Dual Polynomials of the Multi-Indexed (q-)Racah Orthogonal Polynomials

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§ 1. Introduction

Quantum Mechanics vs Orthogonal Polynomials

- Orthogonal polynomials often appear in quantum mechanical systems.

  harmonic oscillator ➔ **Hermite** polynomial
  hydrogen atom ➔ Legendre polynomial (associated Legendre "polynomial")
  Laguerre polynomial
  Pöschl-Teller potential ➔ **Jacobi** polynomial

- By using the known properties of orthogonal polynomials, we can investigate quantum mechanical systems.
- Conversely, by using quantum mechanical systems, we can investigate unknown properties of new orthogonal polynomials.

  ← physicist's approach to orthogonal polynomials

- We consider quantum mechanical systems in one dimension.

$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x)$ $\|\phi_n\| < \infty$ $(n = 0, 1, \ldots)$ $0 = \mathcal{E}_0 < \mathcal{E}_1 < \mathcal{E}_2 < \cdots$

eigenvalue problem of the Schrödinger equation

exactly solvable : $\{\mathcal{E}_n\}$ and $\{\phi_n(x)\}$ are known explicitly

- **(ordinary) orthogonal polynomial**

$P_n(\eta)$: polynomial in η $(n=0,1,\ldots)$ $\deg P_n = n$ ➔ a complete set

orthogonal with respect to appropriate inner product

⇔ three term recurrence relations

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta)$$

- **Bochner's theorem**

(ordinary) orthogonal polynomials satisfying

(iii)

2nd order differential equation (with polynomial coefficients)

(i) (ii)

⇔ Hermite, Laguerre, Jacobi, Bessel polynomials

↰ non positive definite inner product

- To avoid this No-Go theorem

(i) 2nd → higher ➔ **Krall** polynomials

(ii) differential → difference

➔ **Askey-Wilson**, ***q*-Racah** polynomials etc.

➔ generalizations of Bochner's theorem

Askey-scheme of hypergeomtric orthogonal polynomials

(iii) ordinary → non ordinary : in spite of missing degrees, a complete set

➔ exceptional or multi-indexed polynomials

Gómez-Ullate–Kamran–Milson
arXiv:0807.3939

(i) 2nd → higher (ii) differential → difference (iii) ordinary → non ordinary

- **Quantum Mechanical Systems**

We consider three kind of systems (Hamiltonians).

- **oQM** : ordinary Quantum Mechanics
- **idQM** : discrete QM with imaginary shifts
- **rdQM** : discrete QM with real shifts

with R.Sasaki

QM system	dynamical variable	Schrödinger equation	order	examples of orthogonal polynomials
oQM	continuous	differential eq.	2^{nd}	Hermite, Laguerre, Jacobi
idQM	continuous	difference eq.	2^{nd}	MP, Wilson, Askey-Wilson etc.
rdQM	discrete	difference eq.	2^{nd}	Hahn, Racah, *q*-Racah etc.

(ii) oQM → idQM, rdQM

Askey-scheme of hypergeomtric orthogonal polynomials

(iii) deform oQM, idQM, rdQM systems by Darboux transformations

multi-indexed orthogonal polynomials

(i) We have considered 2^{nd} order so far, but can think of higher orders.

Today's talk : dual (*q*-)Racah multi-indexed polynomials, which satisfy higher order difference equations, and exactly solvable rdQM systems

- Hamiltonian : hermitian operator

Forms of Hamiltonians (2nd order case)

oQM $p=-i\frac{d}{dx}$

$$\mathcal{H}=p^2+U(x)$$

idQM $p=-i\frac{d}{dx}$ $\gamma=1,\log q$

$$\mathcal{H}=\sqrt{V(x)}\,e^{\gamma p}\sqrt{V^*(x)}+\sqrt{V^*(x)}\,e^{-\gamma p}\sqrt{V(x)}-V(x)-V^*(x)$$

rdQM $x=0,1,\ldots\quad(x=0,1,\ldots,N)$

$$\mathcal{H}_{x,y}=-\sqrt{B(x)D(x+1)}\,\delta_{x+1,y}-\sqrt{B(x-1)D(x)}\,\delta_{x-1,y}+\bigl(B(x)+D(x)\bigr)\delta_{x,y}$$

$$\mathcal{H}=(\mathcal{H}_{x,y})\qquad (e^{\pm\partial})_{x,y}=\delta_{x\pm1,y}$$

$$\mathcal{H}=-\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)}-\sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)}+B(x)+D(x)$$

