circle pattern with the same periods (m, n) and the same monodromies $(\vec{u}, \vec{v})$.

- This reduces the problem to a finite-dimensional one.

- A biperiodic circle pattern in the plane projects down to a circle pattern on a flat torus.

[Mathematica]

Towards integrability



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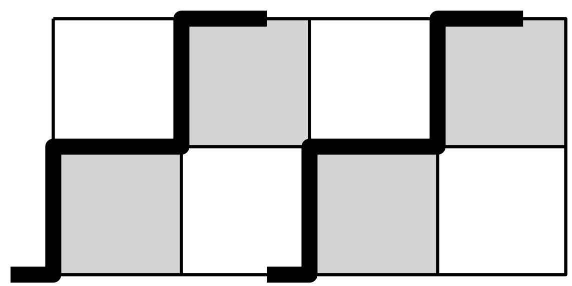
Theorem (R., 2018). *For $m \times n$ biperiodic SGCPs, the sum of the turning angles along a zigzag loop is invariant under Miquel dynamics.*

Theorem (Glutsyuk-R., 2018). *For 2×2 biperiodic SGCPs, the relative motion under Miquel dynamics of a vertex with respect to another corresponds to translation on an elliptic curve.*

Towards integrability



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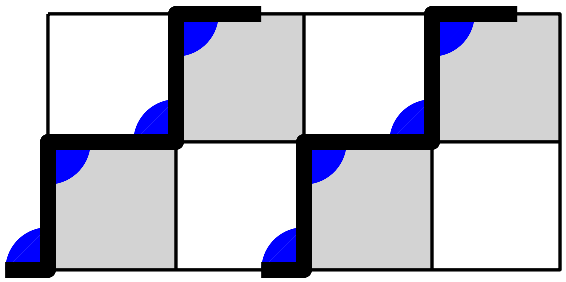

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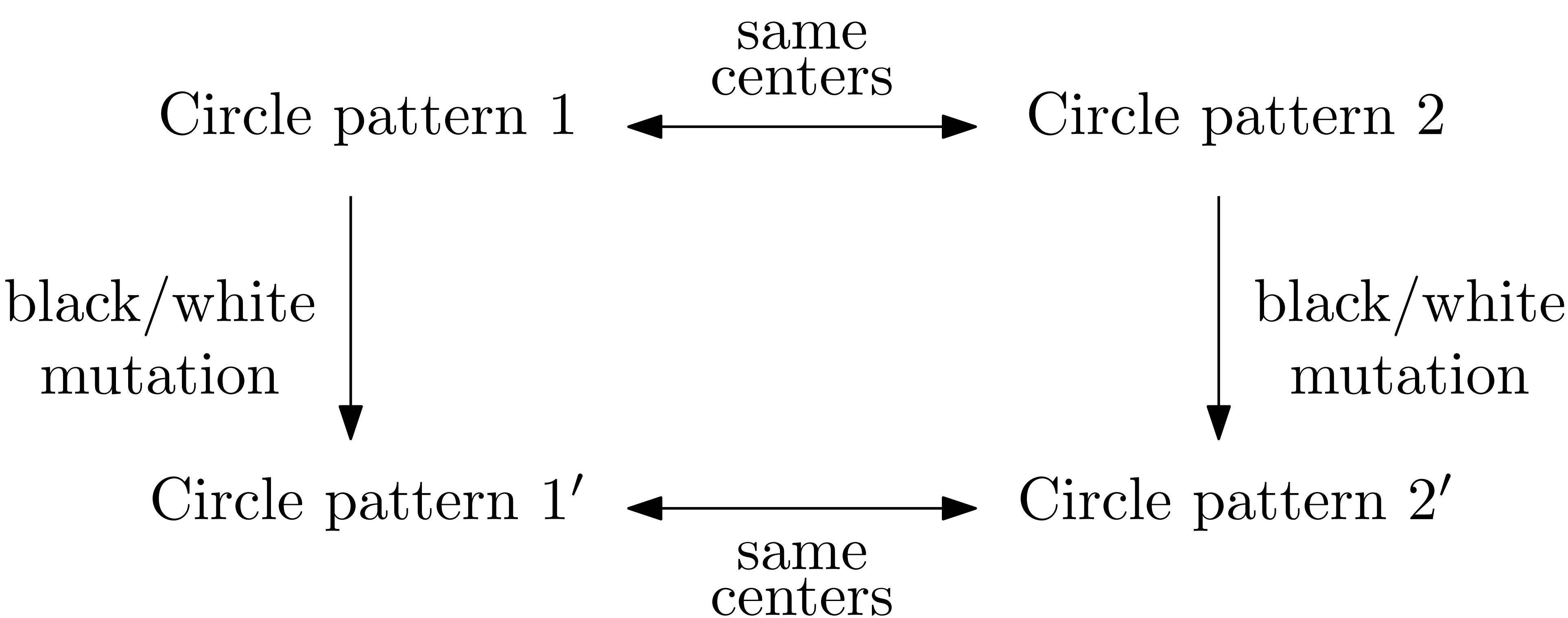
Key observation: study the evolution of circle centers rather than vertices (intersection points).

