Ultradiscretization with parity variables for nonlinear oscillator and its conserved quantity

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(traditional) Ultradiscretization (UD)

For a given difference equation for x_n,
1. Replace x_n with $\exp\frac{X_n}{\epsilon}$ $(\epsilon > 0)$.
2. Take the limit $\lim_{\epsilon\to+0} \epsilon \log\ (*)$.

From the formula

$$\lim_{\epsilon\to+0} \epsilon \log\left(e^{\frac{X}{\epsilon}} + be^{\frac{Y}{\epsilon}}\right) = \max(X,Y) \quad (X,Y\in\mathbb{R}),$$

we obtain a piecewise-linear equation for X_n.

- Under proper restriction, the resulting equation is regarded a Cellular Automaton (CA).
- This CA (may) inherits some essential properties of the original equation.

First and typical example:
Discrete Lotka--Volterra equation → UD → The box-and-ball system

Negative Difficulty in Ultradiscretization

The difference equation must be subtraction-free.

$$\lim_{\epsilon\to+0} \epsilon \log\left(e^{\frac{X}{\epsilon}} - e^{\frac{Y}{\epsilon}}\right) = ?$$

Negative Difficulty

Its solutions must be definite-sign.

$$x_n = \exp\frac{X_n}{\epsilon} > 0$$

For $x_{n+1} = -ax_n + b\ (a,b>0)$, we assume $x_n > 0$ and move the negative term, and apply UD:

$$x_{n+1} + ax_n = b$$
$$\exp\frac{X_{n+1}}{\epsilon} + \exp\frac{A}{\epsilon}\exp\frac{X_n}{\epsilon} = \exp\frac{B}{\epsilon}$$
$$\epsilon\log\left(\exp\frac{X_{n+1}}{\epsilon} + \exp\frac{A+X_n}{\epsilon}\right) = \epsilon\log\exp\frac{B}{\epsilon}$$
$$\to \max(X_{n+1}, A+X_n) = B$$

If $A=1, B=0, X_n=2$, $\max(X_{n+1},3)=0$ has no solution.
For these parameters, x_{n+1} should be negative:

$$x_{n+1} + e^{3/\epsilon} = e^{0/\epsilon}$$

Ultradiscretization with parity variables (p-UD)

Introduce *parity* variable, $\xi_n = x_n/|x_n|$ (the sign of x_n), and *Amplitude* X_n by replacing $|x_n|$ by $\exp\frac{X_n}{\epsilon}$.
That is, $x_n = \xi_n \exp\frac{X_n}{\epsilon}$.

For $x_{n+1} = -ax_n + b\ (a,b>0)$, we consider four cases,

(i) $\xi_{n+1} = 1, \xi_n = 1$:
$\exp\frac{X_{n+1}}{\epsilon} + \exp\frac{A}{\epsilon}\exp\frac{X_n}{\epsilon} = \exp\frac{B}{\epsilon} \to \max(X_{n+1}, A+X_n) = B$

(ii) $\xi_{n+1} = 1, \xi_n = -1$:
$\exp\frac{X_{n+1}}{\epsilon} = \exp\frac{A}{\epsilon}\exp\frac{X_n}{\epsilon} + \exp\frac{B}{\epsilon} \to X_{n+1} = \max(A+X_n, B)$

(iii) $\xi_{n+1} = -1, \xi_n = 1$:
$\exp\frac{A}{\epsilon}\exp\frac{X_n}{\epsilon} = \exp\frac{X_{n+1}}{\epsilon} + \exp\frac{B}{\epsilon} \to A+X_n = \max(X_{n+1}, B)$

(vi) $\xi_{n+1} = -1, \xi_n = -1$:
$0 = \exp\frac{X_{n+1}}{\epsilon} + \exp\frac{A}{\epsilon}\exp\frac{X_n}{\epsilon} + \exp\frac{B}{\epsilon} \to -\infty = \max(X_{n+1}, A+X_n, B)$

All terms are positive

We consider this set of equations as UD analogue of $x_{n+1} = -ax_n + b$.

Time evolution of p-UD equation

We put $A=1, B=0$.

- Assume the pair $(\xi_n, X_n) = (1,2)$ is given.

