Canonical forms for discrete Painlevé equations

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Derivation of discrete Painlevé equations

The deautonomisation method

- Start with a QRT mapping
- Assume coefficients are functions of indep. variable
- Determine dependence with singularity confinement

Process greatly simplified if

canonical forms of QRT are known

The Quispel, Roberts and Thompson mapping

Two 3×3 matrices, $A_{0,1}$ symmetric or asymmetric

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}$$

Introduce vector $\vec{X} = (x^2, x, 1)$

and construct $\vec{F} \equiv (f_1, f_2, f_3)$ and $\vec{G} \equiv (g_1, g_2, g_3)$:

$\vec{F} = (\vec{X}\widetilde{A}_0) \times (\vec{X}\widetilde{A}_1)$ and $\vec{G} = (\vec{X}A_0) \times (\vec{X}A_1)$,

QRT mapping:

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}$$

Invariant relation biquadratic in x and y:

$$\alpha x_n^2 y_n^2 + \beta x_n^2 y_n + \gamma x_n^2 + \delta x_n y_n^2 + \epsilon x_n y_n + \zeta x_n + \kappa y_n^2 + \lambda y_n + \mu = 0$$

$\alpha \equiv \alpha_0 + K\alpha_1$ etc. where K integration constant

For given K, solution x_n in terms of elliptic functions

A first classification of canonical forms

First classification of canonical QRT forms:

based upon already existing forms for dPs

Examples (from $\mathrm{dP_I}$ or $\mathrm{dP_{II}}$)

$$x_{n+1} + x_{n-1} = \frac{f_1(x_n)}{f_2(x_n)}$$

obtained with the matrix

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From $q\mathrm{P_{III}}$

$$x_{n+1} x_{n-1} = \frac{f_1(x_n)}{f_3(x_n)}$$

obtained with the matrix

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so on

The situation becomes complicated for higher dPs

Systematic approach to canonical forms

Work with the matrices A_0 and A_1

Number of parameters: 18

Number of *genuine* parameters: 8

(In symmetric case: 5 (from 12))

Generic A_1 (after homographic transf. for x, y):

$$A_1 = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & \epsilon & 0 \\ \gamma & 0 & 1 \end{pmatrix}$$

Deautonomisation→elliptic dP associated to group $\mathrm{E}_8^{(1)}$

Study all possible reductions of A_1

From now on work with a matrix

with zeros on the upper left corner

After transformations

$$A_1 = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & \epsilon & \zeta \\ \kappa & \lambda & \mu \end{pmatrix}$$

We obtain 8 classes

(VIII) $$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -(z^2 + 1/z^2) & 0 \\ 1 & 0 & (z^2 - 1/z^2)^2 \end{pmatrix}$$

(VI′) $$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & z + 1/z & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(VII) $$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & -2z^2 \\ 1 & -2z^2 & z^4 \end{pmatrix}$$

(V) $$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2z \\ 1 & 2z & 0 \end{pmatrix}$$

Classes VII and VIII are associated to $\mathrm{E}_8^{(1)}$

Classes V and VI associated to $\mathrm{E}_7^{(1)}$

(IV) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(II) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(III) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(I) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(IX) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

All previously known

But two asymmetric cases were missing

(X) $$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By deautonomisation q-Ps associated to $\mathrm{E}_7^{(1)}$

Also

(XI) $$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

Gives dPs associated to the group $\mathrm{E}_7^{(1)}$

With 'elliptic' case

(XII) $$A_1 = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & \epsilon & 0 \\ \gamma & 0 & 1 \end{pmatrix}$$

we have 12 classes in all

From canonical form XI

$$(x_{n+1}-y_n^2)(x_n-y_n^2) = \frac{y_n^6+\beta y_n^5+(\delta+\gamma)y_n^4+\epsilon y_n^3+(\kappa+\zeta)y_n^2+\lambda y_n+\mu}{y_n^2+\beta y_n+\gamma}$$

$$\frac{y_n y_{n-1}+x_n}{y_n+y_{n-1}} = -\frac{x_n^3+(\delta+\gamma)x_n^2+(\kappa+\zeta)x_n+\mu}{\beta x_n^2+\epsilon x_n+\lambda}$$

Rerwrite as

$$(x_{n+1}-y_n^2)(x_n-y_n^2) = \frac{\prod_{i=1}^6(y_n-a_i)}{(y_n-p)(y_n-q)}$$

$$\frac{y_n y_{n-1}+x_n}{y_n+y_{n-1}} = -\frac{C(x_n)}{Q(x_n)}$$

where a_i are 6 parameters and $p+q=\sum_{i=1}^6 a_i$

C, Q are respectively cubic and quadratic, polynomials

Introduce $S(y_n) = \prod_{i=1}^6(y_n-a_i)$ and from identity $S(\eta)=C(\eta^2)+\eta Q(\eta^2)$ obtain C and Q