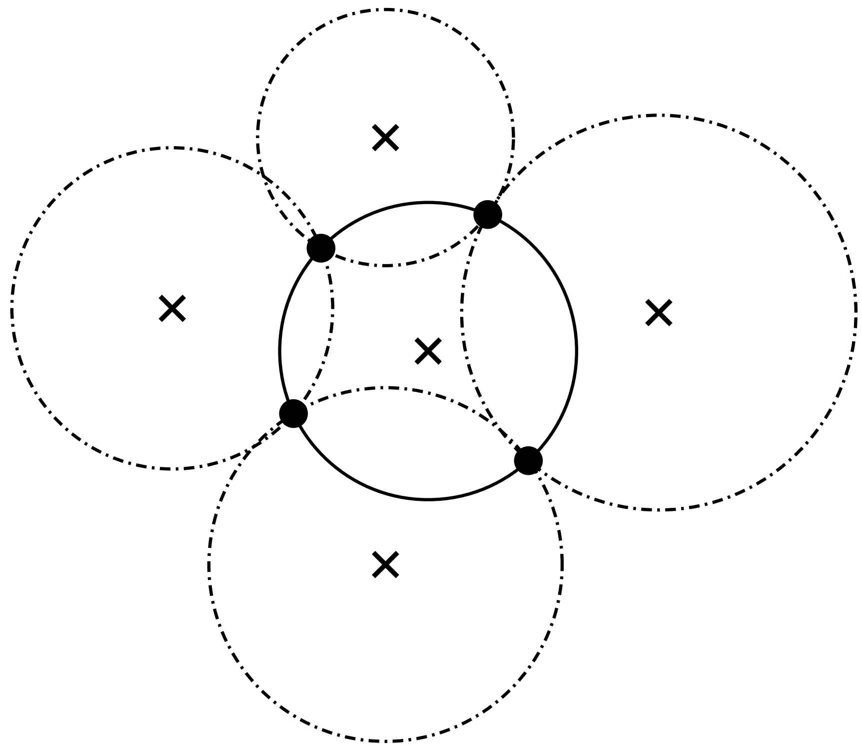
[Geogebra]

- Two-parameter family of circle patterns with given circle centers (pick one vertex freely).



- Miquel dynamics induces a dynamics on circle centers.

Miquel dynamics on centers



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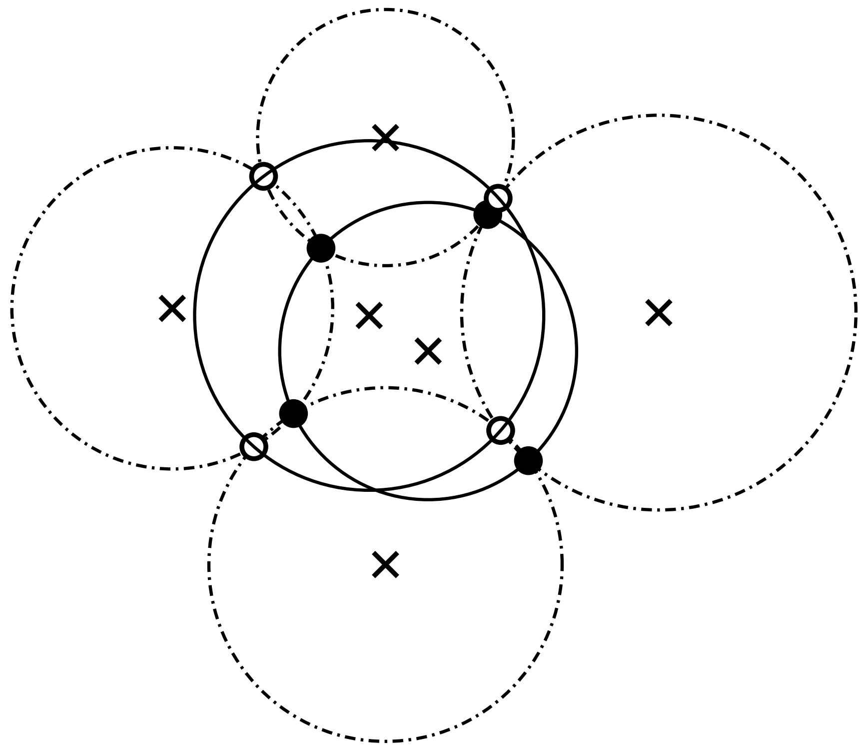

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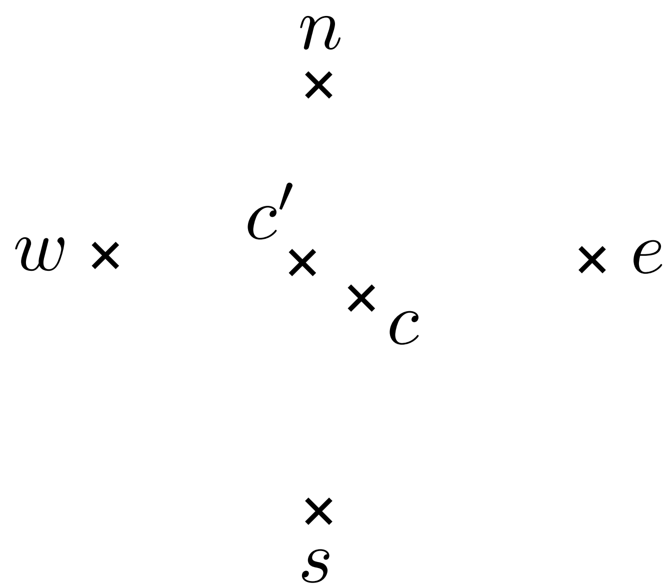
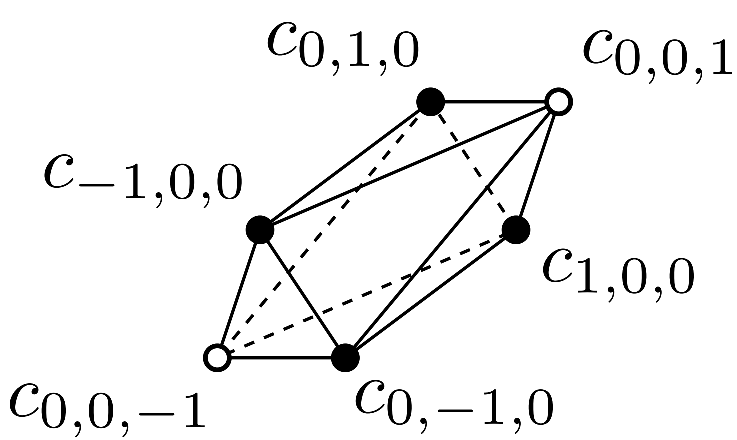