From $\xi_n = 1$, we have
(i) $\max(X_{n+1}, 1+2) = 0$ or (iii) $1+2 = \max(X_{n+1}, 0)$.
Only (iii) has a solution $X_{n+1} = 3$, which means $\xi_{n+1} = -1$.
That is, we obtain the unique solution:

$$(\xi_n, X_n) = (1,2) \to (\xi_{n+1}, X_{n+1}) = (-1,3).$$

- Assume the pair $(\xi_n, X_n) = (1,-1)$ is given. We have

(i) $\max(X_{n+1}, 1-1) = 0$ or (iii) $1-1 = \max(X_{n+1}, 0)$.
Both equations have solutions $X_{n+1} \le 0$, which means $\xi_{n+1} = \pm 1$.
That is, we obtain indeterminate solutions:

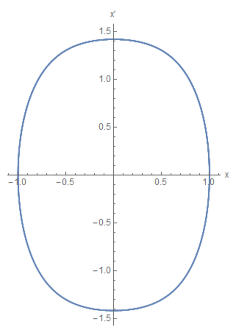
$$(\xi_n, X_n) = (1,-1) \to \xi_{n+1} = \pm 1, X_{n+1} \le 0.$$

Our study

We give the p-UD analogue of a nonlinear equation with conserved quantity (CQ) and examine the behavior of ultradiscretized solutions and CQ.

Hard spring equation

$$\frac{d^2x}{dt^2} + ax + bx^3 = 0 \quad (a,b>0, x=x(t))$$



Conserved quantity

$$H(t) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

Their integrable discrete analogue is given in a Japanese book, `Discrete and Ultradiscrete Systems' by Prof. Hirota and Prof. Takahashi.

Discrete hard spring equation

Discrete Hard spring equation

$$x_{n+1} - 2x_n + x_{n-1} + 2\delta^2 c_1(x_{n+1} + x_{n-1}) + 4\delta^2 c_2 x_n + 2\delta^2 c_3 x_n^2 (x_{n+1} + x_{n-1}) = 0$$

$$\Leftrightarrow x_{n+1} = \frac{2x_n - x_{n-1} - 2\delta^2 c_1 x_{n-1} - 4\delta^2 c_2 x_n - 2\delta^2 c_3 x_n^2 x_{n-1}}{1 + 2\delta^2 c_1 + 2\delta^2 c_3 x_n^2}$$

Note: $t = n\delta,\ x(t) = x(n\delta) = x_n,\ \ c_1 + c_2 = \frac{a}{4},\ c_3 = \frac{b}{4}$

Conserved quantity

$$2\delta^2 H_n = x_n^2 - 2x_n x_{n-1} + x_{n-1}^2 + 2\delta^2 c_1 (x_n^2 + x_{n-1}^2) + 2\delta^2 c_2 x_n x_{n-1} + 2\delta^2 c_3 x_n^2 x_{n-1}^2$$

This is an integrable discretization!

P-ultradiscrete hard spring equation (forward schime)

Notation: $c_i = e^{\frac{\alpha_i}{\epsilon}},\ \delta = e^{\frac{\Delta}{\epsilon}},\ \widehat{\alpha_i} = \alpha_i + 2\Delta,$
$\xi_n := \frac{x_n}{|x_n|},\ e^{\frac{X_n}{\epsilon}} := |x_n|,\ A_n := \max(0, \widehat{\alpha_1}, 2X_n + \widehat{\alpha_3})$

$\widehat{\alpha_2} > 0$

(a) $\xi_n\xi_{n-1} = 1$
$$\begin{cases}\xi_{n+1} = -\xi_n (= -\xi_{n-1})\\ X_{n+1} = \max(X_n - A_n + \widehat{\alpha_2}, X_{n-1})\end{cases}$$

(b) $\xi_n\xi_{n-1} = -1,\ X_n + \widehat{\alpha_2} = X_{n-1} + A_n$
$$\begin{cases}\xi_{n+1} = \pm 1\\ X_{n+1} \le X_{n-1}\end{cases}$$

(c) $\xi_n\xi_{n-1} = -1,\ X_n + \widehat{\alpha_2} \ne X_{n-1} + A_n$
$$\begin{cases}\xi_{n+1} = \begin{cases}\xi_{n-1} & (X_n + \widehat{\alpha_2} > X_{n-1} + A_n)\\ \xi_n & (X_n + \widehat{\alpha_2} < X_{n-1} + A_n)\end{cases}\\ X_{n+1} = \max(X_n - A_n + \widehat{\alpha_2}, X_{n-1})\end{cases}$$

$\widehat{\alpha_2} < 0$

(a) $\xi_n\xi_{n-1} = 1,\ X_n - A_n = X_{n-1}$
$$\begin{cases}\xi_{n+1} = \pm 1\\ X_{n+1} \le X_{n-1}\end{cases}$$

