Cluster algebra and generalized q-Painlevé VI systems of type A (arXiv:1810.03252)

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Higher order generalizations of the q-Painlevé equations (we call it q-Painlevé systems) has been studied from some points of view.

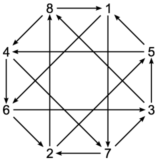
- Birational representations of extended affine Weyl groups
  - Type $(A_{m-1} + A_{n-1})^{(1)}$ [Kajiwara-Noumi-Yamada 02]
  - Type $D^{(1)}_{2n+3}$ (q-Sasano system) [Masuda 15]
- Cluster mutations
  - Somos-type Y-system [Hone-Inoue 14]
- Compatibility conditions of Lax pairs
  - q-Garnier system [Sakai 05], [Nagao-Yamada 18]
  - Similarity reduction of lattice q-UC hierarchy [Tsuda 10]
  - Similarity reduction of q-Drinfeld-Sokolov hierarchy [S 15], [S 17]

In this talk we present the following result.

1. A birational representation of $\widetilde{W}((A_{2n+1} + A_1 + A_1)^{(1)})$ is formulated with the aid of cluster mutations.
2. This extended affine Weyl group provides the latter three q-Painlevé systems (arising from Lax pairs) as translations.

We recall the derivation of Jimbo-Sakai's q-P_{VI} from cluster mutations.
Consider a skew-symmetric matrix corresponding to the q-P_{VI} quiver (cf. [Okubo 15])

$$\Lambda = (\lambda_{i,j})_{i,j=1}^{8} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$



Example

A mutation $\mu_1 : (\Lambda, \boldsymbol{y}) \mapsto (\Lambda', \boldsymbol{y}')$ *is given explicitly by*

$$\lambda'_{5,6} = \lambda'_{5,7} = \lambda'_{8,6} = \lambda'_{8,7} = 1, \quad \lambda'_{6,5} = \lambda'_{7,5} = \lambda'_{6,8} = \lambda'_{7,8} = -1,$$
$$\lambda'_{1,i} = -\lambda_{1,i}, \quad \lambda'_{i,1} = -\lambda_{i,1} \quad (i \neq 1), \quad \lambda'_{i,j} = \lambda_{i,j} \quad (\textit{otherwise}),$$
$$y'_5 = y_5(1 + y_1), \quad y'_6 = \frac{y_6}{1 + \frac{1}{y_1}}, \quad y'_7 = \frac{y_7}{1 + \frac{1}{y_1}}, \quad y'_8 = y_8(1 + y_1),$$
$$y'_1 = \frac{1}{y_1}, \quad y'_i = y_i \quad (\textit{otherwise}).$$

We can formulate a birational representation of $\widetilde{W}(A_3^{(1)})$ with the aid of cluster mutations.

Fact (cf. [Bershtein-Gavrylenko-Marshakov 17])

Let

$$r_0 = (1,2)\,\mu_1\,\mu_2, \quad r_1 = (5,6)\,\mu_5\,\mu_6, \quad r_2 = (3,4)\,\mu_3\,\mu_4, \quad r_2 = (7,8)\,\mu_7\,\mu_8,$$
$$\pi = (1,5,3,7)\,(2,6,4,8),$$

where μ_k stands for a mutation at a vertex k and (i,j) a permutation of vertices i and j. Then the q-P_{VI} quiver is invariant under their actions. Moreover they satisfy

$$r_i^2 = 1, \quad (r_i\,r_{i+1})^3 = 1, \quad (r_i\,r_j)^2 = 1 \quad (j \neq i, i \pm 1), \quad \pi^4 = 1, \quad r_i\,\pi = \pi\,r_{i+1},$$

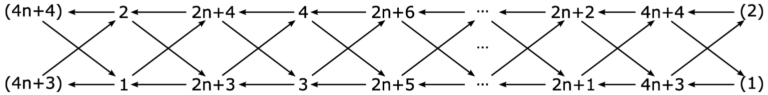
where $r_{i+4} = r_i$.

Fact (cf. [Okubo 15])

A translation $T = (\pi\,r_0\,r_2)^2$ of the extended affine Weyl group provides q-P_{VI}.

In this talk we consider an extension of this previous work.