- parameters : $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots)\qquad q^{\boldsymbol{\lambda}}=(q^{\lambda_1},q^{\lambda_2},\ldots)$

If needed, we write **λ**-dependence : $f=f(\boldsymbol{\lambda}),f(x)=f(x;\boldsymbol{\lambda})$ etc.

§ 2. Muti-Indexed Polynomials

Deformation of quantum mechanical systems

original **exactly solvable system**

$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x)$

$\phi_n(x)$: described by ordinary orthogonal polynomial $P_n(\eta)$

Darboux transformation

type

$\mathcal{D} = \{(\mathfrak{t}_1, d_1), \ldots, (\mathfrak{t}_M, d_M)\}$: label of seed solutions

degree of polynomial part

briefly $\mathcal{D} = \{d_1, \ldots, d_M\}$: multi-index set

deformed **exactly solvable system**

$\mathcal{H}_\mathcal{D}\phi_{\mathcal{D}\,n}(x) = \mathcal{E}_n\phi_{\mathcal{D}\,n}(x)$

$\phi_{\mathcal{D}\,n}(x)$: described by multi-indexed orthogonal polynomial $P_{\mathcal{D},n}(\eta)$

$\mathcal{H}_\mathcal{D}$: expressed by denominator polynomial $\Xi_\mathcal{D}(\eta)$

$\phi_{\mathcal{D}\,n}, P_{\mathcal{D},n}, \Xi_\mathcal{D}$: expressed in terms of determinant (Wronskian, Casoratian)

choice of seed solutions ➔

type of seed solution	deformed system
virtual state	iso spectral
eigenstate	state deletion
pseudo virtual state	state addition

- eigenstate : $\phi_{\mathcal{D}\,n}(x)=\psi_{\mathcal{D}}(x)\check{P}_{\mathcal{D},n}(x)$

  multi-indexed polynomial : $\check{P}_{\mathcal{D},n}(x)=P_{\mathcal{D},n}\bigl(\eta(x)\bigr)$

  ➔ polynomial in $\eta=\eta(x)$: sinusoidal coordinate

  ➔ $\psi_{\mathcal{D}}(x)^2$: weight function

- $P_{\mathcal{D},n}(\eta)$ $(n=0,1,\ldots)$ $\deg P_{\mathcal{D},n}(\eta)=e_n\geq n$

  $\{e_0,e_1,\ldots\}=\{0,1,\ldots\}\backslash I$ I : a set of missing degrees

  case (1) $I=\{0,1,\ldots,\ell-1\}$

  case (2) $I\neq\{0,1,\ldots,\ell-1\}$

  The case (1) multi-indexed polynomials were constructed for Laguerre, Jacobi, (Askey-)Wilson, (*q*-)Racah cases.

  $\deg P_{\mathcal{D},n}(\eta)=\ell_{\mathcal{D}}+n$ O-Sasaki : **arXiv:1105.0508, 1203.5868, 1207.5584**

In the following we consider the case (1) polynomials.

Multi-indexed polynomial

← constructed by quantum mechanical formulation and Darboux transformation

$P_{\mathcal{D},n}(\eta)$ $(n=0,1,\ldots)$ case (1) $\deg P_{\mathcal{D},n}(\eta)=\ell_{\mathcal{D}}+n$

$\mathcal{D}=\{d_1,\ldots,d_M\}$: label of system (Darboux transformation)

n : label of polynomial n = number of zeros in the physical region (number of sign changing)

They are not ordinary orthogonal polynomials.

