# Dynamical degrees and singularity patterns

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Ref: T. Mase, R. Willox, A. Ramani, B. Grammaticos, arXiv:1810.02693.

# Outline

Q. Is it possible to rigorously calculate the dynamical degree for a confining equation, based only on the structure of its singularity patterns?

"Equations" in this talk:

- autonomous. (not essential)
- **3**-point mapping: $\mathbf{x_{n+1} = f(x_n, x_{n-1})}$ (essential)
- invertible = can be rationally solved in the opposite direction. (essential)
- "confining" = all singularities are confined.
  $\to \exists$ space of initial conditions only by blow-ups. (essential)
- have unbounded degree growth. (essential)

Background:

- Halburd (2017): Method to calculate the degree sequence from the singularity pattern. Ref: R. G. Halburd, *Proc. A.* 473 (2017), 20160831.
- R-G-W-M (2017): "Express method": method to calculate the degree growth from the singularity pattern. (simplified version of Halburd's method)
  Ref: A. Ramani, B. Grammaticos, R. Willox, T. Mase, *J. Phys. A* 50 (2017), 185203.

This talk: Geometric justification of the express method.

- Introduction.
- Dynamical degrees and singularity patterns.

# Introduction

# Dynamical degree

**Definition (dynamical degree)**

The limit

$$\lambda := \lim_{n\to\infty} (\deg x_n)^{1/n} \ (\geq 1)$$

is called the dynamical degree of an equation.

- $\lambda = \exp$ (algebraic entropy).
- "**deg**": the degree as a rational function of the initial values $(x_0, x_1)$.
- In the autonomous case, the dynamical degree always exists and is invariant under birational coordinate change.
- The equation is integrable $\Leftrightarrow \lambda = 1$.
- It is not easy in general to calculate $\lambda$ for a concrete equation...

Ref : M. P. Bellon, C.-M. Viallet, *Comm. Math. Phys.* 204 (1999), 425–437.

# Singularity pattern

Example:

$$x_{n+1} + x_{n-1} = \frac{1}{x_n^k} \quad (k > 0\text{: even})$$

- $x_{n-1} = \alpha$, $x_n = 0$. ($\alpha$: generic)

$\rightarrow$ $(x_n, x_{n+1}) = (0, \infty)$. The information on the initial value $\alpha$ is lost.

$\rightarrow$ The equation enters a singularity.

- $\varepsilon$: infinitesimal parameter.
  $(x_{n-1}, x_n) = (\alpha, \varepsilon)$

$$\rightarrow \quad x_{n+1} = \varepsilon^{-k} + o\left(\varepsilon^{-k}\right), \quad x_{n+2} = -\varepsilon + o(\varepsilon), \quad x_{n+3} = \alpha + o(1).$$

- $(x_{n+2}, x_{n+3})\Big|_{\varepsilon=0} = (0, \alpha)$ depends on $\alpha$.

$\rightarrow$ This singularity is confined.

- Singularity pattern ($\approx$ leading order of $\varepsilon$):

$$(0, \infty^k, 0) \quad = (0^1, \infty^k, 0^1)$$

Ref : B. Grammaticos, A. Ramani, V. Papageorgiou, *Phys. Rev. Lett.* 67 (1991), 1825–1828.

# Open pattern, cyclic pattern

Example:

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}$$

- Singularities:

$$(0, \infty^2, \infty^2, 0) : \text{open}, \quad (\infty, \infty) : \text{cyclic}.$$

- Cyclic pattern $(\infty, \infty)$:

$$(\infty, \infty, \text{REG}, \infty, \infty, \text{REG}, \cdots)$$

  "REG": some regular value.

- Open pattern $=$ non-cyclic pattern.