Consider the following quiver on a torus (cf. [Inoue-Ishibashi-Ohya 18]).



Let $y_1, \dots, y_{4n+4}$ be coefficients. We define parameters by

$$\alpha_{2i} = y_{2i+1}\, y_{2i+2}, \quad \alpha_{2i+1} = y_{2i+2n+3}\, y_{2i+2n+4} \quad (i = 0, \dots, n), \quad q = \prod_{i=0}^{2n+1} \alpha_i,$$

$$\beta = \prod_{i=0}^{n} \frac{1}{y_{2i+2}\, y_{2i+2n+3}}, \quad \beta' = \prod_{i=0}^{n} y_{2i+1}\, y_{2i+2n+3},$$

and dependent variables by

$$\varphi_{2i} = y_{2i+1}, \quad \varphi_{2i+1} = y_{2i+2n+3} \quad (i = 0, \dots, n).$$

We use a notation for a periodicity

$$\alpha_{i+2n+2} = \alpha_i, \quad \varphi_{i+2n+2} = \varphi_i.$$

Definition (cf. [Bershtein-Gavrylenko-Marshakov 17])

We define reflections $r_0, \dots, r_{2n+1}$ *by*

$$r_{2i} = (2i+1, 2i+2)\, \mu_{2i+2}\, \mu_{2i+1},$$
$$r_{2i+1} = (2i+2n+3, 2i+2n+4)\, \mu_{2i+2n+4}\, \mu_{2i+2n+3}.$$

They act on $(\alpha_i, \beta, \beta'; \varphi_i)$ as

$$r_i(\alpha_{i-1}) = \alpha_i\, \alpha_{i-1}, \quad r_i(\alpha_i) = \frac{1}{\alpha_i}, \quad r_i(\alpha_{i+1}) = \alpha_i\, \alpha_{i+1}, \quad r_i(\alpha_j) = \alpha_j,$$
$$r_i(\beta) = \beta, \quad r_i(\beta') = \beta',$$
$$r_i(\varphi_{i-1}) = \frac{\alpha_i + \varphi_i}{1 + \varphi_i}\varphi_{i-1}, \quad r_i(\varphi_i) = \frac{\varphi_i}{\alpha_i}, \quad r_i(\varphi_{i+1}) = \frac{1 + \varphi_i}{1 + \frac{\varphi_i}{\alpha_i}}\varphi_{i+1}, \quad r_i(\varphi_j) = \varphi_j,$$

for $j \neq i, i \pm 1$ (cf. [Kajiwara-Noumi-Yamada 02]).

Definition (cf. [Inoue-Lam-Pylyavskyy 16], [Masuda-Okubo-Tsuda 18])

We define reflections s_0, s_1, s'_0, s'_1 *by*

$$\begin{aligned}
s_0 &= \mu_1\,\mu_{2n+4}\,\mu_3\,\mu_{2n+6}\cdots\mu_{2n-1}\,\mu_{4n+2}\,\mu_{2n+1}\,(2n+1,4n+4)\\
&\qquad \mu_{2n+1}\,\mu_{4n+2}\,\mu_{2n-1}\cdots\mu_{2n+6}\,\mu_3\,\mu_{2n+4}\,\mu_1,\\
s_1 &= \mu_2\,\mu_{2n+3}\,\mu_4\,\mu_{2n+5}\cdots\mu_{2n}\,\mu_{4n+1}\,\mu_{2n+2}\,(2n+2,4n+3)\\
&\qquad \mu_{2n+2}\,\mu_{4n+1}\,\mu_{2n}\cdots\mu_{2n+5}\,\mu_4\,\mu_{2n+3}\,\mu_2,\\
s'_0 &= \mu_1\,\mu_{2n+3}\,\mu_3\,\mu_{2n+5}\cdots\mu_{2n-1}\,\mu_{4n+1}\,\mu_{2n+1}\,(2n+1,4n+3)\\
&\qquad \mu_{2n+1}\,\mu_{4n+1}\,\mu_{2n-1}\cdots\mu_{2n+5}\,\mu_3\,\mu_{2n+3}\,\mu_1,\\
s'_1 &= \mu_2\,\mu_{2n+4}\,\mu_4\,\mu_{2n+6}\cdots\mu_{2n}\,\mu_{4n+2}\,\mu_{2n+2}\,(2n+2,4n+4)\\
&\qquad \mu_{2n+2}\,\mu_{4n+2}\,\mu_{2n}\cdots\mu_{2n+6}\,\mu_4\,\mu_{2n+4}\,\mu_2.
\end{aligned}$$