➔ They do not satisfy three term recurrence relations.

➔ They satisfy recurrence relations with more terms.

▪ **Recurrence relations**

arXiv:1303.5820, 1410.8236, 1509.08213, 1606.02836, 1804.10352

(a) variable dependent coefficients

$$\sum_{k=-M-1}^{M+1}\mathcal{C}_{\mathcal{D},n+k}R^{[M]}_{n,k}(\eta)P_{\mathcal{D},n+k}(\eta)=0$$

(b) constant coefficients

$$X(\eta)P_{\mathcal{D},n}(\eta)=\sum_{k=-L}^{L}r^{X,\mathcal{D}}_{n,k}P_{\mathcal{D},n+k}(\eta)$$

(a) $R^{[M]}_{n,k}(\eta)$ depends on M only (← structure of Darboux transformation)

(b) various X 's are possible.

Recurrence relations with constant coefficients

$$X(\eta)P_{\mathcal{D},n}(\eta)=\sum_{k=-L}^{L} r_{n,k}^{X,\mathcal{D}} P_{\mathcal{D},n+k}(\eta)$$

$$\deg \Xi_{\mathcal{D}}(\eta)=\ell_{\mathcal{D}}$$
$$\deg P_{\mathcal{D},n}(\eta)=\ell_{\mathcal{D}}+n$$

What X gives these relations?

➔Answer : $X(\eta)=X^{\mathcal{D},Y}(\eta)$ $\quad Y(\eta)$: arbitrary polynomial in η $(\neq 0)$

$$\deg X(\eta)=L=\ell_{\mathcal{D}}+\deg Y(\eta)+1$$

various X 's are possible

oQM (Laguerre, Jacobi) $X(\eta)=\int_0^{\eta} dy\, \Xi_{\mathcal{D}}(y)Y(y)$ $\quad \frac{dX(\eta)}{d\eta}=\Xi_{\mathcal{D}}(\eta)Y(\eta)$

idQM (Wilson, Askey-Wilson)

Miki-Tsujimoto : **arXiv:1410.0183**

$$X(\eta)=I[\Xi_{\mathcal{D}}Y](\eta)$$

I : polynomial of degree n $\mapsto$ polynomial of degree $n+1$

$$\frac{\check{X}(x-i\frac{\gamma}{2})-\check{X}(x+i\frac{\gamma}{2})}{\eta(x-i\frac{\gamma}{2})-\eta(x+i\frac{\gamma}{2})}=\check{\Xi}_{\mathcal{D}}(x)\check{Y}(x)$$

$\check{\Xi}_{\mathcal{D}}(x)=\Xi_{\mathcal{D}}\bigl(\eta(x)\bigr)$ $\quad \check{X}(x)=X\bigl(\eta(x)\bigr)$ $\quad \check{Y}(x)=Y\bigl(\eta(x)\bigr)$

rdQM (Racah, q-Racah) **arXiv:1804.10352**

$$X(\eta)=I_{\boldsymbol{\lambda}+M\boldsymbol{\delta}}[\Xi_{\mathcal{D}}Y](\eta)$$

$I_{\boldsymbol{\lambda}}$: polynomial of degree n $\mapsto$ polynomial of degree $n+1$

$$\frac{\check{X}(x)-\check{X}(x-1)}{\eta(x;\boldsymbol{\lambda}+M\boldsymbol{\delta})-\eta(x-1;\boldsymbol{\lambda}+M\boldsymbol{\delta})}=\check{\Xi}_{\mathcal{D}}(x)Y\bigl(\eta(x;\boldsymbol{\lambda}+(M-1)\boldsymbol{\delta})\bigr)$$

$\check{X}(x)=X\bigl(\eta(x;\boldsymbol{\lambda}+M\boldsymbol{\delta})\bigr)$

$\check{\Xi}_{\mathcal{D}}(x)=\Xi_{\mathcal{D}}\bigl(\eta(x;\boldsymbol{\lambda}+(M-1)\boldsymbol{\delta})\bigr)$