$n, w, s, e, c, c' \in \mathbb{C}$

Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh 2018)**.**

$$\frac{(c-w)(s-c')(e-n)}{(w-s)(c'-e)(n-c)} = -1$$

$$\frac{(c_{0,0,-1}-c_{-1,0,0})(c_{0,-1,0}-c_{0,0,1})(c_{1,0,0}-c_{0,1,0})}{(c_{-1,0,0}-c_{0,-1,0})(c_{0,0,1}-c_{1,0,0})(c_{0,1,0}-c_{0,0,-1})} = -1$$

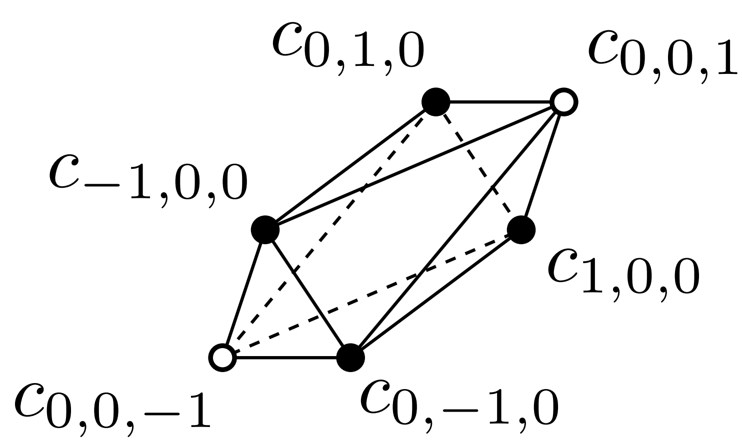


- Cauchy initial data: positions of the white centers at time -1 and of the black centers at time 0.

- Compute iteratively the positions of all centers using the above equation on each elementary octahedron.

The discrete Schwarzian KP equation

$$\frac{(c_{0,0,-1}-c_{-1,0,0})(c_{0,-1,0}-c_{0,0,1})(c_{1,0,0}-c_{0,1,0})}{(c_{-1,0,0}-c_{0,-1,0})(c_{0,0,1}-c_{1,0,0})(c_{0,1,0}-c_{0,0,-1})} = -1$$



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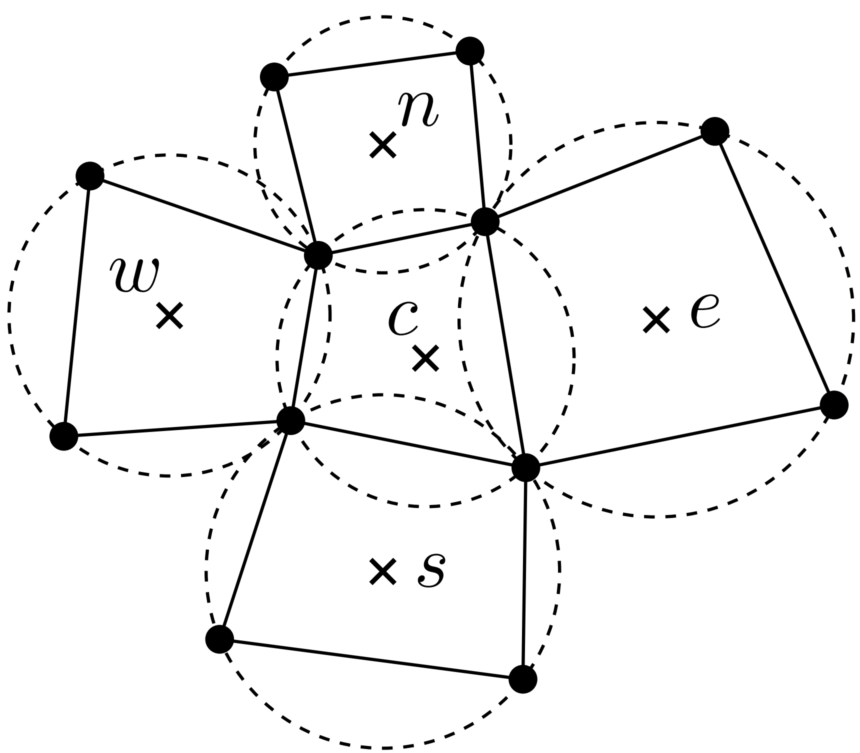
- Governs discrete Laplace-Darboux transformations (Doliwa) and the pentagram map (Schief).

- Geometric interpretation given by Clifford's circle theorem (Konopelchenko-Schief).

- Equation χ_2 in the Adler-Bobenko-Suris classification of integrable equations of octahedron type.