They act on $(\alpha_i, \beta, \beta')$ as

$$s_k(\alpha_i) = \alpha_i, \quad s_k(\beta') = \beta' \quad (k=0,1), \quad s_0(\beta) = \frac{1}{q^2\beta}, \quad s_1(\beta) = \frac{1}{\beta},$$

$$s'_l(\alpha_i) = \alpha_i, \quad s'_l(\beta) = \beta \quad (l=0,1), \quad s'_0(\beta') = \frac{1}{\beta'}, \quad s'_1(\beta') = \frac{q^2}{\beta'}.$$

They also act on (φ_i) as

$$s_0(\varphi_{2i}) = \frac{\varphi_{2i+1}}{\alpha_{2i+1}} \frac{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \varphi_{2i+2k} \frac{\alpha_{2i+2k+1}}{\varphi_{2i+2k+1}})(1+\varphi_{2i+2j})}{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \varphi_{2i+2k+2} \frac{\alpha_{2i+2k+3}}{\varphi_{2i+2k+3}})(1+\varphi_{2i+2j+2})},$$

$$s_0(\varphi_{2i+1}) = \alpha_{2i+1}\, \varphi_{2i+2} \frac{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \frac{\alpha_{2i+2k+3}}{\varphi_{2i+2k+3}} \varphi_{2i+2k+4})(1+\frac{\alpha_{2i+2j+3}}{\varphi_{2i+2j+3}})}{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \frac{\alpha_{2i+2k+1}}{\varphi_{2i+2k+1}} \varphi_{2i+2k+2})(1+\frac{\alpha_{2i+2j+1}}{\varphi_{2i+2j+1}})},$$

$$s_1(\varphi_{2i}) = \alpha_{2i}\, \varphi_{2i+1} \frac{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \frac{\alpha_{2i+2k+2}}{\varphi_{2i+2k+2}} \varphi_{2i+2k+3})(1+\frac{\alpha_{2i+2j+2}}{\varphi_{2i+2j+2}})}{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \frac{\alpha_{2i+2k}}{\varphi_{2i+2k}} \varphi_{2i+2k+1})(1+\frac{\alpha_{2i+2j}}{\varphi_{2i+2j}})},$$

$$s_1(\varphi_{2i+1}) = \frac{\varphi_{2i+2}}{\alpha_{2i+2}} \frac{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \varphi_{2i+2k+1} \frac{\alpha_{2i+2k+2}}{\varphi_{2i+2k+2}})(1+\varphi_{2i+2j+1})}{\sum_{j=0}^{n}(\prod_{k=0}^{j-1} \varphi_{2i+2k+3} \frac{\alpha_{2i+2k+4}}{\varphi_{2i+2k+4}})(1+\varphi_{2i+2j+3})},$$

$$s'_0(\varphi_i) = \frac{1}{\varphi_{i+1}} \frac{\sum_{j=0}^{2n+1} \prod_{k=0}^{j-1} \frac{1}{\varphi_{i+k+2}}}{\sum_{j=0}^{2n+1} \prod_{k=0}^{j-1} \frac{1}{\varphi_{i+k}}},$$

$$s'_1(\varphi_i) = \frac{\alpha_i}{\frac{\varphi_{i+1}}{\alpha_{i+1}}} \frac{\sum_{j=0}^{2n+1} \prod_{k=0}^{j-1} \frac{\varphi_{i+k}}{\alpha_{i+k}}}{\sum_{j=0}^{2n+1} \prod_{k=0}^{j-1} \frac{\varphi_{i+k+2}}{\alpha_{i+k+2}}}.$$