For $x\in\mathbb{Z}_{\geq 0}$ $\quad \check{X}(x)=\sum_{j=1}^{x}\bigl(\eta(j;\boldsymbol{\lambda}+M\boldsymbol{\delta})-\eta(j-1;\boldsymbol{\lambda}+M\boldsymbol{\delta})\bigr)\check{\Xi}_{\mathcal{D}}(j)Y\bigl(\eta(j;\boldsymbol{\lambda}+(M-1)\boldsymbol{\delta})\bigr)$

§ 3. Dual Polynomials

$\mathcal{H}_{\mathcal{D}}\phi_{\mathcal{D}\,n}(x)=\mathcal{E}_n\phi_{\mathcal{D}\,n}(x)$

Multi-indexed (*q*-)Racah polynomials ← described by rdQM

$\check{P}_{\mathcal{D},n}(x)=P_{\mathcal{D},n}\big(\eta(x)\big)$ $\quad\eta(0)=0\quad P_{\mathcal{D},n}(0)=1\quad \deg P_{\mathcal{D},n}(\eta)=\ell_{\mathcal{D}}+n$

$x=0,1,\ldots,N$: variable(coordinate)

$n=0,1,\ldots,N$: label of polynomial

finite system

n = number of sign changing

- orthogonality $\displaystyle\sum_{x=0}^{N}\frac{\psi_{\mathcal{D}}(x)^2}{\check{\Xi}_{\mathcal{D}}(1)}\check{P}_{\mathcal{D},n}(x)\check{P}_{\mathcal{D},m}(x)=\frac{\delta_{nm}}{d_{\mathcal{D},n}^2}$

normalized eigenvector: $\hat{\phi}_{\mathcal{D}\,n}(x)=\frac{d_{\mathcal{D},n}}{\sqrt{\check{\Xi}_{\mathcal{D}}(1)}}\psi_{\mathcal{D}}(x)\check{P}_{\mathcal{D},n}(x)$ ➔ $\displaystyle\sum_{x=0}^{N}\hat{\phi}_{\mathcal{D},n}(x)\hat{\phi}_{\mathcal{D},m}(x)=\delta_{nm}$

- 2nd order difference equation ← Schrödinger equation for polynomial part

$\widetilde{\mathcal{H}}_{\mathcal{D}}\check{P}_{\mathcal{D},n}(x)=\mathcal{E}_n\check{P}_{\mathcal{D},n}(x)\qquad \mathcal{E}_0=0$

$$\mathcal{E}_n\check{P}_{\mathcal{D},n}(x)=(\cdots)\check{P}_{\mathcal{D},n}(x+1)+(\cdots)\check{P}_{\mathcal{D},n}(x)+(\cdots)\check{P}_{\mathcal{D},n}(x-1)$$

recurrence relations for *x*

- recurrence relations with constant coefficients

$$\check{X}(x)\check{P}_{\mathcal{D},n}(x)=\sum_{k=-\min(L,n)}^{\min(L,N-n)}r_{n,k}^{X,\mathcal{D}}\check{P}_{\mathcal{D},n+k}(x)\qquad(x,n=0,1,\ldots,N)$$

recurrence relations for *n*

orthogonality: $\sum_{x=0}^{N}\hat{\phi}_{\mathcal{D},n}(x)\hat{\phi}_{\mathcal{D},m}(x)=\delta_{nm}$ $\Leftrightarrow$ $\sum_{n=0}^{N}\hat{\phi}_{\mathcal{D},n}(x)\hat{\phi}_{\mathcal{D},n}(x)=\delta_{xy}$

Dual multi-indexed (*q*-)Racah polynomials

arXiv:1805.00345

dual polynomial : exchange *x* and *n*

$$\check{Q}_{\mathcal{D},x}(n)=\frac{\check{P}_{\mathcal{D},n}(x)}{\check{P}_{\mathcal{D},0}(x)}\qquad \check{Q}_{\mathcal{D},x}(0)=1\qquad \hat{\phi}_{\mathcal{D}\,n}(x)=\frac{\phi_{\mathcal{D}\,0}(x)}{\sqrt{\check{\Xi}_{\mathcal{D}}(1)}}d_{\mathcal{D},n}\check{Q}_{\mathcal{D},x}(n)$$