(b) $\xi_n\xi_{n-1} = 1,\ X_n - A_n \ne X_{n-1}$
$$\begin{cases}\xi_{n+1} = \begin{cases}\xi_n & (X_n - A_n > X_{n-1})\\ -\xi_n & (X_n - A_n < X_{n-1})\end{cases}\\ X_{n+1} = \max(X_n - A_n, X_{n-1})\end{cases}$$

(c) $\xi_n\xi_{n-1} = -1$
$$\begin{cases}\xi_{n+1} = \xi_n (= -\xi_{n-1})\\ X_{n+1} = \max(X_n - A_n, X_{n-1})\end{cases}$$

P-ultradiscrete analogue of the conserved quantity

Notation: $q_n := \frac{H_n}{|H_n|},\ \ e^{\frac{Q_n}{\epsilon}} := |H_n|$

$\xi_n\xi_{n-1} = 1$

$F^a(n) := X_n + X_{n-1},$

$G^a(n) := \max\begin{pmatrix}2X_n + A_{n-1},\\ 2X_{n-1} + A_n,\\ X_n + X_{n-1} + \widehat{\alpha_2}\end{pmatrix}$

(a) $F^a(n) > G^a(n)$
$$\begin{cases}q_n = -1\\ Q_n = F^a(n) - 2\Delta\end{cases}$$

(b) $F^a(n) < G^a(n)$
$$\begin{cases}q_n = +1\\ Q_n = G^a(n) - 2\Delta\end{cases}$$

(c) $F^a(n) = G^a(n)$
$$\begin{cases}q_n = \pm 1\\ Q_n \le X_n + X_{n-1} - 2\Delta\end{cases}$$

$\xi_n\xi_{n-1} = -1$

$F^b(n) := X_n + X_{n-1} + \widehat{\alpha_2},$

$G^b(n) := \max\begin{pmatrix}2X_n + A_{n-1},\\ 2X_{n-1} + A_n,\\ X_n + X_{n-1}\end{pmatrix}$

(a) $F^b(n) > G^b(n)$
$$\begin{cases}q_n = -1\\ Q_n = F^b(n) - 2\Delta\end{cases}$$

(b) $F^b(n) < G^b(n)$
$$\begin{cases}q_n = +1\\ Q_n = G^b(n) - 2\Delta\end{cases}$$

(c) $F^b(n) = G^b(n)$
$$\begin{cases}q_n = \pm 1\\ Q_n \le X_n + X_{n-1} + \widehat{\alpha_2} - 2\Delta\end{cases}$$

Behavior of solution and conserved quantity

Solution (1) ---eventually periodic---

$\widehat{\alpha_2} < 0,\ \xi_1\xi_0 = 1,\ X_1 > X_0,\ \ A_1 = \widehat{\alpha_1},\ \ X_1 - \widehat{\alpha_1} > X_0$

We write $\xi_0 = \xi_1 = \xi$.
We obtain the unique solution as follows.

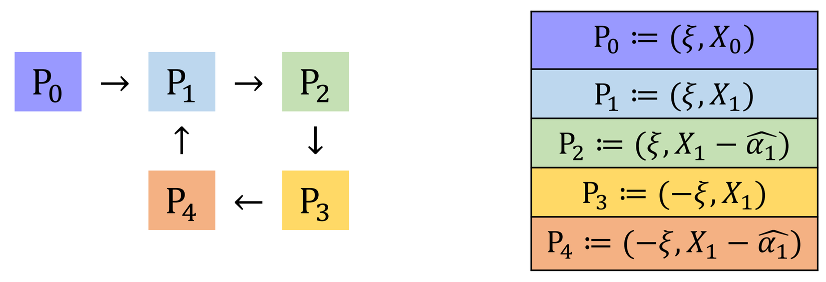
Notation: $\chi_n = (\xi_n, X_n)$

- $\chi_0 := (\xi, X_0)$
- $\chi_1 := (\xi, X_1)$
- $\chi_2 = (\xi, X_1 - \widehat{\alpha_1})$
- $\chi_3 = (-\xi, X_1)$
- $\chi_4 = (-\xi, X_1 - \widehat{\alpha_1})$
- $\chi_5 = (\xi, X_1)$
- $\chi_6 = (\xi, X_1 - \widehat{\alpha_1})$
- $\chi_7 = (-\xi, X_1)$
- $\chi_8 = (-\xi, X_1 - \widehat{\alpha_1})$

⋮

- We do not meet indeterminate-type evolution.
- The solution is eventually four-periodic.