Definition

We define rotations π, π' by

$$\begin{aligned}\pi &= (1, 2n+3, 3, 2n+5, \ldots, 2n+1, 4n+3, 1)\\ &\qquad (2, 2n+4, 4, 2n+6, \ldots, 2n+2, 4n+4, 2),\\ \pi' &= (1, 2n+4, 3, 2n+6, \ldots, 2n+1, 4n+4, 1)\\ &\qquad (2, 2n+3, 4, 2n+5, \ldots, 2n+2, 4n+3, 2).\end{aligned}$$

They act on $(\alpha_i, \beta, \beta'; \varphi_i)$ as

$$\pi(\alpha_i) = \alpha_{i+1}, \quad \pi(\beta) = \frac{1}{q\beta}, \quad \pi(\beta') = \beta', \quad \pi(\varphi_i) = \varphi_{i+1},$$
$$\pi'(\alpha_i) = \alpha_{i+1}, \quad \pi'(\beta) = \beta, \quad \pi'(\beta') = \frac{q}{\beta'}, \quad \pi'(\varphi_i) = \frac{\alpha_{i+1}}{\varphi_{i+1}}.$$

Theorem ([Okubo-S 18], [Masuda-Okubo-Tsuda 18])

The birational transformations $r_0, \ldots, r_{2n+1}, s_0, s_1, s_0', s_1', \pi, \pi'$ *satisfy the fundamental relations of an extended affine Weyl group of type* $(A_{2n+1} + A_1 + A_1)^{(1)}$

$$
\begin{aligned}
&r_i^2 = 1, \quad (r_i\, r_{i+1})^3 = 1, \quad (r_i\, r_j)^2 = 1 \quad (j \neq i, i \pm 1), \\
&s_k^2 = 1, \quad (s_l')^2 = 1, \quad (s_k\, s_l')^2 = 1, \\
&\pi^{2n+2} = 1, \quad (\pi')^{2n+2} = 1, \quad \pi\pi' = \pi'\pi, \quad \pi^2 = (\pi')^2, \\
&(r_i\, s_k)^2 = 1, \quad (r_i\, s_l')^2 = 1, \\
&r_i\, \pi = \pi\, r_{i+1}, \quad r_i\, \pi' = \pi' r_{i+1}, \\
&s_k\, \pi = \pi\, s_{k+1}, \quad s_k\, \pi' = \pi' s_k, \quad s_l'\, \pi = \pi\, s_l', \quad s_l'\, \pi' = \pi' s_{l+1}',
\end{aligned}
$$

where

$$
r_{i+2n+2} = r_i, \quad s_{k+2} = s_k, \quad s_{l+2}' = s_l'.
$$

Remark

The fundamental relation $(s_k\, s_l')^2 = 1$ *was shown by R. Inoue, T. Lam and P. Pylyavskyy in 2016.*

Definition

We define translations T_1, T_2, T_3 and a quasi-translation T_4 by

$$T_1 = s_1' \, s_1 \, \pi' \pi^{-1}, \quad T_2 = (r_2 \, r_3 \ldots r_{2n+1} \, \pi')^2, \quad T_3 = r_1 \, r_2 \ldots r_{2n+1} \, s_1' \, \pi',$$
$$T_4 = (r_0 \, r_2 \ldots r_{2n} \, \pi')^2.$$

They act on $(\alpha_i, \beta, \beta')$ as

$$T_1(\beta) = q\,\beta, \quad T_1(\beta') = q\,\beta',$$
$$T_2(\alpha_0) = \frac{\alpha_0}{q}, \quad T_2(\alpha_n) = q\,\alpha_n, \quad T_2(\alpha_{n+1}) = \frac{\alpha_{n+1}}{q}, \quad T_2(\alpha_{2n+1}) = q\,\alpha_{2n+1},$$
$$T_3(\alpha_0) = q\,\alpha_0, \quad T_3(\alpha_1) = \frac{\alpha_1}{q}, \quad T_3(\beta) = q\,\beta,$$
$$T_4(\alpha_{2i}) = \frac{1}{\alpha_{2i+1}\,\alpha_{2i+2}\,\alpha_{2i+3}}, \quad T_4(\alpha_{2i+1}) = \alpha_{2i+1}\,\alpha_{2i+2}\,\alpha_{2i+3}\,\alpha_{2i+4}\,\alpha_{2i+5}.$$

Note that iterative action of T_4 becomes a translation as

$$T_4^m(\alpha_{2i}) = \frac{\alpha_{2i}}{q}, \quad T_4^m(\alpha_{2i+1}) = q\,\alpha_{2i+1} \quad (n = 2m-1),$$
$$T_4^{2m+1}(\alpha_{2i}) = \frac{\alpha_{2i}}{q^2}, \quad T_4^{2m+1}(\alpha_{2i+1}) = q^2\,\alpha_{2i+1} \quad (n = 2m).$$

Theorem ([Okubo-S 18])

These translations gives the known three higher order q-Painlevé systems.

