### Ultradiscretization with parity variables for nonlinear oscillator and its conserved quantity

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#### [Traditional] Ultradiscretization (UD)

- For a given difference equation for $x_n$:  
  1. Replace $x_n$ with $\exp \log (\pm x_n)$.
  2. Take the limit $\lim_{\alpha \to 0}$.

- From the formula:  
  $$\lim_{\alpha \to 0} \frac{\alpha}{\log(\pm(\alpha(\pm \Delta x + \alpha\pm x_n)))} = \max(x, x_n)$$

- We obtain a piecewise-linear equation for $x_n$.

- Under proper restriction, the resulting equation is regarded a Cellular Automaton (CA).

- This CA has some essential properties of the original equation.

#### First and typical example: Discrete Lotka-Volterra equation

- The box-and-ball system

#### Negative Difficulty in Ultradiscretization

- The difference equation must be subtraction-free.  
  $$\dfrac{\alpha}{\log(\pm x_n)} = 0$$

- Its solutions must be definite-sign,  
  $$x_n = \exp \log (\pm x_n)$$

- For $x_0 = -\alpha$, it becomes $x_n = 0$.

- We assume $x_n > 0$ and move the negative term, and employ UD:

  - $x_n = x_0 - \alpha$, \( \Delta x = 0 \)

- $\exp \log(\pm(\alpha(\pm \Delta x + \alpha x_n))) = 0$

- $\alpha = 0$

#### Ultradiscretization with parity variables (p-UD)

- Introduce parity variables, \( q_n, \bar{q}_n \) (the sign of \( x_n \)) and an amplitude $A_n$, replacing $x_n$ by $x_n = e^{\alpha q_n}$.

- That is,  
  $$x_n = \pm A_n e^{\alpha q_n}$$

- For $x_0 = -\alpha$, we consider four cases,
  - (i) \( \Delta x = 0 \) \( A_n = \alpha \),  
    $$\exp \log(\pm(\alpha(\pm \Delta x + \alpha \pm x_n))) = A_n$$
  - (ii) \( \Delta x = 0 \) \( A_n = 0 \),  
    $$\exp \log(\pm(\alpha(\pm \Delta x + \alpha x_n))) = \pm \alpha$$
  - (iii) \( \Delta x = 0 \) \( A_n = 1 \),  
    $$\exp \log(\pm(\alpha(\pm \Delta x + \alpha \pm x_n))) = 1$$
  - (iv) \( \Delta x = 0 \) \( A_n = -1 \),  
    $$\exp \log(\pm(\alpha(\pm \Delta x + \alpha x_n))) = -1$$

- We consider this set of equations as UD analogue of $x_n = -\alpha x_0 + \Delta x$.

#### Time evolution of p-UD equation

- We put $A_n = 1, \bar{q}_n = 0$.

- Assume the pair \((x_0, q_n) = (1.2)\) is given.

- From $x_0 = 1$, the variable $x_n$ becomes  
  - $\Delta x = 1, \alpha = 0$

- Only if $\Delta x$ is solution $x_n = 1$ (which means $q_n = 1$).

- That is, $x_n$ obtains the unique solution:

  - $x_n = 1, q_n = 1$

- Both equations have solutions $x_n = 1, \alpha = 1$.  

- $q_n = 1, q_n = 1$

- $x_n = 0, q_n = 0$

- We obtain indeterminate solutions:

  - $x_n = 0, q_n = -1$

- $x_n = 1, q_n = 1$

- $x_n = 1, q_n = -1$

#### Our study

- We give the p-UD analogue of a nonlinear equation with conserved quantity (CQ) and examine the behavior of ultradiscretized solutions and CQ.

#### Hard spring equation

- $$\frac{d^2 x}{dt^2} + \alpha x + \beta x^3 = 0$$

- $$M(x) = 1 + \frac{d^2 x}{dt^2} + \frac{d^2 x}{dt^2}$$

- Their integrable discrete analogues is given in a Japanese book, `Discrete and Ultradiscrete System` by Prof. Hirota and Prof. Takahashi.

#### Discrete hard spring equation

- **Conserved quantity**

  - $$H(\Delta x, \Delta q_n) = \frac{1}{2} \Delta q_n^2 + \frac{1}{2} \Delta x^2 + \frac{1}{2} \Delta x^2$$

- Their integrable discrete analogues is given in a Japanese book, `Discrete and Ultradiscrete System` by Prof. Hirota and Prof. Takahashi.

#### Summary by diagram (1) —periodic type

- Solution $\phi(0) = 0, \phi(0) > 0, \phi(0) \geq 0, \phi(0) \neq 0$

- Solution $\phi(0) = 0$.

- Solution $\phi(0) > 0$.

- Solution $\phi(0) \geq 0$.

#### Conclusion

- We constructed the ultradiscretized analogue of the hard-spring equation and its conserved quantity.

- We have studied

  - all initial values for $\phi(0) = 0$ and $\phi(1), q(0, \phi(0)) \neq 0$ for $\phi(0) > 0$.

- The behavior of solutions and ultradiscretized conserved quantity is categorized as follows.

- **Non-periodic**

  - One solution.

  - Parity variable.

- **Non-periodic and complicated**

  - Two solutions.

  - Parity variable.

#### Reference