# Dynamical degrees and singularity patterns

$$x_{n+1} + x_{n-1} = \frac{1}{x_n^k} \quad (k \geq 2\text{: even})$$

- Open pattern: $(0, \infty^k, 0) \qquad = (0^1, 0^{-k}, 0^1)$
- Dynamical degree: $\lambda^2 - k\lambda + 1 = 0. \qquad 1\lambda^2 - k\lambda + 1 = 0.$

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}$$

- Open pattern: $(0, \infty^2, \infty^2, 0) \qquad = (0^1, 0^{-2}, 0^{-2}, 0^1)$
- Dynamical degree: $\lambda^3 - 2\lambda^2 - 2\lambda + 1 = 0. \qquad 1\lambda^3 - 2\lambda^2 - 2\lambda + 1 = 0.$

$$x_{n+1}x_{n-1} = \frac{x_n + 1}{x_n^2}$$

- Open pattern: $(-1, 0, \infty^2, 0, -1) \quad = (-1 + 0^1, 0^1, \infty^2, 0^1, -1 + 0^1)$
- Dynamical degree: $\lambda = 1$.
- Each pair of singular values gives a polynomial of $\lambda$:

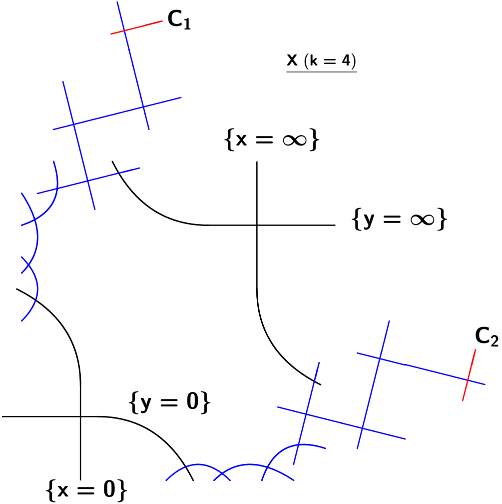
| pair of values | $(-1,$ ↓ | $0,$ ↓ | $\infty^2,$ ↓ | $0,$ ↓ | $-1)$ ↓ | |
|---|---|---|---|---|---|---|
| $0$ & $\infty$ | | $+1\lambda^2$ | $-2\lambda$ | $+1$ | | $= (\lambda - 1)^2$ |
| $0$ & $-1$ | $+1\lambda^4$ | $-1\lambda^3$ | | $-1\lambda$ | $+1$ | $= (\lambda^2 + \lambda + 1)(\lambda - 1)^2$ |
| $-1$ & $\infty$ | $+1\lambda^4$ | | $-2\lambda^2$ | | $+1$ | $= (\lambda + 1)^2(\lambda - 1)^2$ |

Q. Can we rigorously calculate $\lambda$ in this way?

Dynamical degrees and singularity patterns

$$\varphi\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad (x, y) \mapsto \left( y, -x + \frac{1}{y^k} \right).$$

Blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ $4k$ times, we can construct the space of initial conditions $X$.




- $\varphi\colon X \to X$: automorphism
- $\varphi_*\colon \operatorname{Pic} X \to \operatorname{Pic} X$: linear
- $\lambda = \max$(eigenvalues of $\varphi_*$)

Ref : T. Takenawa, *J. Phys. A* 34 (2001), 10533–10545.

Motion of curves:

- Blue curves: permuted
- $\{x = \infty\} \leftrightarrow \{y = \infty\}$
- $\{y = 0\} \to C_1 \to C_2 \to \{x = 0\}$

$\updownarrow$

Open pattern $(0, \infty^k, 0)$

- $P_X := \{F \in \operatorname{Pic} X \mid \exists m > 0 \colon \varphi_*^m F = F\}$: cyclic part.
- $\overline{\varphi}_* \colon \operatorname{Pic} X / P_X \to \operatorname{Pic} X / P_X$. $\lambda = \max($eigenvalues of $\overline{\varphi}_*)$.

In this case,

- $P_X = \operatorname{span}($Blue curves, $\{x = \infty\}, \{y = \infty\})$.
- $[C_1], [C_2]$: basis of $\operatorname{Pic} X / P_X$.
- $H_y$: the total transform of $\{y = \text{const}\} \subset \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} H_y &= kC_1 \mod P_X & (\leftarrow y = \infty) \\ &= \{y = 0\} + C_2 \mod P_X & (\leftarrow y = 0) \\ &= (\varphi_*^{-1} + \varphi_*)C_1. \end{aligned}$$

→ $(\overline{\varphi}_*^2 - k\overline{\varphi}_* + 1)[C_1] = 0$.

- $\overline{\varphi}_*[C_1] = [C_2] \notin \operatorname{span}[C_1]$.

→ The minimal polynomial of $\overline{\varphi}_*$ is $\mu_{\overline{\varphi}_*}(t) = t^2 - kt + 1$.

