Space of initial conditions for some 4D Painlevé systems

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Aim

The notion of the Space of Initial Conditions for Painlevé equations was introduced by [Okamoto 77] in the continuous case and by [Sakai 01] in the discrete case, where an equation gives a flow on its SIC. The SIC is non-compact in the continuous case and compact in the discrete case. The cohomology group of the SIC gives information on the symmetries of the equation.

In recent years, research on 4D Painlevé systems has been progressed mainly from the viewpoint of isomonodromic deformation of linear equations, while the space of initial conditions was known only for few equations, f.g. [Tsuda-T 2009]. The difficulty lies in the part of using higher dimensional algebraic geometry.

Finding the space of initial conditions for the 4D Painlevé equations is an important and challenging problem.

In this talk, we construct the SIC by using discrete symmetries.

Blowing up in higher dimensional space

Blowing up along a subvariety $V \subset U \simeq \mathbb{C}^N$ of dimension $N-k$, $k \geq 2$, written as

$$x_1 - h_1(x_{k+1}, \dots x_N) = \cdots = x_k - h_k(x_{k+1}, \dots x_N) = 0,$$

where h_i's are holomorphic functions, is a birational morphism $\pi : X = \{U_i\} \to U$ such that the coordinates of U_i is given by

$$\begin{aligned} &(u_1^{(i)}, u_2^{(i)}, \dots, u_N^{(i)}) \\ =& \left(\frac{x_1 - h_1}{x_i - h_i}, \dots, \frac{x_{i-1} - h_{i-1}}{x_i - h_i}, x_i - h_i, \frac{x_{i+1} - h_{i+1}}{x_i - h_i}, \dots, \frac{x_k - h_k}{x_i - h_i}, x_{k+1}, \dots x_N \right). \end{aligned}$$

The morphism π does not have inverse on the hypersurface $u_i^{(i)} = x_i - h_i = 0$ in U_i. Such a hypersurface is called the exceptional divisor. The exceptional divisor is locally a direct product $V \times \mathbb{P}^{k-1}$.

Example 1

When $U = \mathbb{C}^4$ and $V = \{x_1 = x_2 = 0\} \simeq \mathbb{C}^2$,

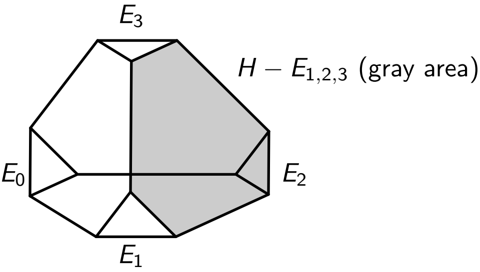
$$(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_4^{(1)}) = \left(x_1, \frac{x_2}{x_1}, x_3, x_4 \right)$$
$$(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, u_4^{(2)}) = \left(\frac{x_1}{x_2}, x_2, x_3, x_4 \right).$$

and the exceptional divisor E is described as $u_1^{(1)} = 0$ in U_1 and $u_2^{(2)} = 0$ in U_2. E is locally a direct product $\mathbb{C}^2 \times \mathbb{P}^1 \ni (x_3, x_4, u_1^{(2)}) = \left(x_3, x_4, \frac{1}{u_2^{(1)}} \right)$.

Pseudo-automorphisms

Example 2

Let $f : \mathcal{X} \to \mathcal{X}$ be the standard Cremona transformation of $\mathbb{P}^3$:
$[x_0 : x_1 : x_2 : x_3] \to [x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1}] = [x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2]$, which is not defined on lines $x_i = x_j = 0$ $(i \neq j)$. Let $\mathcal{X}$ be a rational variety obtained by blowing up $\mathbb{P}^3$ at four points $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$, $(0:0:0:1)$. Still f is not defined on the proper (strict) transform of 6 lines. However, the image of these lines are lines themselves. This is a simple example of a pseudo-automorphism, i.e. an automorphism except subvarieties of codimension 2 at least.



Basic properties of pseudo-automorphisms

A rational map φ from a smooth projective variety $\mathcal{X}$ to itself is called algebraically stable or *1-regular* if $(\varphi^*)^n = (\varphi^n)^*$ holds for all $n \geq 1$. Let $I(\varphi)$ denote the set of indeterminate points of φ.

Proposition 1 (Bedford-Kim 2008, Bayraktar 2012)

A rational map φ from a smooth projective variety $\mathcal{X}$ to itself is algebraically stable if and only if there does not exist a positive integer k and a divisor D on $\mathcal{X}$ such that $\varphi(D \setminus I(\varphi)) \subset I(\varphi^k)$.