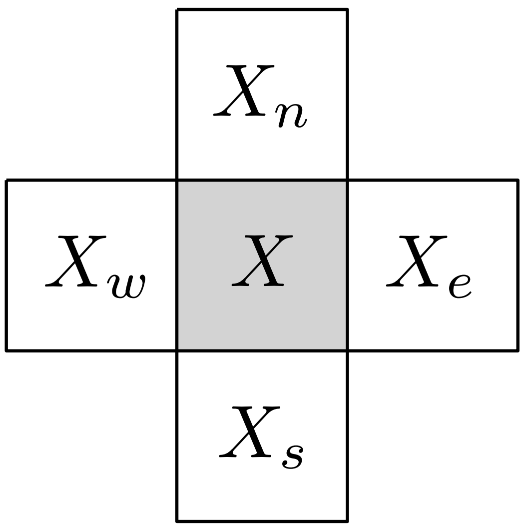
X variables for circle centers...

- Given centers of a circle pattern, associate an X variable to each face.



$$X = -\frac{(c-n)(c-s)}{(c-w)(c-e)}$$

… evolve like a Y-system



black mutation $\quad X \longmapsto X^{-1}$

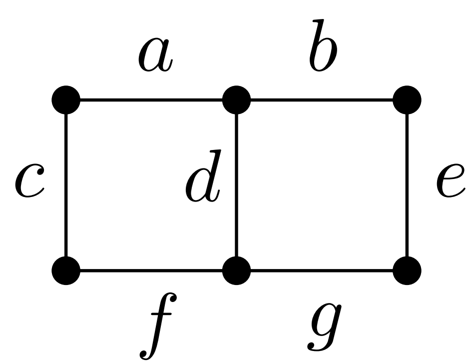
white mutation $\quad X \longmapsto X\frac{(1+X_n)(1+X_s)}{(1+X_w^{-1})(1+X_e^{-1})}$

- Transformation rule for coefficient variables in a cluster algebra.

The dimer model

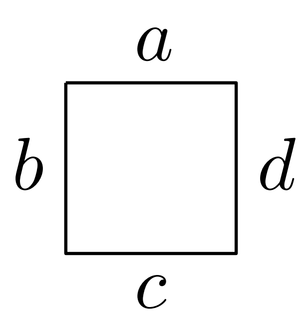
- Statistical mechanical model for random perfect matchings on graphs.

- Setting: planar graph with all faces of even degree. Each edge carries a positive real weight.



Dimer X variables

- Given a graph with edge weights, associate an X variable to each face by taking the alternating product of edge weights around the face.



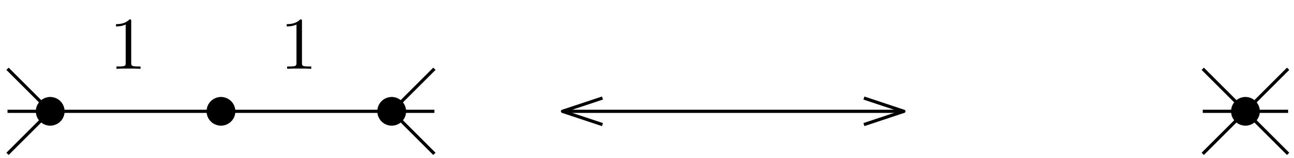
$$X = \frac{ac}{bd}$$

- If two collections of edge weights on a graph induce the same X variables on the faces, then the two collections of edge weights define the same probabilistic model.

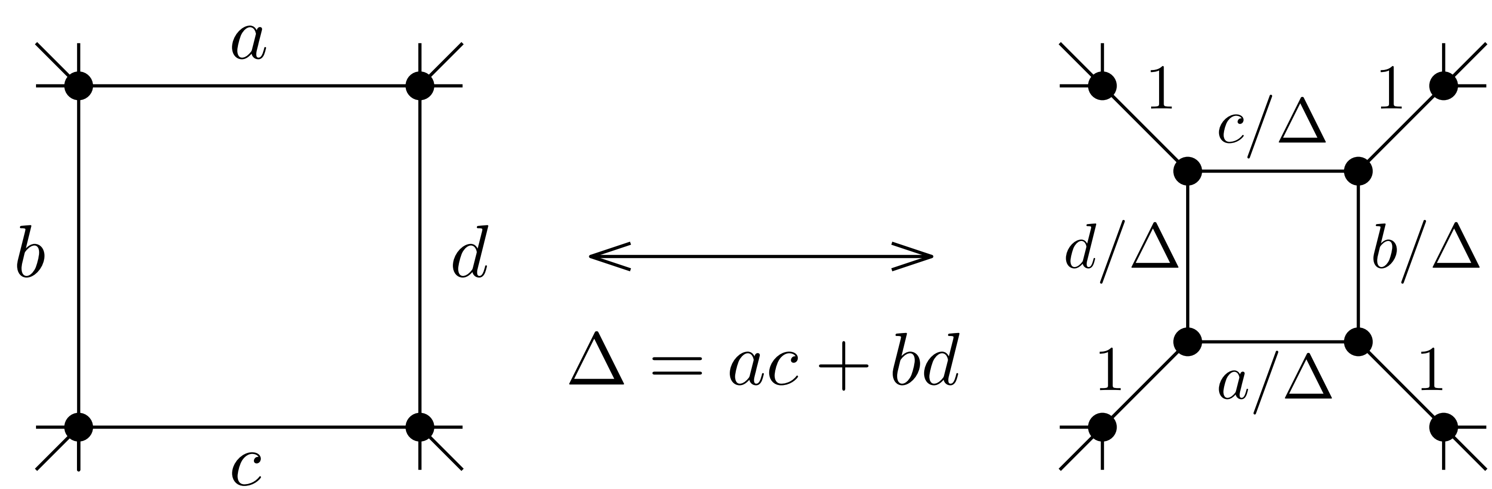
Dimer local moves

- Two local moves preserving the probabilistic model.