- T_1: *q-DS system q-$P_{(n+1,n+1)}$*
- T_2: *Sakai's q-Garnier system.*
- T_4: *Tsuda's q-LUC system*

Conjecture

- T_3: *Nagao-Yamada's variation of the q-Garnier system*

T_1: q-DS system q-$P_{(n+1,n+1)}$

The following lemma is obtained in the course of proof of the fundamental relations.

Lemma

We have

$$s_1'\,s_1(\varphi_{2i-1}) = s_1\,s_1'(\varphi_{2i-1}) = \frac{1+\frac{\alpha_{2i}}{\varphi_{2i}}}{1+\frac{\alpha_{2i-2}}{\varphi_{2i-2}}}\,\frac{\alpha_{2i+1}}{\varphi_{2i+1}}\,\frac{S_{2i-2}\,S'_{2i}+\frac{\alpha_{2i-2}}{\varphi_{2i-2}}\,\alpha_{2i-1}\,S'_{2i-2}\,S_{2i}}{S_{2i}\,S'_{2i+2}+\frac{\alpha_{2i}}{\varphi_{2i}}\,\alpha_{2i+1}\,S'_{2i}\,S_{2i+2}},$$

$$s_1'\,s_1(\varphi_{2i}) = s_1\,s_1'(\varphi_{2i}) = \frac{\alpha_{2i}\,\alpha_{2i+1}}{\frac{\varphi_{2i}}{\alpha_{2i}}}\,\frac{S_{2i+2}\,S'_{2i}}{S_{2i}\,S'_{2i+2}},$$

where

$$S_{2i} = \sum_{j=0}^{n}\left(\prod_{k=0}^{j-1}\frac{\alpha_{2i+2k}}{\varphi_{2i+2k}}\varphi_{2i+2k+1}\right)\left(1+\frac{\alpha_{2i+2j}}{\varphi_{2i+2j}}\right),$$

$$S'_{2i} = \sum_{j=0}^{n}\left(\prod_{k=0}^{j-1}\frac{\varphi_{2i+2k}}{\alpha_{2i+2k}}\,\frac{\varphi_{2i+2k+1}}{\alpha_{2i+2k+1}}\right)\left(1+\frac{\varphi_{2i+2j}}{\alpha_{2i+2j}}\right).$$

T_1: q-DS system q-$P_{(n+1,n+1)}$

Let a_i, b_i $(i = 1, \ldots, n+1)$ be parameters and t an independent variable defined by

$$\frac{a_i}{b_i} = \alpha_{2i-1}, \quad \frac{b_i}{a_{i+1}} = \alpha_{2i}, \quad q^{n-1} t \prod_{i=1}^{n+1} a_i b_i = \beta, \quad t = \beta'.$$

Also let f_i, g_i $(i = 0, \ldots, n)$ be dependent variables defined by

$$f_i = \frac{1+\varphi_{2i}}{1+\varphi_0} \prod_{j=i}^{n} \varphi_{2j+1}\, \varphi_{2j+2}, \quad \frac{g_i}{b_i} = -\frac{1}{\varphi_{2i}}.$$

Note that

$$f_0 = t, \quad g_0 = \frac{1}{q^{\frac{n-2}{2}}\, t\, g_1 \ldots g_n}, \quad b_0 = q\, b_{n+1}.$$

T_1: q-DS system q-$P_{(n+1,n+1)}$

Theorem ([Okubo-S 18])