Summary by diagram (1) ---periodic type---



We calculate the p-ultradiscretized conserved quantity (q_n, Q_n) from possible pairs

$$(\chi_{n-1}, \chi_n) = (P_0, P_1), (P_1, P_2), (P_2, P_3), (P_3, P_4), (P_4, P_1).$$

We obtain

$$(q_n, Q_n) = (q_1, Q_1).$$

Hence, (q_n, Q_n) is actually the conserved quantity for this soution.

Solution (2) ---non-periodic but `summarizable'---

$\widehat{\alpha_2} < 0,\ \xi_1\xi_0 = 1,\ X_1 > X_0,\ \ A_1 = 0$

From the initial values $\chi_0 := (\xi, X_0)$ and $\chi_1 := (\xi, X_1)$,
→ unique $\chi_2 = (\xi, X_1)$
→ χ_3 is not unique: $\xi_3 = \pm\xi,\ X_3 \le X_1$.
We consider the following cases.

(i) $\chi_3 = (\xi, X_1)$
Again, (ξ_4, X_4) is not unique.

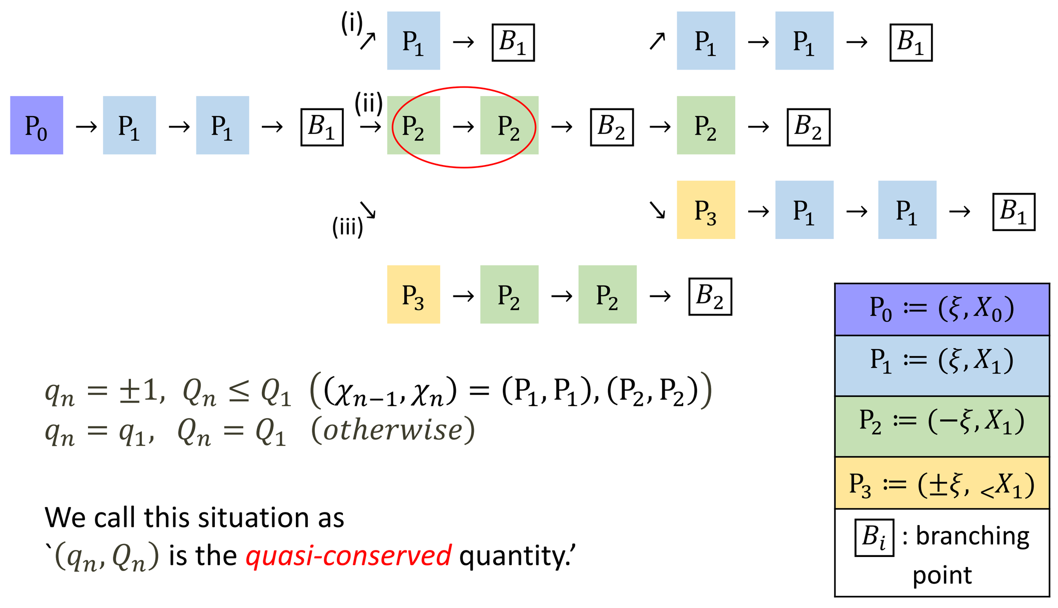
(ii) $\chi_3 = (-\xi, X_1)$
Unique $\chi_4 = (\xi_4, X_4) = (-\xi, X_1)$, and (ξ_5, X_5) is not unique.
→ introduce new cases

(iii) $\chi_3 = (\xi, Y)$ *or* $(-\xi, Y)$ where $Y < X_1$
Unique $\chi_4 = (-\xi, X_1)$ and $\chi_5 = (-\xi, X_1)$, and (ξ_6, X_6) is not unique.

Independent of Y and ξ_3

Summary by diagram (2) ---Bounded type---

$\widehat{\alpha_2} < 0,\ \xi_1\xi_0 = 1,\ X_1 > X_0,\ \ A_1 = 0$



$q_n = \pm 1,\ Q_n \le Q_1\ \ ((\chi_{n-1}, \chi_n) = (P_1, P_1), (P_2, P_2))$
$q_n = q_1,\ \ Q_n = Q_1\ \ (otherwise)$

We call this situation as
`(q_n, Q_n) is the *quasi-conserved* quantity.'

