The discrete two-dimensional Toda equation gives nice formulae for reverse plane partitions

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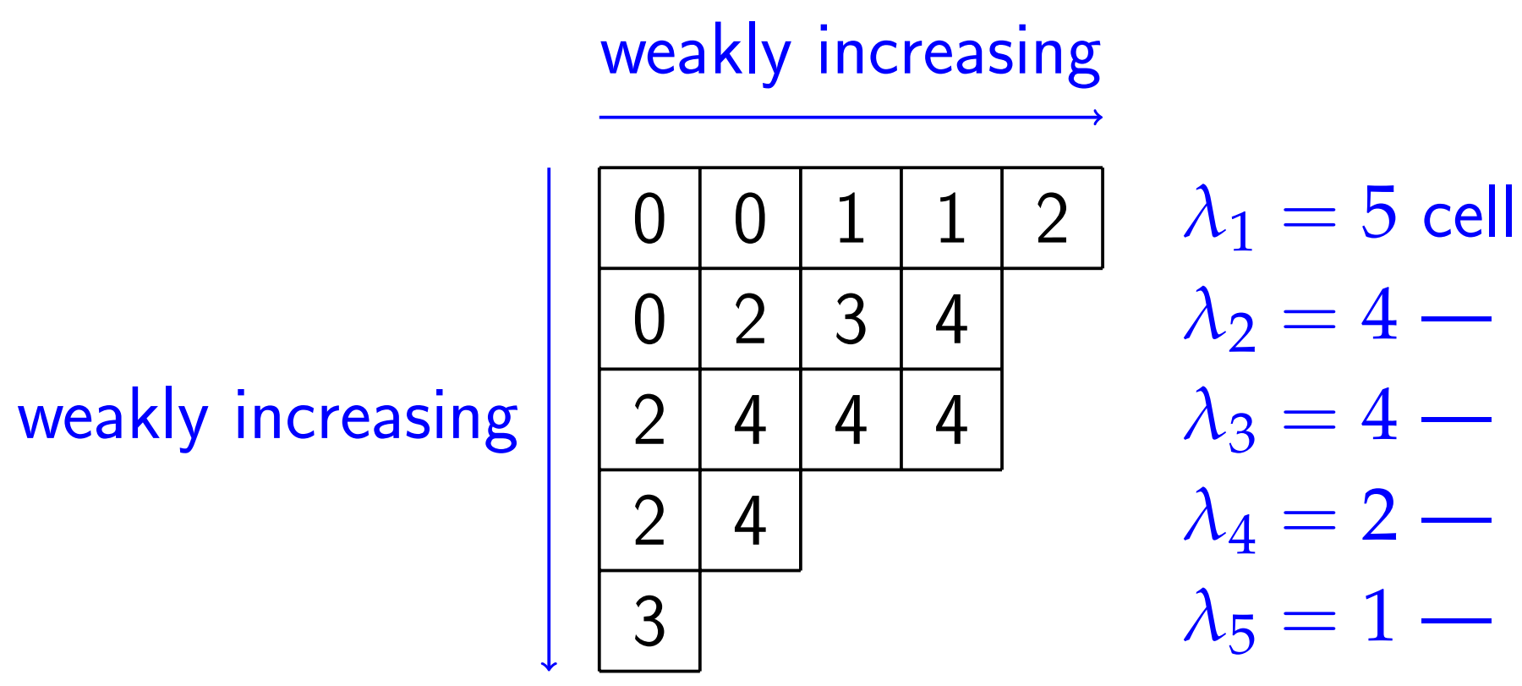
Abstract: We clarify a close connection between reverse plane partitions (RPPs) and an integrable dynamical system called the discrete two-dimensional (2D) Toda equation. We show that a nice partition function (with a product expression) can be obtained from each non-vanishing solution to the discrete 2D Toda equation. As an example we derive a partition function which generalizes MacMahon's formula and Gansner's hook formula from a specific solution to the discrete 2D Toda equation.

Remark: A similar result for rectangular-shaped plane partitions is discussed in [K1,K2] where biorthogonal polynomials are used to find a multiplicative partition function.

Definition

A reverse plane partition (RPP) of shape λ is a filling of a Young diagram λ with nonnegative integers which weakly increase along rows and columns.