If we set $\overline{f}_i = T_1(f_i)$ *and* $\overline{g}_i = T_1(g_i)$*, then they satisfy*

$$f_i\,\overline{f}_i = q\,t\,\frac{F_i\,F_{i+1}\left(\frac{b_i}{\overline{g}_i}-1\right)(\overline{g}_i - a_{i+1})}{F_{n+1}F_1\left(\frac{b_0}{\overline{g}_0}-1\right)(\overline{g}_0 - a_1)}, \quad g_i\,\overline{g}_i = \frac{F_{i+1}\,G_i}{F_i\,G_{i+1}} \quad (i=1,\dots,n),$$

where

$$F_i = \sum_{j=1}^{i-1} f_j + t\sum_{j=i}^{n} f_j + t,$$

$$G_i = \sum_{j=i}^{n}\prod_{k=i}^{j-1} b_k\,a_{k+1}\,\frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l}\,f_j + q^{\frac{n}{2}}\,t\prod_{k=i}^{n} b_k\,a_{k+1}$$
$$+\,q^n t\sum_{j=1}^{i-1}\frac{b_{n+1}\,a_1\prod_{k=1}^{n} b_k\,a_{k+1}}{\prod_{k=j}^{i-1} b_k\,a_{k+1}}\,\frac{\prod_{l=j+1}^{n} g_l}{\prod_{l=1}^{j-1} g_l}\,f_j.$$

T_2: q-Garnier system

Fact ([S 2017])

The system q-$P_{(n+1,n+1)}$ is derived from a Lax pair

$$T_{q,z}^{-1}(\psi) = M\,\psi, \quad T_{q,t}^{-1}(\psi) = B\,\psi.$$

Moreover, the action of T_2 on (f_i, g_i) is derived from the compatibility condition of

(1)
$$T_{q,z}^{-1}(\psi) = M\,\psi, \quad T_2(\psi) = \Gamma\,\psi.$$

Here the matrices M, B and Γ are expressed as

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & & & & \\ & X_{2,2} & X_{2,3} & & & \\ & & X_{3,3} & & & \\ & & & \ddots & & \\ & & & & X_{n,n} & X_{n,n+1} \\ z\,X_{n+1,1} & & & & & X_{n+1,n+1} \end{pmatrix}, \quad (X = M, B, \Gamma),$$

where each block is a 2×2 matrix.

T_2: q-Garnier system

Via a q-Laplace transformation $(z, T_{q,z}^{-1}) \to (T_{q,z}, z)$, system (1) is reduced to the one with 2×2 matrices

(2) $$T_{q,z}(\Psi) = \mathcal{A}\,\Psi, \quad T_2(\Psi) = \mathcal{B}\,\Psi,$$

with

$$\mathcal{A} = M_{n+1,1}^{-1}\,(z\,I - M_{n+1,n+1})\,M_{n,n+1}^{-1}\,(z\,I - M_{n,n})\dots M_{1,2}^{-1}\,(z\,I - M_{1,1})\,,$$
$$\mathcal{B} = \Gamma_{1,1} + \Gamma_{1,2}\,M_{1,2}^{-1}\,(z\,I - M_{1,1})\,.$$

Lemma

The compatibility condition of system (1) *implies the one of* (2)*.*

Remark

The transformation from system (1) *to* (2) *is suggested by Y. Yamada.*

T_2: q-Garnier system

System (2) is equivalent to the Lax pair of the q-Garnier system (of inverse direction).

Theorem ([Okubo-S 18])

The matrices $\mathcal{A}$ and $\mathcal{B}$ satisfy the following properties.

1. $\mathcal{A} = \mathcal{A}_0 + z\,\mathcal{A}_1 + \ldots + z^{n+1}\mathcal{A}_{n+1}$,
$$\mathcal{A}_{n+1} = (-a_1)^{n+1}\begin{pmatrix} t^{-1} & 0 \\ * & 1 \end{pmatrix},\ \mathcal{A}_0 \sim \begin{pmatrix} q^{-\frac{n}{2}}\,t^{-1} & 0 \\ 0 & q^{\frac{n}{2}}\,a_1\,b_1\ldots a_{n+1}\,b_{n+1} \end{pmatrix}.$$
2. $$\det\mathcal{A} = \frac{a_1^{2n+2}}{t}(z-1)\left(z-\frac{b_1}{a_1}\right)\left(z-\frac{a_2}{a_1}\right)\left(z-\frac{b_2}{a_1}\right)\ldots\left(z-\frac{a_{n+1}}{a_1}\right)\left(z-\frac{b_{n+1}}{a_1}\right).$$
3. $$\det\mathcal{B} = \begin{cases} t^{\frac{1}{n+1}}\,a_1^2\left(z-\frac{b_m}{a_1}\right)\left(z-\frac{b_{n+1}}{a_1}\right) & (n=2m-1) \\ t^{\frac{1}{n+1}}\,a_1^2\left(z-\frac{a_m}{a_1}\right)\left(z-\frac{b_{n+1}}{a_1}\right) & (n=2m) \end{cases}.$$