Notation: $X_n = {}_{<}Y \Leftrightarrow X_n = Z, Z < Y$

Solution (3) ---non-periodic and complicated---

$\widehat{\alpha_2} > 0, \xi_1\xi_0 = 1, X_1 = X_0, A_1 = 2X_0 + \widehat{\alpha_3},\ 2X_0 + \widehat{\alpha_3} = \widehat{\alpha_2}$

From the initial values $\chi_0 = \chi_1 = (\xi, X_0)$,
→ unique $\chi_2 = (-\xi, X_0)$
→ χ_3 is not unique: $\xi_3 = \pm\xi,\ X_3 \le X_0$.

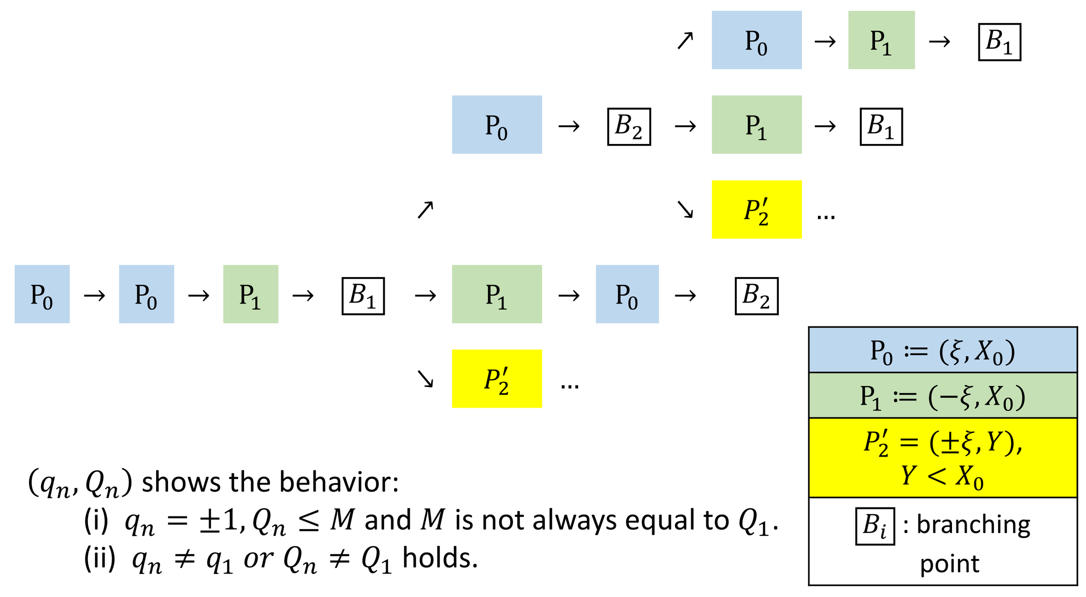
(i) $\chi_3 = (\xi, X_0)$
Again, χ_4 is not unique.

(ii) $\chi_3 = (-\xi, X_0)$
Unique $\chi_4 = (\xi, X_0)$, and χ_5 is not unique.

(iii) $\chi_3 = (\xi, Y)$ *or* $(-\xi, Y)$ where $Y < X_1$
The successive solution depends on χ_3.
(We have to introduce further cases.)

Summary by diagram (3) ---Unbounded type---

$\widehat{\alpha_2} > 0, \xi_1\xi_0 = 1, X_1 = X_0, A_1 = 2X_0 + \widehat{\alpha_3},\ 2X_0 + \widehat{\alpha_3} = \widehat{\alpha_2}$



(q_n, Q_n) shows the behavior:
(i) $q_n = \pm 1, Q_n \le M$ and M is not always equal to Q_1.
(ii) $q_n \ne q_1$ *or* $Q_n \ne Q_1$ holds.

Conclusion

We constructed p-ultradiscrete analogue of the hard-spring equation and its conserved quantity.

We have studied:
all initial values for $\widehat{\alpha_2} < 0$ and $\xi_1\xi_0 = 1,\ X_1 \ge X_0$ for $\widehat{\alpha_2} > 0$.

The behavior of solutions and ultradiscretized conserved quantity is categorized as follows.

Solutions	Ultradiscretized CQ
Exactly or eventually four-periodic	Conserved $\forall n, q_n = q_1, Q_n = Q_1$
Non-periodic but summarized by `bounded diagram'	`quasi-conserved' $\exists n, q_n = \pm 1, Q_n \le Q_1$
Non-periodic with `unbounded diagram'	Not always conserved

We need...
Further analysis for non-periodic cases.
Study on solutions of the original difference equation.

Reference

Shin Isojima, Hirotaka Toyama: Ultradiscrete analogues of the hard-spring equation and its conserved quantity, Jpn. J. Ind. Appl. Math., to appear (DOI: 10.1007/s13160-018-0329-5)