Example: An RPP $\pi \in \mathrm{RPP}(\lambda, 4)$ of shape $\lambda = (5,4,4,2,1)$



(1)

Notations

- $\lambda = (\lambda_1, \lambda_2, \dots)$: (integer) partition, or a Young diagram.
- $\lambda' = (\lambda'_1, \lambda'_2, \dots)$: the Young diagram conjugate with λ.
- $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$: the legnth of the hook of a cell (i,j) in λ.
- $\mathrm{RPP}(\lambda, c)$: the set of RPPs of shape λ with parts $\le c$.

Nice formulae for RPPs

- MacMahon's formula [M] (for RPPs of rectangular shape):

$$\sum_{\pi \in \mathrm{RPP}((b^a),c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \tag{2}$$

  where $|\pi| := \sum_{i,j} \pi_{i,j}$.
- Gansner's hook formula [G] (for RPPs with unbounded parts):

$$\sum_{\pi \in \mathrm{RPP}(\lambda,\infty)} q^{|\pi|} = \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_\lambda(i,j)}} \tag{3}$$

References:

[K1] S. Kamioka, *Plane partitions with bounded size of parts and biorthogonal polynomials*, arXiv:1508.01674.

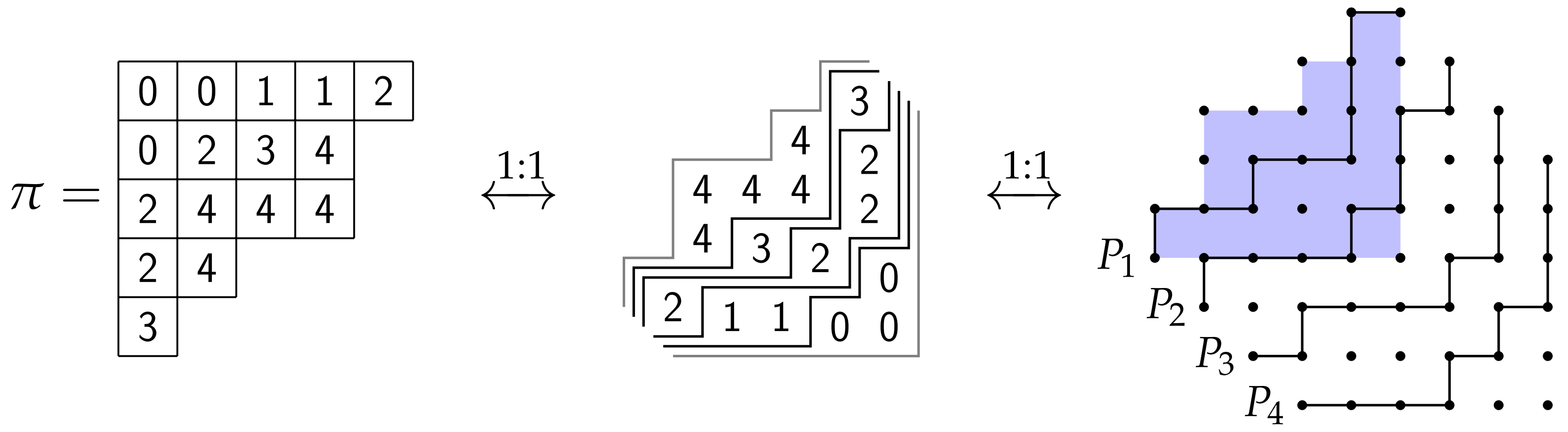
[K2] S. Kamioka, *A triple product formula for plane partitions derived from biorthogonal polynomials*, DMTCS proc. **BC, 2016** (2016), 671–682.

[M] P. A. MacMahon, Combinatory Analysis, volume 2, Cambridge University Press, Cambridge, 1916.

[G] E. R. Gansner, *The Hillman–Grassl correspondence and the enumeration of reverse plane partitions*, J. Combin. Theory Ser. A **30** (1981), 71–89.

RPPs $\pi \in \mathrm{RPP}(\lambda, c)$ are in one-to-one correspondence with c-tuples of non-intersecting paths $(P_1, \dots, P_c)$ on a lattice graph determined from the Young diagram λ.