Especially, a pseudo-automorphism is algebraically stable.

Proposition 2 (Dolgachev-Ortland 1988)

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties and φ a pseudo-isomorphism from $\mathcal{X}$ to $\mathcal{Y}$. Then φ acts on the Néron-Severi bi-lattice as an automorphism preserving the intersections.

Direct product of $\mathrm{P_{IV}}$: a trivial case

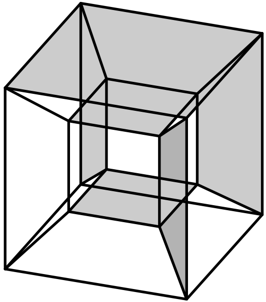
The system

$$\begin{aligned}
\frac{dq_1}{dt_1} &= q_1(2p_1 - q_1 - (a_0^{(1)} + a_1^{(1)} + a_2^{(1)})t_1) + a_2^{(1)} \\
\frac{dp_1}{dt_1} &= -p_1(p_1 - 2q_1 - (a_0^{(1)} + a_1^{(1)} + a_2^{(1)})t_1) - a_1^{(1)} \\
\frac{dq_2}{dt_2} &= q_2(2p_2 - q_2 - (a_0^{(2)} + a_1^{(2)} + a_2^{(2)})t_2) + a_2^{(2)} \\
\frac{dp_2}{dt_2} &= -p_2(p_2 - 2q_2 - (a_0^{(2)} + a_1^{(2)} + a_2^{(2)})t_2) - a_1^{(2)} \\
&\left(\frac{dq_1}{dt_2} = \frac{dp_1}{dt_2} = \frac{dq_2}{dt_1} = \frac{dp_2}{dt_1} = 0\right)
\end{aligned}$$

is the direct product of the fourth Painlvé equation and itself.

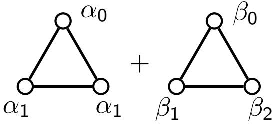
The space of initial conditions and the symmetry

Since the space of initial conditions of P_{IV} is obtained by 8 successive blowups from $\mathbb{P}^1 \times \mathbb{P}^1$, the SIC for this system is obtained by 16 successive blowups from $(\mathbb{P}^1)^4$ whose center manifolds are two-dimensional.



The centers of 6 blow-ups are shown as gray parallelograms. Centers of other 10 blow-ups appear in the exceptional divisor of blow-ups along these centers. Note that some centers intersect with each other, and so the variety depends on the order of blowups. However, the varieties are pseudo-isomorphic with each other.

Moreover, its symmetry (group of Bäcklund transformations) becomes the affine Weyl group of type $A_2^{(1)} + A_2^{(1)}$ extended by its Dynkin diagram, i.e. direct product of the affine Weyl group of type $A_2^{(1)}$ extended by its Dynkin diagram and itself with exchange of them.



All the Bäcklund tranformations acts on the SIC as isomorphisms.

Noumi-Yamada's $A_5^{(1)}$ equation

Singularity confinement after Grammaticos-Ramani

Strategy of constructing SIC: use discrete symmetry.

Let us consider a birational map $\mathbb{C}^4 \to \mathbb{C}^4; (q_1, q_2, p_1, p_2) \mapsto (\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)$:

$$\begin{cases} \bar{q}_1 &= -q_1 - p_2 + a q_2^{-1} + t_1 \\ \bar{p}_1 &= q_2 \\ \bar{q}_2 &= -q_2 - p_1 + a q_1^{-1} + t_2 \\ \bar{p}_2 &= q_1 \end{cases},$$

where a, t_1, t_2 are fixed parameters. This system is turned out to be the autonomous version of a Bäcklund transformation of the $A_5^{(1)}$ member of Noumi-Yamada's $A_n^{(1)}$ higher order Painlevé systems. Let us compactify the phase space to $(\mathbb{P}^1)^4$, where $\mathbb{P}^1$ is the Riemann sphere. Then, we find the following singularity sequence: (ε is a small parameter, $|\varepsilon| \ll 1$)

$$\begin{aligned} (\varepsilon, p_1, q_2, p_2)\text{: 3 dim} &\to (-p_2 + a q_2^{-1} + t_1, q_2, a\varepsilon^{-1}, \varepsilon)\text{: 2 dim} \\ &\to (p_2 - a q_2^{-1}, a\varepsilon^{-1}, -a\varepsilon^{-1}, -p_2 + a q_2^{-1} + t_1)\text{: 1 dim} \\ &\to (-\varepsilon, -a\varepsilon^{-1}, q_2', p_2 - a q_2^{-1})\text{: 2 dim} \\ &\to (q_1'', p_1'', q_2'', -\varepsilon)\text{: 3 dim}, \end{aligned}$$

where only the principal terms of the Laurent series are written.