T_4: q-LUC system

Lemma

The transformation T_4 act on (φ_i) as

$$T_4(\varphi_{2i}) = \frac{1}{\alpha_{2i+2}\,\alpha_{2i+3}} \frac{(1+\varphi_{2i+1})(\alpha_{2i+3}+\varphi_{2i+3})}{(1+\varphi_{2i+3})(\alpha_{2i+1}+\varphi_{2i+1})}\,\varphi_{2i+2},$$

$$T_4(\varphi_{2i+1}) = \alpha_{2i+1}\,\alpha_{2i+2} \frac{\{1+T_4(\varphi_{2i+2})\}\left\{\frac{1}{\alpha_{2i+1}\,\alpha_{2i+2}\,\alpha_{2i+3}} + T_4(\varphi_{2i})\right\}}{\{1+T_4(\varphi_{2i})\}\left\{\frac{1}{\alpha_{2i+3}\,\alpha_{2i+4}\,\alpha_{2i+5}} + T_4(\varphi_{2i+2})\right\}}\,\varphi_{2i+3}.$$

Remark

The above system is not suitable as a q-Painlevé system because T_4 act on the parameters as

$$T_4(\alpha_{2i}) = \frac{1}{\alpha_{2i+1}\,\alpha_{2i+2}\,\alpha_{2i+3}}, \quad T_4(\alpha_{2i+1}) = \alpha_{2i+1}\,\alpha_{2i+2}\,\alpha_{2i+3}\,\alpha_{2i+4}\,\alpha_{2i+5}.$$

T_4: q-LUC system

Let $t=(t_0,t_1)$ be a 2-tuples of independent variables defined by

$$t_0=\prod_{i=0}^{n}\frac{1}{\varphi_{2i}}=\frac{1}{\beta^{\frac{1}{2}}(\beta')^{\frac{1}{2}}}\prod_{i=0}^{n}\frac{1}{\alpha_{2i}^{\frac{1}{2}}},\quad t_1=\prod_{i=0}^{n}\frac{\alpha_{2i+1}}{\varphi_{2i+1}}=\frac{\beta^{\frac{1}{2}}}{(\beta')^{\frac{1}{2}}}\prod_{i=0}^{n}\alpha_{2i}^{\frac{1}{2}}\,\alpha_{2i+1}.$$

Also let $c=(c_0,\dots,c_{2n+1})$ be a tuple of parameters defined by

$$c_{2i}=t_0^{\frac{1}{n+1}}t_1^{\frac{1}{n+1}}\alpha_{2i},\quad c_{2i+1}=t_0^{-\frac{1}{n+1}}t_1^{-\frac{1}{n+1}}\alpha_{2i+1}.$$

Then we have

$$T_4(t_0)=q\,t_0,\quad T_4(t_1)=q\,t_1,$$

and

$$T_4^{m}(c_{2i})=c_{2i},\quad T_4^{m}(c_{2i+1})=c_{2i+1}\quad(n=2m-1),$$
$$T_4^{2m+1}(c_{2i})=c_{2i},\quad T_4^{2m+1}(c_{2i+1})=c_{2i+1}\quad(n=2m).$$

We now regard each φ_i as a dependent variable on (t,c) and denote it by $\varphi_i(t;c)$.