Example: For the RPP (1) of shape $\lambda = (5,4,4,2,1)$



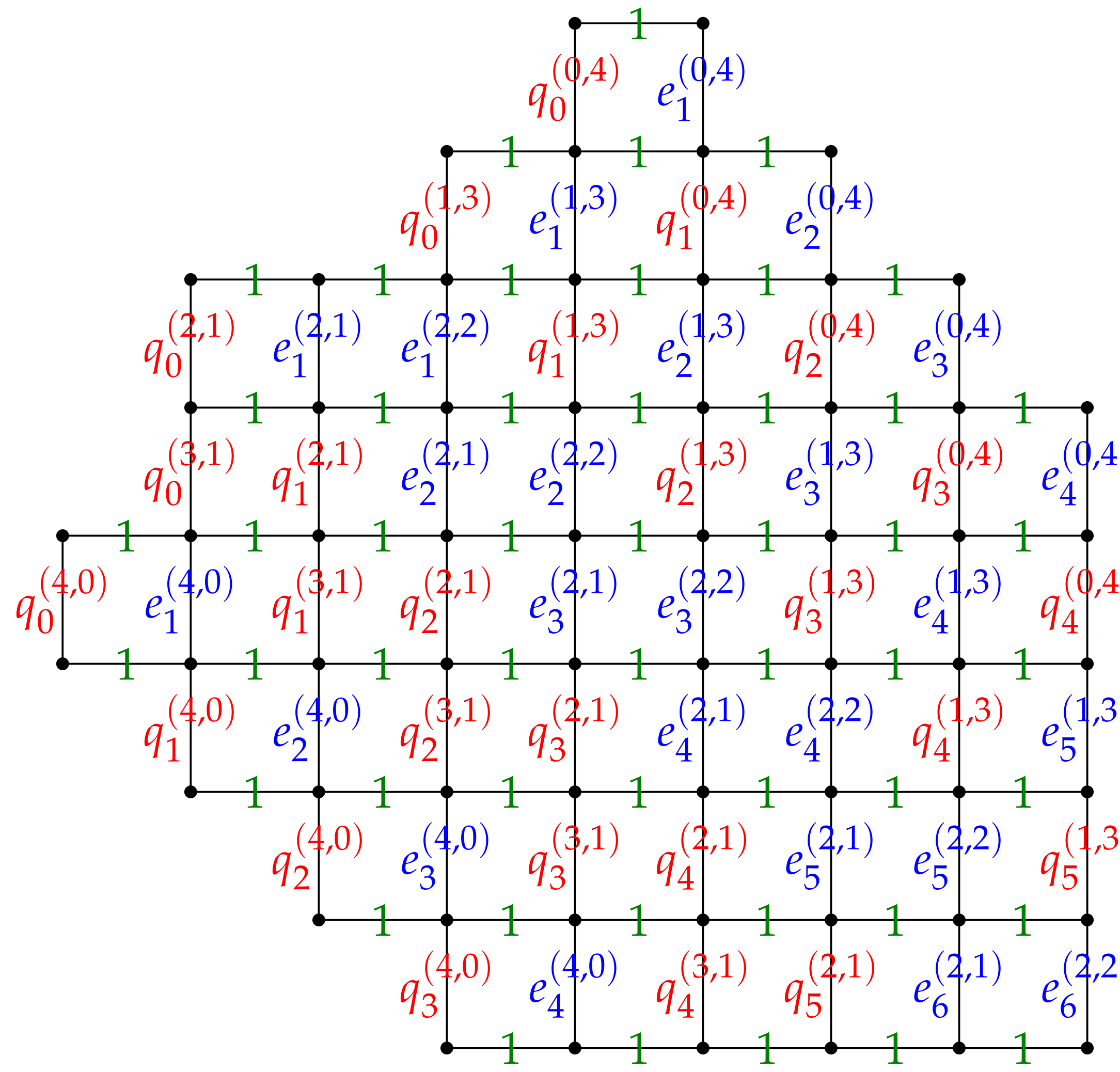
Weight by lattice paths

For the lattice graph we assume an edge-weight function w determined from the Young diagram λ, see Example below. We then define the weight $v(\pi)$ of an RPP $\pi \in \mathrm{RPP}(\lambda, c)$ as follows. Let $(P_1, \dots, P_c) \overset{1:1}{\longleftrightarrow} \pi$ by the one-to-one correspondence. Then

$$v(\pi) := \prod_{k=1}^{c} w(P_k) = \prod_{k=1}^{c} \prod_{e \in P_k} w(e) \tag{4}$$

that is the product of the weights of all the edges passed by $P_1, \dots, P_c$.