→ The dynamical degree $\lambda = \dfrac{k + \sqrt{k^2 - 4}}{2}$.

- The coefficient $k$ of $kC_1$ corresponds to the exponent of $\infty^k$ in the open pattern.

# General procedure to calculate $\lambda$

- $\mathbf{X}$: space of initial conditions. $\varphi\colon \mathbf{X} \to \mathbf{X}$: automorphism.
- $\mathbf{P_X} := \{\mathbf{F} \in \operatorname{Pic}\mathbf{X} \mid \exists \mathbf{m} > 0\colon \varphi_*^{\mathbf{m}}\mathbf{F} = \mathbf{F}\}$: cyclic part.
- $\overline{\varphi}_*\colon \operatorname{Pic}\mathbf{X}/\mathbf{P_X} \to \operatorname{Pic}\mathbf{X}/\mathbf{P_X}$. $\lambda = \max(\text{eigenvalues of } \overline{\varphi}_*)$.
- It is sufficient to calculate $\mu_{\overline{\varphi}_*}(\mathbf{t})$.

- $\mathbf{H_y}$: the total transform of $\{\mathbf{y} = \text{const}\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.
- Using singularity patterns, express $\mathbf{H_y}$ for many $\mathbf{y}$'s as

$$\mathbf{H_y} = \underbrace{\sum_{\mathbf{m},\mathbf{j}} \mathbf{a_{mj}}\varphi_*^{\mathbf{m}}\mathbf{C_j}}_{(\uparrow\, \mathbf{y} = \alpha_1)} = \underbrace{\sum_{\mathbf{m},\mathbf{j}} \mathbf{b_{mj}}\varphi_*^{\mathbf{m}}\mathbf{C_j}}_{(\uparrow\, \mathbf{y} = \alpha_2)} = \cdots \mod \mathbf{P_X}$$

and obtain a relation

$$\psi(\overline{\varphi}_*)\mathbf{F} = \mathbf{0} \text{ in } \operatorname{Pic}\mathbf{X}/\mathbf{P_X}$$

for some polynomial $\psi(\overline{\varphi}_*)$ and $\mathbf{F} \in (\operatorname{Pic}\mathbf{X}/\mathbf{P_X}) \setminus \{\mathbf{0}\}$.

$\rightarrow$ At least one factor of $\psi(\mathbf{t})$ appears in $\mu_{\overline{\varphi}_*}(\mathbf{t})$.

- The coefficients $\mathbf{a_{mj}}, \mathbf{b_{mj}}, \cdots$ correspond to the exponents in open patterns.
- In the case of $\mathbf{3}$-point mappings, all contracted curves in $\mathbb{P}^1 \times \mathbb{P}^1$ have the form $\{\mathbf{y} = \alpha\}$.

# Questions

Basic idea:

- $\psi(\overline{\varphi}_*)\mathbf{F} = \mathbf{0}$ in $\mathbf{Pic}\,\mathbf{X}/\mathbf{P_X}$.
- At least one factor of $\psi(\mathbf{t})$ appears in $\mu_{\overline{\varphi}_*}(\mathbf{t})$.

Q:

1. Which factor of $\psi(\mathbf{t})$ appears in $\mu_{\overline{\varphi}_*}(\mathbf{t})$?
2. How do we check $\mathbf{F} \neq \mathbf{0}$ in $\mathbf{Pic}\,\mathbf{X}/\mathbf{P_X}$?

A:

1. Classification of $\mu_{\overline{\varphi}_*}(\mathbf{t})$ is useful.
   (Next page)
2. There are several useful techniques.
   (See Poster)

Poster:
P43: "Dynamical degrees and singularity patterns (2)"

## Classification of $\mu_{\overline{\varphi}_*}(\mathbf{t})$

### Theorem [Diller-Favre]

The Jordan normal form of $\varphi_*$ can only take one of the following three forms:

(a) $\begin{bmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_\rho \end{bmatrix}$

$\nu_j$: roots of unity
**deg $x_n$**: bounded

(b) $\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & \nu_1 & & \\ & & & & \ddots & \\ & & & & & \nu_{\rho-3} \end{bmatrix}$

$\nu_j$: roots of unity
$\deg x_n \sim n^2$

(c) $\begin{bmatrix} \lambda & & & & \\ & 1/\lambda & & & \\ & & \nu_1 & & \\ & & & \ddots & \\ & & & & \nu_{\rho-2} \end{bmatrix}$