- Contraction of degree 2 vertices:

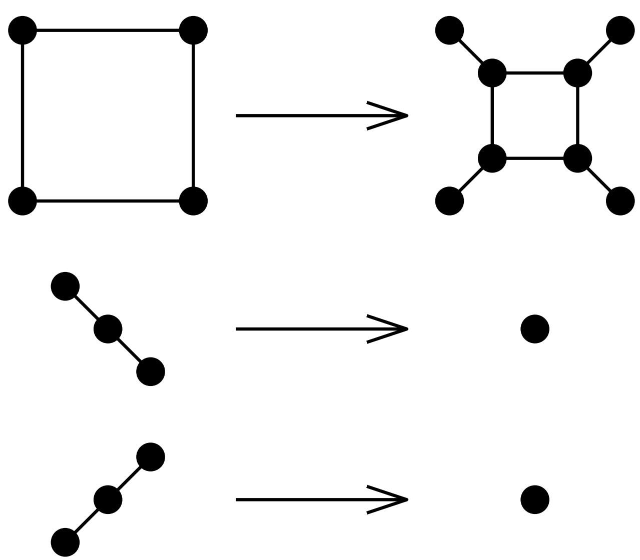


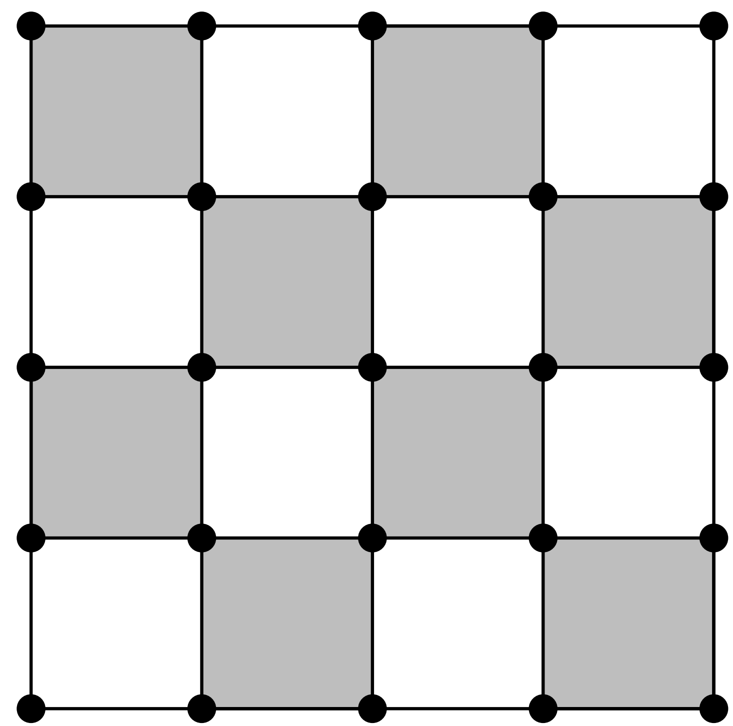
- Urban renewal:



Goncharov-Kenyon dimer dynamics

- Start with $\mathbb{Z}^2$ with some edge weights. At even (resp. odd) times, perform an urban renewal on each white (resp. black) face followed by the contraction of all the degree 2 vertices. Get $\mathbb{Z}^2$ with different edge weights.





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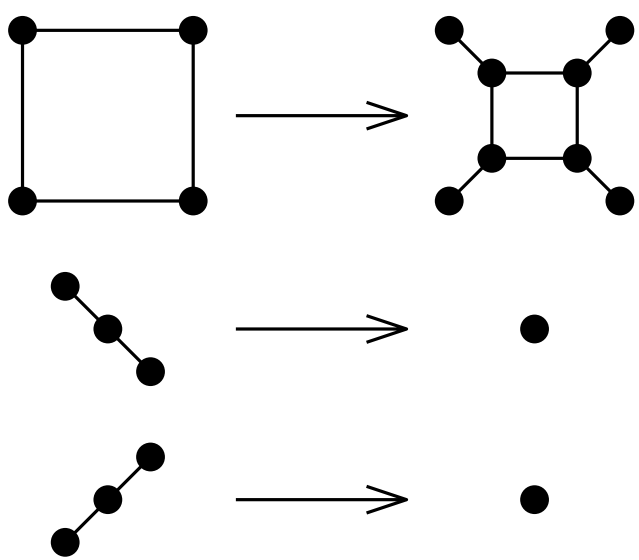