- orthogonality $$\sum_{n=0}^{N}\frac{d_{\mathcal{D},n}^2}{\check{\Xi}_{\mathcal{D}}(1)}\check{Q}_{\mathcal{D},x}(n)\check{Q}_{\mathcal{D},y}(n)=\frac{\delta_{xy}}{\phi_{\mathcal{D}\,0}(x)^2}$$

- 2nd order difference equation of $\check{P}_{\mathcal{D},n}(x)$ ➔ three term recurrence relations of $\check{Q}_{\mathcal{D},x}(n)$ — recurrence relations for label *x*

  recurrence relations for variable *x*

  ➔ ordinary orthogonal polynomial

  $\check{Q}_{\mathcal{D},x}(n)=Q_{\mathcal{D},x}(\mathcal{E}_n)$ $\deg Q_{\mathcal{D},x}(\mathcal{E})=x$

- *2L+1* term recurrence relations of $\check{P}_{\mathcal{D},n}(x)$ ➔ *2L*-th order difference equation of $\check{Q}_{\mathcal{D},x}(n)$ — recurrence relations for variable *n*

  recurrence relations for label *n*

  ➔ Krall type

- correspondence $x\leftrightarrow n,\ \eta(x)\leftrightarrow\mathcal{E}_n,\ P_{\mathcal{D},n}(\eta)\leftrightarrow Q_{\mathcal{D},x}(\mathcal{E}),\ \frac{\phi_{\mathcal{D}\,0}(x)}{\phi_{\mathcal{D}\,0}(0)}\leftrightarrow\frac{d_{\mathcal{D},n}}{d_{\mathcal{D},0}}$

§ 4. Exactly Solvable rdQM

discrete QM with real shifts **rdQM**

dynamical variable x : $x = 0, 1, \ldots, N$ finite system

Hamiltonian $\mathcal{H}$: hermitian matrix (real symmetric matrix)

2nd order case

$$\mathcal{H}_{x,y} = -\sqrt{B(x)D(x+1)}\,\delta_{x+1,y} - \sqrt{B(x-1)D(x)}\,\delta_{x-1,y} + \bigl(B(x)+D(x)\bigr)\delta_{x,y}$$

tri-diagonal matrix (Jacobi matrix)

$B(x) > 0,\ D(x) > 0$ at boundaries $D(0) = 0$ $B(N) = 0$

$$\mathcal{H} = -\sqrt{B(x)}\,e^{\partial}\sqrt{D(x)} - \sqrt{D(x)}\,e^{-\partial}\sqrt{B(x)} + B(x) + D(x)$$

- Schrödinger equation

$$\mathcal{H}\phi_n(x) = \mathcal{E}_n\phi_n(x) \quad (n = 0, 1, \ldots, N) \quad 0 = \mathcal{E}_0 < \mathcal{E}_1 < \cdots < \mathcal{E}_N$$

eigenvector $\|\phi_n\| < \infty$ (trivial for finite systems)

diagonalization of matrices

new exactly solvable rdQM systems

arXiv:1805.00345

For each $X(\eta) = X^{\mathcal{D},Y}(\eta)$

$Y(\eta)$: non-negative coefficients

$\deg X(\eta) = L = \ell_{\mathcal{D}} + \deg Y(\eta) + 1$

➔ $\mathcal{H}_{\mathcal{D}}^{X\ \text{dual}} = \big(\mathcal{H}_{\mathcal{D};x,y}^{X\ \text{dual}}\big)_{0\le x,y\le N}$: real symmetric

$$\mathcal{H}_{\mathcal{D}}^{X\ \text{dual}} = \sum_{k=-\min(L,x)}^{\min(L,N-x)} r_{x,k}^{X,\mathcal{D}} \frac{d_{\mathcal{D},x}}{d_{\mathcal{D},x+k}} e^{k\partial}$$