$\lambda > 1$, $|\nu_j| = 1$
$\deg x_n \sim \lambda^n$


Ref :
- J. Diller, C. Favre, *Amer. J. Math.* 123 (2001), 1135–1169.
- T. Mase, *J. Integrable Syst.* 3 (2018), xyy010. (Nonautonomous case)


### Corollary

In Cases (b) and (c), the minimal polynomial $\mu_{\overline{\varphi}_*}(\mathbf{t})$ is

(b) $\mu_{\overline{\varphi}_*}(\mathbf{t}) = (\mathbf{t}-1)^2$.

(c) $\mu_{\overline{\varphi}_*}(\mathbf{t}) = \mu_\lambda(\mathbf{t})$: minimal polynomial for $\lambda$ over $\mathbb{Q}$.

In particular, no cyclotomic polynomial other than $(\mathbf{t}-1)$ appears in $\mu_{\overline{\varphi}_*}(\mathbf{t})$ as a factor.

# Useful proposition to show $F \neq 0$

- $\Phi_k(t)$: $k$-th cyclotomic polynomial.
- $\Phi_1(t) = t - 1$, $\Phi_2(t) = t + 1$, $\Phi_3(t) = t^2 + t + 1$, $\Phi_4(t) = t^2 + 1$, $\cdots$

**Proposition (See Poster)**

Let $\varphi\colon X \to X$ be an automorphism on the space of initial conditions $X$ for a mapping with unbounded degree growth. Let $C_1, \ldots, C_m \subset X$ be irreducible curves that appear in the non-cyclic patterns for $\varphi$, let $a_1, \ldots, a_m > 0$ and consider

$$F = \left[\sum_{j=1}^{m} a_j C_j\right] \in \operatorname{Pic} X / P_X.$$

Then,

$$(\overline{\varphi}_*^{m_0} - 1) \left(\prod_{j=1}^{\ell} \Phi_{k_j}(\overline{\varphi}_*^{m_j})\right) F \neq 0 \text{ in } \operatorname{Pic} X / P_X$$

for $\ell \geq 0$, $k_j \geq 2$ and $m_j \geq 1$ ($j = 0, 1, \ldots, \ell$). In particular,

$$F \neq 0, \quad (\overline{\varphi}_* - 1) F \neq 0 \text{ in } \operatorname{Pic} X / P_X.$$

- Any positive linear combination of open curves is nonzero in $\operatorname{Pic} X / P_X$.

# Example 1: $x_{n+1} + x_{n-1} = x_n + \dfrac{1}{x_n^k}$ ($k > 0$: even)

- Open pattern:

$$\begin{array}{cccccccccc} ( & 0, & & \infty^k, & & \infty^k, & & 0 & & ) \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ \{y=0\} = & C_0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & \{x=0\} \end{array}$$

- Cyclic pattern: $(\infty, \infty)$

$\to$ $\{y = \infty\} \in P_X$.

- Express $H_y$ for the singular values $y = \infty, 0$:

$$\begin{aligned} H_y &= kC_1 + kC_2 \mod P_X & (\leftarrow y = \infty) \\ &= C_0 + C_3 \mod P_X. & (\leftarrow y = 0) \end{aligned}$$

$\to$ $(\overline{\varphi}_*^3 - k\overline{\varphi}_*^2 - k\overline{\varphi}_* + 1)[C_0] = (\overline{\varphi}_*^2 - (k+1)\overline{\varphi}_* + 1)(\overline{\varphi}_* + 1)[C_0] = 0$.

- $[C_0] \neq 0$ in $\operatorname{Pic} X / P_X$. (by Proposition. See Poster)
- $\mu_{\overline{\varphi}_*}(t)$ cannot have the factor $(t+1)$. (Classification)

$\to$ Case (c). $\lambda^2 - (k+1)\lambda + 1 = 0$. Nonintegrable.

- Correspondence:
  - Open pattern: $(0^1, \infty^k, \infty^k, 0^1)$.
  - Relation: $(1\overline{\varphi}_*^3 - k\overline{\varphi}_*^2 - k\overline{\varphi}_* + 1)[C_0] = 0$.
  - Dynamical degree: $1\lambda^3 - k\lambda^2 - k\lambda + 1 = 0$.