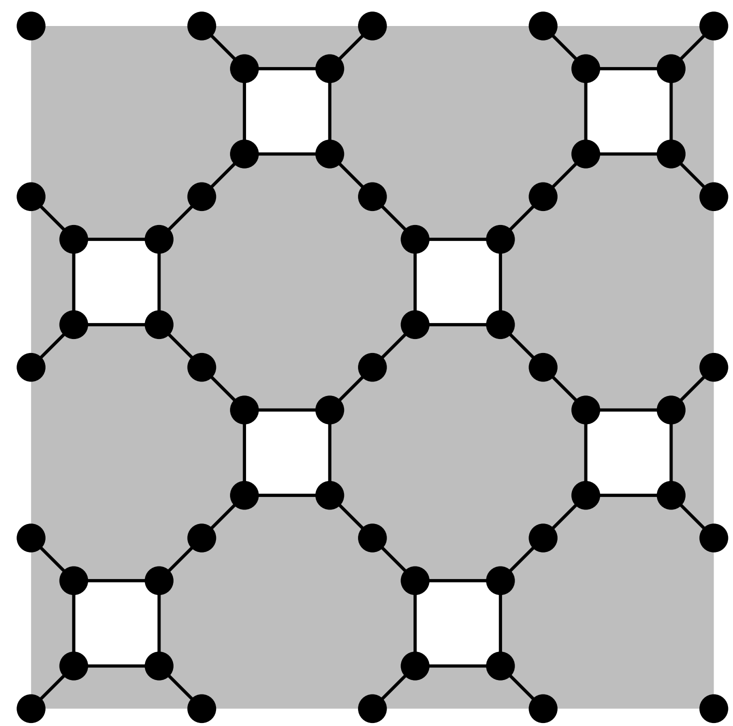

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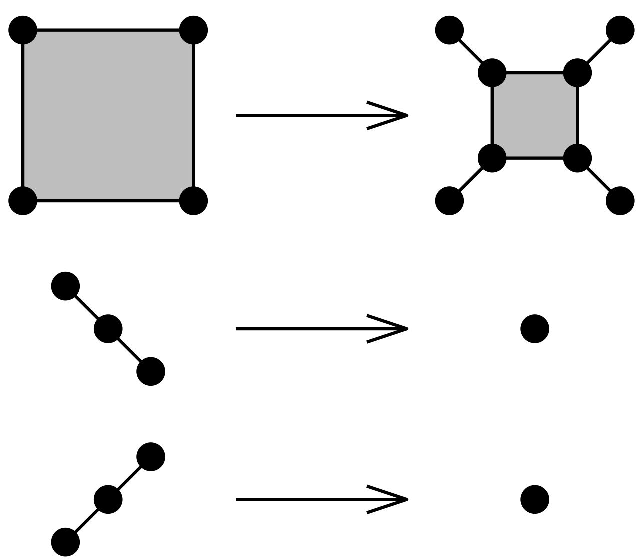

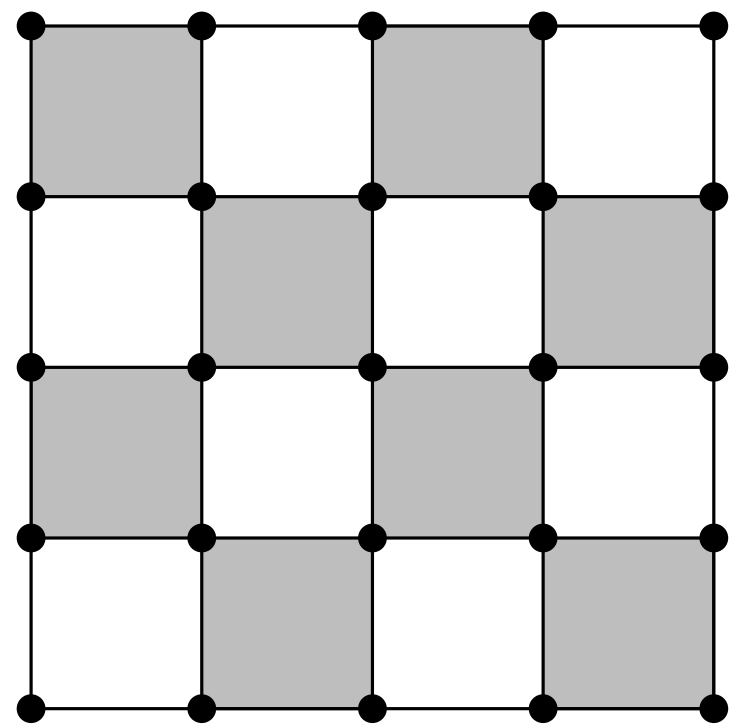

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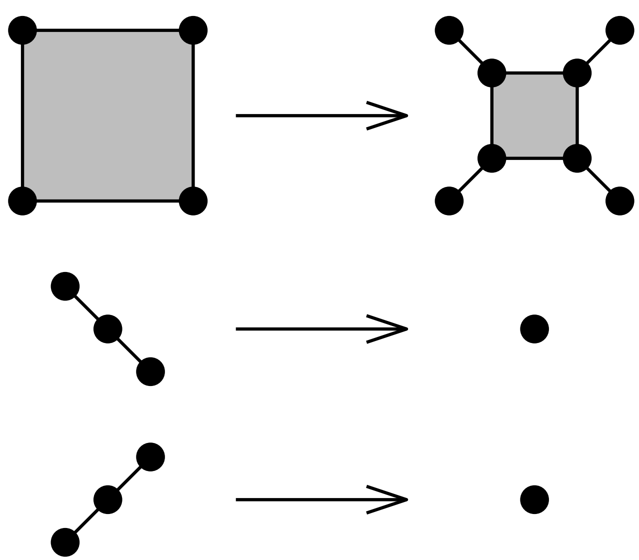