Another singularity sequence is

$$(q_1, p_1, \varepsilon^{-1}, p_2)\text{: 3 dim} \to (-p_2 - q_1 + t_1, \varepsilon^{-1}, -\varepsilon^{-1}, q_1)\text{: 2 dim}$$
$$\to (p_2, -\varepsilon^{-1}, q_2' - p_2 - q_1 + t_1)\text{: 3 dim}$$
$$\to (q_1'', p_1'', \varepsilon^{-1}, p_2'')\text{: Returned}$$

In these two sequences, some 3 dimensional subvarieties are contracted to lower dimensional ones (called "singularity"), along which we want to blow-up. Since in order to eliminate these singularities we need infinitely near blow-ups, we should also consider the following singularity sequences which become bases of the above blow-ups.

$$(q_1, c_1\varepsilon^{-1}, c_2\varepsilon^{-1}, p_2)\text{: 2 dim} \to (q_1', c_1'\varepsilon^{-1}, c_2'\varepsilon^{-1}, p_2')\text{: Returned}$$
$$(q_1, p_1, c_1\varepsilon^{-1}, c_2\varepsilon)\text{: 2 dim} \to (-q_1 + t_1, c_1\varepsilon^{-1}, -c_1\varepsilon^{-1}, q_1)\text{: 1 dim}$$
$$\to (c_2'\varepsilon, -c_1\varepsilon^{-1}, p_2, -q_1 + t_1)\text{: 2 dim}$$
$$\to (q_1', p_1', c_1'\varepsilon^{-1}, c_2'\varepsilon)\text{: Returned}$$

Space of initial conditions

Since the system is symmetric for the exchange of (q_1, p_1) and (q_2, p_2), there is a counterpart of the above sequences. Let X be a rational variety obtained by successive 16 blow-ups from $(\mathbb{P}^1)^4$ along the singularities. Same as the direct product case, all the center manifolds are two-dimensional.

Theorem 1

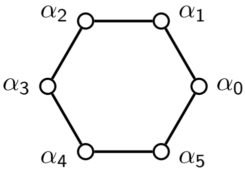
φ can be lifted to a pseudo-automorphism on X (automorphism except finite number of subvarieties of co-dimension higher than two). In the non-autonomous case, it is lifted to a pseudo-isomorphism from $X_{\mathbf{a}}$ to $X_{\varphi(\mathbf{a})}$.

Root system on Nélon-Severi bilattice

Let us define root vectors α_i $(i = 0, 1, \ldots, 5)$ and co-root vectors as

$$
\begin{aligned}
&\alpha_0 = H_{q_1} + H_{p_2} - E_3 - E_4 - E_9 - E_{10}, \quad \alpha_1 = H_{q_2} - E_{15} - E_{16} \\
&\alpha_2 = H_{p_2} - E_5 - E_6, \quad \alpha_3 = H_{q_2} + H_{p_1} - E_1 - E_2 - E_{11} - E_{12} \\
&\alpha_4 = H_{q_1} - E_7 - E_8, \quad \alpha_5 = H_{p_1} - E_{13} - E_{14} \\
&\check{\alpha}_0 = h_{q_2} + h_{p_1} - e_1 - e_2 - e_3 - e_4, \quad \check{\alpha}_1 = h_{p_2} - e_{15} - e_{16} \\
&\check{\alpha}_2 = h_{q_2} - e_5 - e_6, \quad \check{\alpha}_3 = h_{q_1} + h_{p_2} - e_9 - e_{10} - e_{11} - e_{12} \\
&\check{\alpha}_4 = h_{p_1} - e_7 - e_8, \quad \check{\alpha}_5 = h_{q_1} - e_{13} - e_{14}.
\end{aligned}
$$

Then, the pairing $\langle \alpha_i, \check{\alpha}_j \rangle$ defined by the intersection form induces the affine root system of type $A_5^{(1)}$.