# Example 2: $x_{n+1}x_{n-1} = \dfrac{x_n + 1}{x_n^2}$

- Open pattern:

$$
\begin{array}{ccccccccccc}
( & & -1, & & 0, & & \infty^2, & & 0, & & -1 & & ) \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\{y=-1\} = & & C_0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & C_4 & \to & \{x=-1\}
\end{array}
$$

- Cyclic pattern: $(0, \infty^2, 0, \mathrm{REG}, \infty, 0, \infty, \mathrm{REG}, \cdots)$.

$\to$ $\{y = 0\}, \{y = \infty\} \in P_X$.

- Express $H_y$ for the singular values $y = -1, 0, \infty$:

$$H_y = C_0 + C_4 = C_1 + C_3 = 2C_2 \quad \text{mod } P_X.$$

- Each pair of singular values gives a relation:
  - $y = 0$ & $\infty$: $(\overline{\varphi}_*^2 - 2\overline{\varphi}_* + 1)[C_1] = 0$.
  - $y = 0$ & $-1$: $(\overline{\varphi}_*^4 - \overline{\varphi}_*^3 - \overline{\varphi}_* + 1)[C_0] = 0$.
  - $y = -1$ & $\infty$: $(\overline{\varphi}_*^4 - 2\overline{\varphi}_*^2 + 1)[C_0] = 0$.
- Correspondence (for each pair of singular values):

| pair of values | $(-1,$ ↓ | $0,$ ↓ | $\infty^2,$ ↓ | $0,$ ↓ | $-1)$ ↓ | | |
|---|---|---|---|---|---|---|---|
| $0$ & $\infty$ | | $+1\lambda^2$ | $-2\lambda$ | $+1$ | | $\leftrightarrow$ | $(\overline{\varphi}_*^2 - 2\overline{\varphi}_* + 1)[C_1] = 0$ |
| $0$ & $-1$ | $+1\lambda^4$ | $-1\lambda^3$ | | $-1\lambda$ | $+1$ | $\leftrightarrow$ | $(\overline{\varphi}_*^4 - \overline{\varphi}_*^3 - \overline{\varphi}_* + 1)[C_0] = 0$ |
| $-1$ & $\infty$ | $+1\lambda^4$ | | $-2\lambda^2$ | | $+1$ | $\leftrightarrow$ | $(\overline{\varphi}_*^4 - 2\overline{\varphi}_*^2 + 1)[C_0] = 0$ |

- $[C_0] \neq 0$ in $\mathrm{Pic}\, X / P_X$. (by Proposition. See Poster)

$\to$ Case (b). Integrable.

# Example 3: $x_{n+1}x_{n-1} = \dfrac{a(x_n - b)}{x_n - 1} \quad (a, b \neq 0, 1, a \neq b)$

- Open pattern 1:

$$\begin{array}{ccccccccccccc} & ( & 1, & & \infty, & & a, & & 0 & & b & & ) \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & & & \\ \{y=1\} = & & C_0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & C_4 & \to & \{x=b\} \end{array}$$

- Open pattern 2:

$$\begin{array}{ccccccccccccc} & ( & b, & & 0, & & a, & & \infty, & & 1 & & ) \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\ \{y=b\} = & & C'_0 & \to & C'_1 & \to & C'_2 & \to & C'_3 & \to & C'_4 & \to & \{x=1\} \end{array}$$

- No cyclic pattern.
- $\{y = 0\}, \{y = \infty\}, \{y = a\} \in P_X$. (checked by direct calculation)
- Express $H_y$ for the singular values: $y = 1, b, \infty, 0, a$.

$$H_y = C_0 + C'_4 = C'_0 + C_4 = C_1 + C'_3 = C'_1 + C_3 = C_2 + C'_2 \mod P_X$$

$$(y = 1) \quad (y = b) \quad (y = \infty) \quad (y = 0) \quad (y = a)$$

- Using $\boxed{y = \infty} + \boxed{y = 0} = 2\boxed{y = a}$, we have $(\overline{\varphi}_* - 1)^2[C_1 + C'_1] = 0$.
- $[C_1 + C'_1] \neq 0$ in $\mathrm{Pic}\, X/P_X$. (by Proposition. See Poster)

$\to$ $\mu_{\overline{\varphi}_*}(t)$ has the factor $(t - 1)$.