Introduce auxiliary quantity z by $6z=\sum_{i=1}^6 a_i$

$$(x_{n+1}-(Y_n+z)^2)(x_n-(Y_n+z)^2) = \frac{\prod_{i=1}^6(Y_n-b_i)}{(Y_n-2z+r)(Y_n-2z-r)}$$

$$\frac{Y_nY_{n-1}+x_n-z^2}{Y_n+Y_{n-1}+2z} = -\frac{C(x_n,z)}{Q(x_n,z)}-z$$

From singularity confinement find non-autonomous form

$$(x_{n+1} - (Y_n + z_{n+1})^2)(x_n - (Y_n + z_n)^2) = \frac{\prod_{i=1}^{6}(Y_n - b_i)}{(Y_n - z_{n+1} - z_n + r)(Y_n - z_{n+1} - z_n - r)}$$

$$\frac{Y_n Y_{n-1} + x_n - z_n^2}{Y_n + Y_{n-1} + 2z_n} = -\frac{C(x_n, z_n)}{Q(x_n, z_n)} - z_n.$$

Constraint

$$z_{n+2} - 2z_{n+1} + z_n = 0 \tag{1}$$

i.e. $z_n = an + b$

Difference discrete Painlevé equation associated to $\mathrm{E}_7^{(1)}$

From canonical form X

$$(x_{n+1}y_n - y_n^2 - 1)(x_n y_n - y_n^2 - 1) = \frac{\prod_{i=1}^{6}(y_n - a_i)}{(y_n - p)(y_n - q)}$$

$$\frac{y_n + y_{n-1} - x_n}{y_n y_{n-1} - 1} = \frac{C(x_n)}{Q(x_n)}$$

where $pq = \prod_{i=1}^{6} a_i$

Again introduce $S(y_n) = \prod_{i=1}^{6}(y_n - a_i)$ and from $\frac{S(\eta)}{\eta^2} = \eta C(\eta + 1/\eta) - Q(\eta + 1/\eta)$ obtain C, Q

Finally deautonomisation

$$\left(\frac{x_{n+1}Y_n}{z_{n+1}} - \frac{Y_n^2}{z_{n+1}^2} - 1\right)\left(\frac{x_n Y_n}{z_n} - \frac{Y_n^2}{z_n^2} - 1\right) = \frac{\prod_{i=1}^{6}(Y_n - b_i)}{(1 - z_n z_{n+1} Y_n / r)(1 - z_n z_{n+1} 2 Y_n r)}$$

$$\frac{Y_n + Y_{n-1} - z_n x_n}{Y_n Y_{n-1} - z_n^2} = \frac{C(x_n, z_n)}{z_n Q(x_n, z_n)}$$

with constraint

$$z_{n+2} z_n = z_{n+1}^2 \tag{2}$$

Solution: $\log z_n = an + b$

A q-discrete Painlevé equation associated to $\mathrm{E}_7^{(1)}$

Degeneration cascade of the two equations is very rich

Interesting case when the rhs of second eq. is linear

Asymmetric discrete P_V equation

$$\left(\frac{y_{n+1}+y_n-z_{n+1}-z_n}{y_{n+1}+y_n}\right)\left(\frac{y_n+y_{n-1}-z_n-z_{n-1}}{y_n+y_{n-1}}\right) = \frac{\prod_{i=1}^4 (y_n-z_n-(-1)^n b_i)}{\prod_{i=1}^4 (y_n-(-1)^n a_i)}$$

where $z_n = \alpha n + \beta$ and $\sum_{i=1}^4 a_i = \sum_{i=1}^4 b_i = 0$

In the form XI

$$(x_{n+1}-(Y_n+Z)^2)(x_n-(Y_n+Z)^2) = \frac{((Y_n+Z)^2-a^2)((Y_n+Z)^2-b^2)((Y_n-2Z)^2-s^2)}{(Y_n-2Z+r)(Y_n-2Z-r)}$$

$$\frac{Y_nY_{n-1}+x_n-Z^2}{Y_n+Y_{n-1}+2Z} = \frac{x_n+3Z^2-s^2}{6Z}$$

Deautonomised to

$$(x_{n+1} - (Y_n + Z_{n+1})^2)(x_n - (Y_n + Z_n)^2) =$$
$$(Y_n - Z_{n+1} - Z_n + (-1)^n(a_1 - Z_{n+1} + Z_{n-1})))$$
$$\times(Y_n - Z_{n+1} - Z_n + (-1)^n(a_2 + Z_{n+1} - Z_{n-1}))$$
$$\times(Y_n - Z_{n+1} - Z_n - (-1)^n a_3)^{-1}(Y_n - Z_{n+1} - Z_n - (-1)^n a_4)^{-1}$$
$$\times\prod_{i=1}^{4}(Y_n + (Z_{n+1} + Z_n)/2 - (-1)^n b_i)$$

$$\frac{Y_n Y_{n-1} + x_n - Z_n^2}{Y_n + Y_{n-1} + 2Z_n} =$$
$$\frac{x_n + 3Z_n^2 - (Z_{n+1} - Z_{n-1} + a_1 - a_2 + (-1)^n(a_1 + a_2))^2/4}{6Z_n}$$

where $Y_n = y_n - (Z_{n+1} + Z_n)/2$ and $z_n = 3(Z_{n+1} + Z_n)/2$

Many more examples can be found in our paper

On the canonical forms of QRT mappings and discrete Painlevé equations

by A. Ramani, B. Grammaticos, J. Satsuma, T. Tamizhmani

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