T_4: q-LUC system

With the aid of the action of T_4 on c, we define dependent variables and parameters on a lattice by

$$f_{i,n-i+2k}(t) = t_0^{\frac{1}{n+1}} \varphi_{2i}(t; T_4^{-k}(c)), \quad g_{i,n-i+2k-1}(t) = \frac{t_0^{-\frac{1}{n+1}} \varphi_{2i+1}(t; T_4^{-k}(c))}{T_4^{-k}(c_{2i+1})},$$

$$c_{i,n-i+2k} = \frac{T_4^{-k}(c_{2i})}{q^{\frac{1}{n+1}}}, \quad c_{i,n-i+2k-1} = \frac{1}{T_4^{-k}(c_{2i+1})},$$

where the indices i, j of $f_{i,j}(t), g_{i,j}(t), c_{i,j}$ are congruent modulo $n+1$. Note that

$$\prod_{i=0}^{n} f_{i,n-i+2k} = 1, \quad \prod_{i=0}^{n} g_{i,n-i+2k-1} = 1 \quad (k = 0, \ldots, n),$$

$$\frac{c_{i,n-i+2k}\, c_{i+1,n-i+2k+1}}{c_{i+1,n-i+2k}\, c_{i,n-i+2k+1}} = 1, \quad \frac{c_{i,n-i+2k-1}\, c_{i+1,n-i+2k}}{c_{i+1,n-i+2k-1}\, c_{i,n-i+2k}} = 1 \quad (i, k = 0, \ldots, n).$$

We also set

$$\alpha = -t_1^{\frac{1}{n+1}}, \quad \beta = -t_0^{-\frac{1}{n+1}}, \quad \gamma = -t_0^{\frac{1}{n+1}}, \quad \delta = -t_1^{-\frac{1}{n+1}}.$$

T_4: q-LUC system

Theorem ([Okubo-S 18])

The dependent variables $f_{i,n-i+2k}, g_{i,n-i+2k-1}$ *satisfy*

$$\frac{\overline{f}_{i,n-i+2k}}{f_{i+1,n-i+2k+1}} = \frac{c_{i,n-i+2k}}{c_{i,n-i+2k+1}} \frac{(g_{i+1,n-i+2k} - \alpha)(g_{i,n-i+2k+1} - c_{i,n-i+2k+1}\,\beta)}{(g_{i,n-i+2k+1} - \alpha)(g_{i+1,n-i+2k} - c_{i+1,n-i+2k}\,\beta)},$$

$$\frac{\overline{g}_{i,n-i+2k-1}}{g_{i+1,n-i+2k}} = \frac{c_{i+1,n-i+2k-1}}{c_{i+1,n-i+2k}} \frac{(\overline{f}_{i+1,n-i+2k-1} - \overline{\gamma})(\overline{f}_{i,n-i+2k} - c_{i,n-i+2k}\,\delta)}{(\overline{f}_{i,n-i+2k} - \overline{\gamma})(\overline{f}_{i+1,n-i+2k-1} - c_{i+1,n-i+2k-1}\,\delta)},$$

where

$$\overline{f}_{i,n-i+2k} = f_{i,n-i+2k}(q\,t), \quad \overline{g}_{i,n-i+2k-1} = g_{i,n-i+2k-1}(q\,t),$$

$$\overline{\alpha} = q^{\frac{1}{n+1}}\,\alpha, \quad \overline{\beta} = q^{-\frac{1}{n+1}}\,\beta, \quad \overline{\gamma} = q^{\frac{1}{n+1}}\,\gamma, \quad \overline{\delta} = q^{-\frac{1}{n+1}}\,\delta.$$

In this talk we obtain the following object from cluster mutations.

- A birational representation of $\widetilde{W}((A_{2n+1} + A_1 + A_1)^{(1)})$
- The known three q-Painlevé systems
  - q-DS system q-$P_{(n+1,n+1)}$
  - Sakai's q-Garnier system
  - Tsuda's q-LUC system

We want to investigates the following properties of the above q-Painlevé systems from a viewpoint of the Kac-Moody Weyl group or the cluster algebra.

- τ-Functions
- Lax pairs
- Hypergeometric solutions

The following objects are also our next targets.

- Lax pair of q-Sasano system
- q-Painlevé systems of type $(A_{l-1} + A_{m-1} + A_{n-1})^{(1)}$

Thank you for your attention.