$\to$ Case (b). Integrable. (Classification)

# Example 4: $x_{n+1} = \dfrac{(x_n + 3a)x_{n-1} - 2ax_n}{x_n - 3a} \quad (a \neq 0)$

- Open pattern 1:

$$\{y=3a\} = \underset{\substack{\updownarrow \\ 3a,}}{C_0} \to \underset{\substack{\updownarrow \\ \infty,}}{C_1} \to \underset{\substack{\updownarrow \\ a,}}{C_2} \to \underset{\substack{\updownarrow \\ \infty}}{C_3} \to \underset{\substack{\updownarrow \\ -a}}{C_4} \to \underset{\substack{\updownarrow \\ \infty}}{C_5} \to \underset{\substack{\updownarrow \\ -3a}}{C_6} \to \{x=-3a\}$$

$$\left( 3a, \ \infty, \ a, \ \infty \ -a \ \infty \ -3a \right)$$

- Open pattern 2:

$$\{y=-3a\} = C'_0 \to C'_1 \to C'_2 \to C'_3 \to \{x=3a\}$$

$$\left( -3a, \ -a, \ a, \ 3a \right)$$

- No cyclic pattern.
- $\{y=\infty\}, \{y=\pm a\} \in P_X$. (checked by direct calculation)
- Express $H_y$ for the singular values: $y = \infty, 3a, a, -a, -3a$.

$$H_y = C_1 + C_3 + C_5 = C_0 + C'_3 = C_2 + C'_2 = C_4 + C'_1 = C_6 + C'_0 \mod P_X$$

$$(y=\infty) \quad (y=3a) \quad (y=a) \quad (y=-a) \quad (y=-3a)$$

$\to$ $(\overline{\varphi}_*^6 - \overline{\varphi}_*^5 + \overline{\varphi}_*^4 - 2\overline{\varphi}_*^3 + \overline{\varphi}_*^2 - \overline{\varphi}_* + 1)[C_1] = 0$.

- $[C_1] \neq 0$ in $\mathrm{Pic}\, X / P_X$. (by Proposition. See Poster)
- $(\overline{\varphi}_*^6 - \overline{\varphi}_*^5 + \overline{\varphi}_*^4 - 2\overline{\varphi}_*^3 + \overline{\varphi}_*^2 - \overline{\varphi}_* + 1) = (\overline{\varphi}_*^2 + \overline{\varphi}_* + 1)(\overline{\varphi}_*^2 + 1)(\overline{\varphi}_* - 1)^2$.

$\to$ $\mu_{\overline{\varphi}_*}(t)$ has the factor $(t-1)$. (Classification)

$\to$ Case (b). Integrable.

# Summary

- Procedure to calculate $\boldsymbol{\lambda}$, based only on the structure of its singularity patterns: Using singularity patterns, express $\mathbf{H_y}$ for many $\mathbf{y}$'s as

$$\mathbf{H_y} = \sum_{\mathbf{m,j}} \mathbf{a_{mj}} \varphi_*^{\mathbf{m}} \mathbf{C_j} = \sum_{\mathbf{m,j}} \mathbf{b_{mj}} \varphi_*^{\mathbf{m}} \mathbf{C_j} = \cdots \quad \mathbf{mod}\ \mathbf{P_X}$$

  and obtain a relation

$$\psi(\overline{\varphi}_*)\mathbf{F} = \mathbf{0} \text{ in } \mathbf{Pic}\,\mathbf{X}/\mathbf{P_X}$$

  for some polynomial $\psi(\overline{\varphi}_*)$ and $\mathbf{F} \in (\mathbf{Pic}\,\mathbf{X}/\mathbf{P_X}) \setminus \{\mathbf{0}\}$.

$\rightarrow$ At least one factor of $\psi(\mathbf{t})$ appears in $\mu_{\overline{\varphi}_*}(\mathbf{t})$.

- The coefficients $\mathbf{a_{mj}}, \mathbf{b_{mj}}, \cdots$ correspond to the exponents in open patterns.

- This method is rigorous.
- It is not necessary to construct a space of initial conditions, although the method relies on the theory of spaces of initial conditions.
- Classification of $\mu_{\overline{\varphi}_*}(\mathbf{t})$ is useful. There are several other useful techniques. (See Poster)

Poster: P43: "Dynamical degrees and singularity patterns (2)"

# Reference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