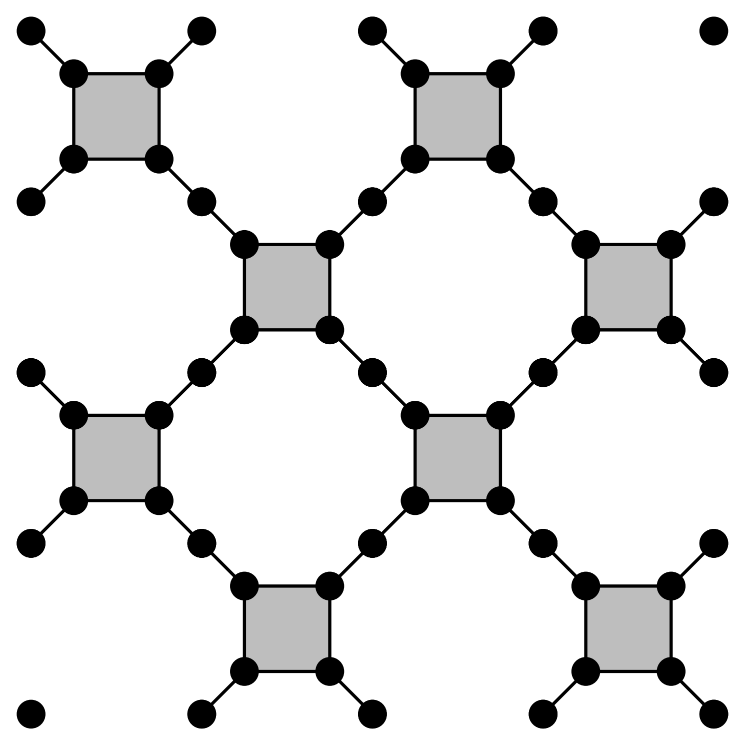

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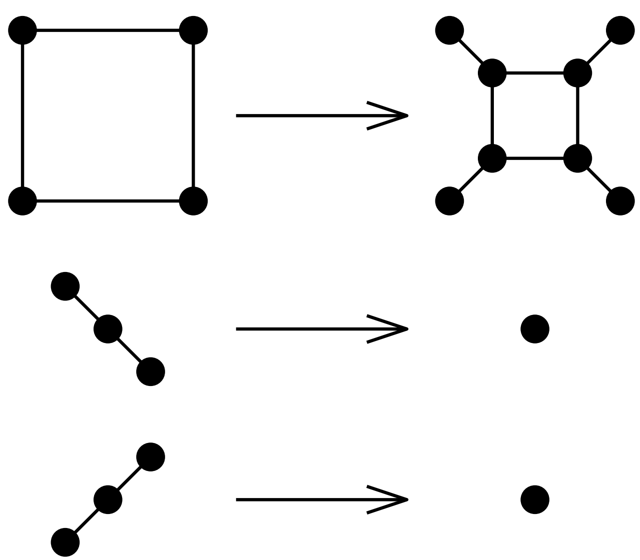

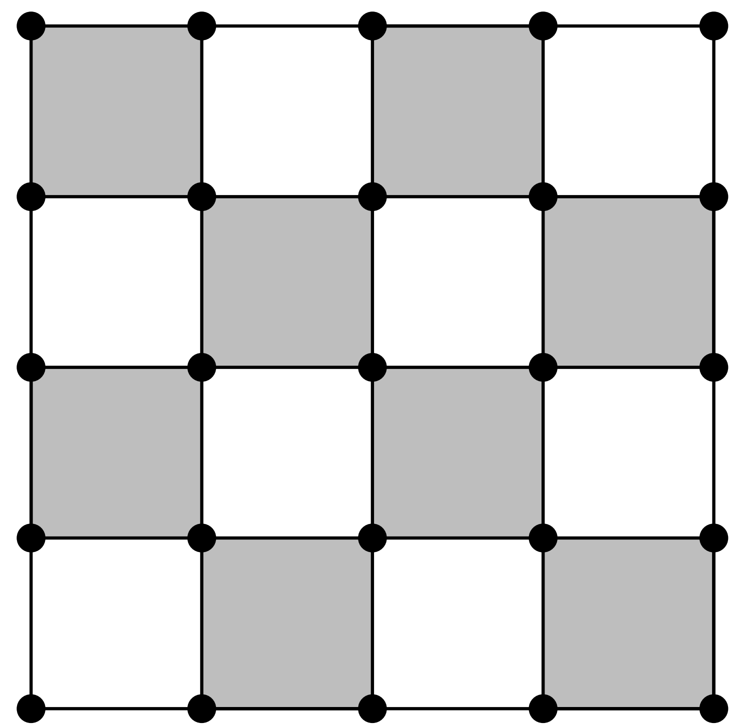

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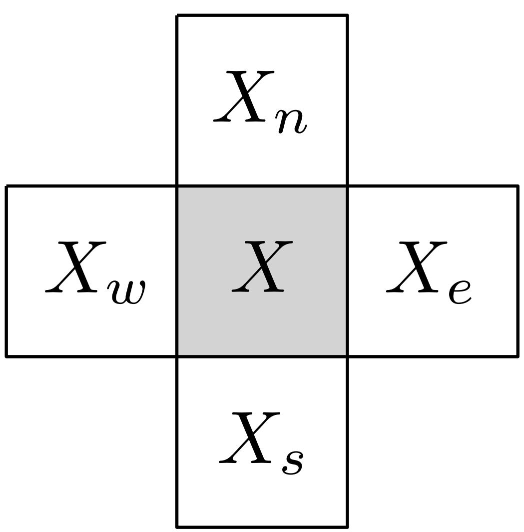
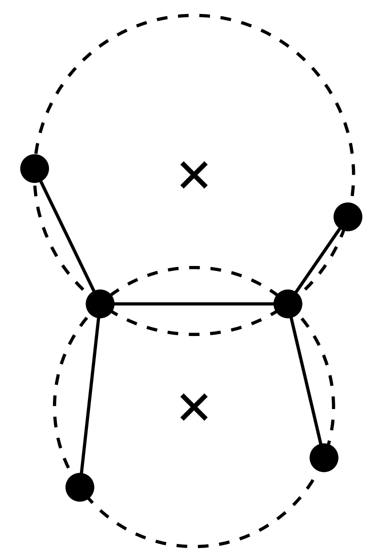