$(e^{k\partial})_{x,y} = \delta_{x+k,y}$

($2L+1$) - diagonal matrix

- eigenvectors $\mathcal{H}_{\mathcal{D}}^{X\ \text{dual}} \phi_{\mathcal{D}\,n}^{\text{dual}}(x) = \mathcal{E}_{\mathcal{D},n}^{X\ \text{dual}} \phi_{\mathcal{D}\,n}^{\text{dual}}(x)$ $\quad (n = 0, 1, \ldots, N)$

$$\phi_{\mathcal{D}\,n}^{\text{dual}}(x) = \frac{d_{\mathcal{D},x}}{d_{\mathcal{D},0}} \check{Q}_{\mathcal{D},n}(x) \qquad \check{Q}_{\mathcal{D},n}(x) = Q_{\mathcal{D},n}(\mathcal{E}_x)$$

$$\mathcal{E}_{\mathcal{D},n}^{X\ \text{dual}} = \check{X}(n) \qquad 0 = \mathcal{E}_{\mathcal{D},0}^{X\ \text{dual}} < \mathcal{E}_{\mathcal{D},1}^{X\ \text{dual}} < \cdots < \mathcal{E}_{\mathcal{D},N}^{X\ \text{dual}}$$

Eigenvectors do not depend on X !

➔ For X_1, X_2 with arbitrary Y_1, Y_2 $\quad \big[\mathcal{H}_{\mathcal{D}}^{X_1\ \text{dual}}, \mathcal{H}_{\mathcal{D}}^{X_2\ \text{dual}}\big] = 0$

- closure relation holds (in brute force way) ➔ creation/annihilation operators
- shape invariance does not hold ➔ first example

§ 5. Summary and Comments

- By using QM systems, we study orthogonal polynomials.
- Recurrence relations for case (1) multi-indexed polynomials (variable dependent coefficients and constant coefficients).
  (known for Laguerre, Jacobi, Wilson, Askey-Wilson)
  (q-)Racah **arXiv:1804.10352**
- Dual multi-indexed (q-)Racah polynomials. **arXiv:1805.00345**
  ordinary orthogonal polynomials,
  higher order difference equations (Krall type).
- New exactly solvable rdQM systems. **arXiv:1805.00345**
  eigenvectors : dual multi-indexed (q-)Racah polynomials
- Future problems :
  - explicit form of $r_{n,k}^{X,\mathcal{D}}$ for general n,k,X,D ?
  - relation among $r_{n,k}^{X,\mathcal{D}}$ for different X ?
  - Is it possible to deform rdQM systems of dual multi-indexed (q-)Racah polynomials by Darboux transformation ?
  - Is it possible to construct "dual polynomials" of multi-indexed (Askey-)Wilson polynomials ?

Ortho. Poly. and QM systems

How to avoid Bochner's theorem

- (i) 2nd order → higher
- (ii) differential → difference
- (iii) ordinary → non-ordinary

QM system	dyn. var.	Schrödinger equation	order	examples of orthogonal polynomials	status
oQM	continuous	differential equation	2nd	Hermite, Laguerre, Jacobi	◎
				multi-indexed version (iii)	○
			higher (i)	Krall version (i)	○
				multi-indexed version (i)(iii)	×
idQM (ii)	continuous	difference equation (ii)	2nd	MP, Wilson, Askey-Wilson etc.	◎
				multi-indexed version (ii)(iii)	△
			higher (i)	Krall type version (i)(ii)	△
				multi-indexed version (i)(ii)(iii)	×
rdQM (ii)	discrete	difference equation (ii)	2nd	Hahn, Racah, *q*-Racah etc.	◎
				multi-indexed version (ii)(iii)	△ ← today's talk
			higher (i)	Krall type version (i)(ii)	△ ← today's talk
				multi-indexed version (i)(ii)(iii)	×

understanding : ◎ > ○ > △ > ×

END