Example: For the shape $\lambda = (5,4,4,2,1)$ the edge-weight function w is looked as



The weights $q_n^{(s,t)}$, $e_n^{(s,t)}$ for vertical edges are taken from:

The two-dimensional (2D) Toda equation [HTI]

$$q_n^{(s,t+1)} + e_n^{(s+1,t)} = q_n^{(s,t)} + e_{n+1}^{(s,t)}, \tag{5a}$$

$$q_n^{(s,t+1)} e_{n+1}^{(s+1,t)} = q_{n+1}^{(s,t)} e_{n+1}^{(s,t)} \tag{5b}$$

for $(s,t) \in \mathbb{Z}^2$ and $n = 0, 1, 2, \dots$ with $e_0^{(s,t)} = 0$.

Main Theorem

Let $q_n^{(s,t)} \neq 0$, $e_n^{(s,t)} \neq 0$ solve the discrete 2D Toda equation (5). Let λ be a Young diagram with a rows and b columns. Let $c \ge 0$. Then

$$\sum_{\pi \in \mathrm{RPP}(\lambda,c)} v(\pi) = \prod_{i=1}^{a} \prod_{k=1}^{c} \frac{q_{c-k}^{(a-i,b)}}{q_{c-k}^{(a-i,b-\lambda_i)}}. \tag{6}$$

Each non-vanishing solution to the discrete 2D Toda eq. gives a weight v for RPPs having a nice formula!

We have the following solution to the discrete 2D Toda eq. (5):

$$q_n^{(s,t)} = [u]_{s+1}^{s+n} (1 - x[u]_1^s [v]_1^{t+n}), \tag{7a}$$

$$e_n^{(s,t)} = x[u]_1^{s+n-1} [v]_1^t (1 - [v]_{t+1}^{t+n}) \tag{7b}$$

involving parameters x, $u_1, u_2, \dots$ and $v_1, v_2, \dots$ where

$$[z]_\alpha^\beta := z_\alpha z_{\alpha+1} \cdots z_\beta. \tag{8}$$

For a Young diagram λ with a rows and b columns let

$$x = q^{h_\lambda(a,b)}, \qquad u_\ell = q^{\lambda_{a-\ell} - \lambda_{a-\ell+1} + 1}, \qquad v_\ell = q^{\lambda'_{b-\ell} - \lambda'_{b-\ell+1} + 1}$$

where $\lambda_i = \lambda_1 = b$ and $\lambda'_i = \lambda'_1 = a$ for $i < 0$.

Then, as an instance of Main Theorem:

Corollary: New nice formula

Let λ be a Young diagram with a rows and b columns. Let $c \ge 0$. Then, the weight

$$v(\pi) = q^{|\pi|} \prod_{(i,j)\in\lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1 - q^{c-i-k+1+\lambda'_{j+k-c-1}}}{1 - q^{c-i-k+1+\lambda'_{j+k-c}}} \tag{9a}$$

for reverse place partitions $\in \mathrm{RPP}(\lambda, c)$ satisfies a nice formula

$$\sum_{\pi \in \mathrm{RPP}(\lambda,c)} v(\pi) = \prod_{(i,j)\in\lambda} \frac{1 - q^{h_\lambda(i,j-c)}}{1 - q^{h_\lambda(i,j)}}. \tag{9b}$$

where $\lambda_i = b$ and $\lambda'_i = a$ for $i < 0$, and $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$.

Remark: The new nice formula (9) refines known nice formulae; the (9) reduces into:

- MacMahon's formula (2) by $\lambda = (b^a)$;
- Gansner's hook formula (3) by $c \to \infty$.

See [K3] for further nice formulae derived from Main Theorem.

[HTI] R. Hirota, S. Tsujimoto and T. Imai, *Difference scheme of soliton equations*, Sūrikaisekikenkyūsho Kōkyūroku, 822, pp. 144–152, 1993.

[K3] S. Kamioka, *Multiplicative partition functions for reverse plane partitions derived from an integrable dynamical system*, Sém. Lothar. Combin. **78B** (2017), Article #29, 12 pp.

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