Nonstandard discretisation of Painlevé type Hamiltonian systems

SIDE 13 conference, 11 – 17 November 2018, Fukuoka, Japan

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Portsmouth harbour, 23 Oct 2018, 6 pm GMT

Abstract

We consider the non-standard discretisation method introduced by Kahan & Li [6], for quadratic vector fields $\dot{x} = F(x)$, given by the formula

$$\frac{\tilde{\mathbf{x}} - \mathbf{x}}{h} = \left(I - \frac{h}{2} DF(x)\right)^{-1} F(x).$$

We apply the method to certain cubic Hamiltonian systems of Painlevé type, investigate the possibility of singularity confinement for the resulting systems and their algebraic entropy, or dynamical degree. This is joint work with Dr. habil. Galina Filipuk (University of Warsaw, Poland) [2].

Introduction

Discretising a differential equation or system of equations is not a unique process. Although many discretisation schemes are available, in particalar for numerical solutions, they often do not preserve certain nice properties of the equations. For integrable equations, in addition to having the original system as a continuous limit, we seek for the discretised system to preserve integrability. The Kahan method was applied to the equations of motion for the Euler top [4] and the Lagrange top [5] by Hirota and Kimura, where it does exactly that, and hence is also known as the Hirota-Kimura method. However, for non-autonomous systems such as, for example the Painlevé equations, the method is in general not successful, apart from some special cases [1].

A cubic Hamiltonian systems related to Painlevé IV

We discretise the following differential system given by a cubic Hamiltonian studied in [7],

$$\begin{aligned} H(z,p,q) &= \frac{1}{3}\left(p^3 + q^3\right) + f(z)pq + \alpha p + \beta q, \quad \alpha, \beta \in \mathbb{C} \\ q' &= p^2 + f(z)q + \alpha, \quad p' = -q^2 - f(z)p - \beta, \end{aligned} \tag{1}$$

When $f(z) = cz + d$ is linear, $c \neq 0$, the system is related to the Painlevé IV equation, and in case $c = 0$, $f(z) = \text{const.}$, it has elliptic solutions. We apply the Kahan method to obtain the following non-autonomous discrete system:

$$\begin{aligned} x_{n+1} &= \frac{2(f_{n+1} - 2)y_n^2 + 4\beta y_n - (f_n + 2)(f_{n+1} + 2)x_n - 2\alpha f_n - 4\alpha}{-4 + f_n^2 - 4x_n y_n} \\ y_{n+1} &= \frac{2(f_{n+1} + 2)x_n^2 + 4\alpha x_n - (f_n - 2)(f_{n+1} - 2)y_n - 2\beta f_n + 4\beta}{-4 + f_n^2 - 4x_n y_n}. \end{aligned} \tag{2}$$

We will perform a type of singularity confinement test, imposing a condition on f_n, and determine the dynamical degree of the discrete map in this case.

Singularity confinement

Singularities in the discrete system (2) occur when the denominator vanishes, i.e. when $4x_n y_n = (f_n + 2)(f_n - 2)$. We apply a singularity confinement test in the form

$$x_n = \frac{f_n}{2} - \sigma + \epsilon, \quad y_n = \frac{f_n}{2} + \sigma + \epsilon, \quad \sigma \in \{1, -1\}, \quad |\epsilon| \ll 1.$$

$$\begin{aligned} x_{n+1} &= \frac{2(2 + \alpha + \beta) + f_n(\alpha - \beta + f_n + 2f_{n+1})}{2\epsilon f_n} + O(1) \\ y_{n+1} &= -\frac{2(2 + \alpha + \beta) + f_n(\alpha - \beta + f_n + 2f_{n+1})}{2\epsilon f_n} + O(1) \end{aligned}$$

The singularity reappears periodically, unless f_n satisfies the condition

$$\alpha - \beta + f_n - f_{n+1} = 0 \quad \Longrightarrow \quad f_n = (\alpha - \beta)n + \gamma, \quad \gamma \in \mathbb{C},$$

that is, when $f(z) = (\alpha - \beta)z + d$ is a linear function in the original system (1), corresponding to the case when it is related to the Painlevé IV equation. But although (1) is integrable for any $f(z) = cz + d$, we only find singularity confinement in the discrete system for a special value of c.

Dynamical degree

Algebraic entropy was introduced by Hietarinta & Viallet [3] as a measure of complexity for a discrete map, defined by

$$\lim_{n \to \infty} \frac{\log d_n}{n},$$

where the d_n denote the degrees of rational iterates under the discrete map. Zero algebraic entropy is associated with integrable discrete maps. The dynamical degree λ is the limit factor of degree growth of the rational iterates under the discrete map so essentially just the exponential of the algebraic entropy.

Using MATHEMATICA, we obtain the dynamical degree of the discrete map (2):

- when $f_n = cn, \quad c \neq \alpha - \beta, \quad c \neq 0$, the dynamical degree is 2
- when $f_n = (\alpha - \beta)n, \quad \alpha \neq \beta$, the dynamical degree λ is $1 < \lambda < 2$, where $\lambda \approx 1.839\ldots$ is the largest real root of the polynomial $\lambda^4 - 2\lambda^3 + 1$. This is found by generating the sequence of degrees d_n of the rational iterates: $1, 2, 4, 8, 15, 28, 52, 96, 177, 326, 600, 1104, 2031, \ldots$, obeying the recursive law

$$d_n = 2d_{n-1} - d_{n-4}, \quad d_0 = 1, \quad d_n = 0, \quad n < 0.$$

Conclusions

Due to singularity confinement in the case $f_n = (\alpha - \beta)n$, cancellations occur in the iteration of the discrete map leading to a reduced dynamical degree compared to the generic case, but not enough for the map to be integrable. This corresponds to the case $f(z) = (\alpha - \beta)z$ in the Hamiltonian system (1). When $f(z) = cz$ the system (1) is related to the fourth Painlevé equation, but only for the special case $c = \alpha - \beta$ do we get singularity confinement. The question remains how one can further modify the Kahan discretisation method to obtain an integrable discrete map.

Acknowledgements

TK would like to thank the oragnisers of SIDE 13 for their generous financial support to attend the conference. Furthermore, travel support by the QJMAM fund (Quaterly Journal of Mechanics and Applied Mathematics) provided by the IMA (Institute of Mathematics and its Applications, U.K.) is greatly acknowledged. The research was conducted in part supported by an LMS (London Mathematical Society) 'Research in Pairs' grant for TK to visit Galina Filipuk in Warsaw, for which we are grateful.

QJMAM THE QUARTERLY JOURNAL OF MECHANICS & APPLIED MATHEMATICS — Institute of mathematics & its applications — LONDON MATHEMATICAL SOCIETY EST. 1865 — UNIVERSITY OF PORTSMOUTH

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