black urban renewals $$X \longmapsto X^{-1}$$

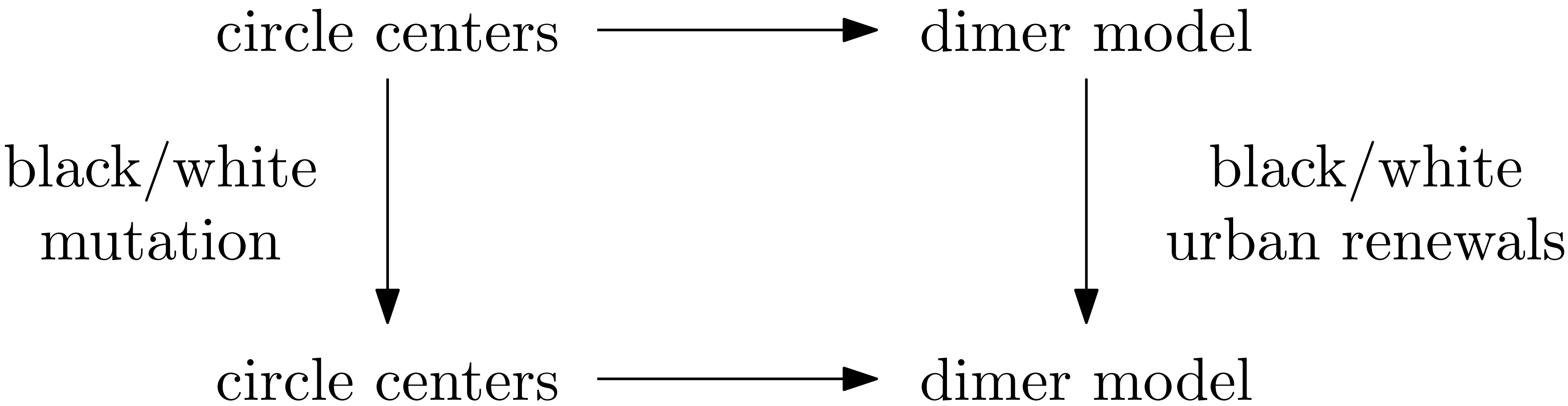
white urban renewals $$X \longmapsto X \frac{(1+X_n)(1+X_s)}{(1+X_w^{-1})(1+X_e^{-1})}$$

From circle centers to dimers

- Starting from a circle pattern realization of a graph G with positive X variables, define the dimer weight on each edge of G as the distance between the two centers separated by the edge.



- Extends the map defined in the isoradial case (all the circles have the same radius) by Kenyon in 2002.



Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh 2018). *The diagram commutes in the sense that the two dimer models produced by* $\rightarrow\downarrow$ *and* $\downarrow\rightarrow$ *have the same* X *variables.*

- Isoradial case: the two dimers models produced by $\rightarrow\downarrow$ and $\downarrow\rightarrow$ have the same edge weights [R. 2018].

From dimers to circle centers

- Impossible to recover the circle centers from the dimer model in the infinite planar case.

- To a dimer model on the torus (infinite biperiodic case), associate its spectral curve (algebraic curve).

Theorem (Kenyon-Lam-R.-Russkikh 2018). *For graphs on the torus, the correspondence*

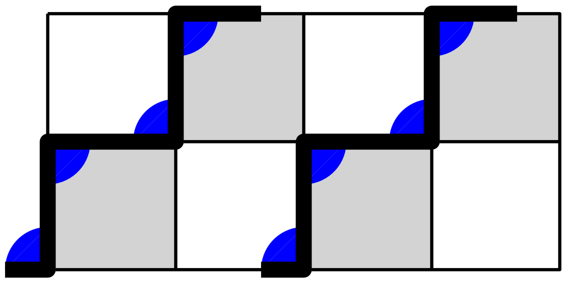
$$(\textit{dimer model}, \textit{point on its spectral curve}) \leftrightarrow \{\textit{circle centers}\}$$

is bijective if we quotient out by the appropriate symmetries.

Miquel's integrals of motion

- The integrals of motion of the dimer dynamics on the torus are given by the coefficients of the polynomial defining the spectral curve (Goncharov-Kenyon).

- Integrals of motion of Miquel dynamics on biperiodic circle centers: dimer integrals of motion together with the two sums of turning angles along zigzag paths.



N. Affolter, Miquel dynamics, Clifford lattices and the dimer model, arXiv:1808.04227 (2018).

A. Glutsyuk and S. Ramassamy, A first integrability result for Miquel dynamics, J. Geom. Phys., 130 (2018), 121-129.

R. Kenyon, W. Y. Lam, S. Ramassamy and M. Russkikh, Dimers and circle patterns, arXiv:1810.05616 (2018).

S. Ramassamy, Miquel dynamics for circle patterns, Int. Math. Res. Not. (2018), published electronically.

THANK YOU !