Theorem 2

The mapping w_{α_i} *defined by*

$$w_{\alpha_i}(D) := D - 2\frac{\langle D, \check{\alpha}_i\rangle}{\langle \alpha_i, \check{\alpha}_i\rangle}\alpha_i, \quad w_{\alpha_i}(d) := d - 2\frac{\langle \alpha_i, d\rangle}{\langle \alpha_i, \check{\alpha}_i\rangle}\check{\alpha}_i$$

for $D \in H^2(X_{\mathbf{a}}, \mathbb{Z})$ *and* $d \in H_2(X_{\mathbf{a}}, \mathbb{Z})$ *coincides with the Bäcklund transformation given by Noumi-Yamada.*

The push-forward action of φ on the root lattice is

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_4 + \alpha_5, -\alpha_5, \alpha_0 + \alpha_5, \alpha_1 + \alpha_2, -\alpha_2, \alpha_2 + \alpha_3)$$

and hence that of φ^4 is a translation

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) + \delta(0, -1, 1, 0, -1, 1).$$

Corollary 1

The dynamical degree of φ^n *grows quadratically as* $n \to \infty$*.*

4D Garnier system

$$\frac{dq_i}{ds_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{dp_i}{ds_j} = -\frac{\partial H_j}{\partial q_i} \quad (i, j = 1, 2)$$

with

$$\begin{aligned}
s_1(s_1-1)H_1 =& \Big(q_1(q_1-1)(q_1-s_1) - \frac{s_1(s_1-1)}{s_1-s_2} q_1 q_2\Big) p_1^2 \\
& + 2q_1q_2\Big(q_1 + \frac{s_1(s_2-1)}{s_1-s_2}\Big) p_1 p_2 + q_1q_2\Big(q_2 - \frac{s_2(s_1-1)}{s_1-s_2}\Big) p_2^2 \\
& - \Big\{(\kappa_0 - d) q_1(q_1-1) + \kappa_1 q_1(q_1 - s_1) + \theta_1(q_1-1)(q_1-s_1) \\
& \quad + \theta_2 q_1\Big(q_1 + \frac{s_1(s_2-1)}{s_1-s_2}\Big) - \theta_1 \frac{s_1(s_1-1)}{s_1-s_2} q_2\Big\} p_1 \\
& + \Big((2_0 + \kappa_\infty) q_1 q_2 + \theta_2 q_1 \frac{s_2(s_1-1)}{s_1-s_2} - \theta_1 q_2 \frac{s_1(s_2-1)}{s_1-s_2}\Big) p_2 \\
& + a_0(a_0 + \kappa_\infty) q_1 \\
s_2(s_2-1)H_2 =& \{\text{replacing as } q_1 \leftrightarrow q_2,\ p_1 \leftrightarrow p_2,\ s_1 \leftrightarrow s_2,\ \theta_1 \leftrightarrow \theta_2 \text{ in } H_1\},
\end{aligned}$$

and $d = 2\alpha_0 + \kappa_0 + \kappa_1 + \kappa_\infty + \theta_1 + \theta_2$

[Kimura 1990 except for S_0 below, Tsuda 2003 for S_0]
Actions on parameters:

	$\bar{\kappa}_0$	$\bar{\kappa}_1$	$\bar{\kappa}_\infty$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{a}_0$
w_{κ_0}	$-\kappa_0$	κ_1	κ_∞	θ_1	θ_2	$a_0 + \kappa_0$
w_{κ_1}	κ_0	$-\kappa_1$	κ_∞	θ_1	θ_2	$a_0 + \kappa_1$
w_{κ_∞}	κ_0	κ_1	$-\kappa_\infty$	θ_1	θ_2	$a_0 + \kappa_\infty$
w_{θ_1}	κ_0	κ_1	κ_∞	$-\theta_1$	θ_2	$a_0 + \theta_1$
w_{κ_2}	κ_0	κ_1	κ_∞	θ_1	$-\theta_2$	$a_0 + \theta_2$
S_0	$d - \kappa_0$	$d - \kappa_1$	$-\kappa_\infty$	$-\theta_1$	$-\theta_2$	$-a_0$
σ_1	κ_1	κ_0	κ_∞	θ_1	θ_2	a_0
σ_2	κ_0	κ_∞	κ_1	θ_1	θ_2	a_0
σ_3	κ_0	κ_1	θ_1	κ_∞	θ_2	a_0
σ_4	κ_0	κ_1	κ_∞	θ_2	θ_1	a_0

where $d = 2a_0 + \kappa_0 + \kappa_1 + \kappa_\infty + \theta_1 + \theta_2$.

Actions on dependent and independent variables:

	$\bar{q}_1$	$\bar{q}_2$	$\bar{p}_1$	$\bar{p}_2$	$\bar{s}_1$	$\bar{s}_2$
w_{κ_0}	q_1	q_2	$p_1 - \frac{\kappa_0}{s_1 Q_{12}^s}$	$p_2 - \frac{\kappa_0}{s_2 Q_{12}^s}$	s_1	s_2
w_{κ_1}	q_1	q_2	$p_1 - \frac{\kappa_1}{Q_{12}}$	$p_2 - \frac{\kappa_1}{Q_{12}}$	s_1	s_2
w_{κ_∞}	q_1	q_2	p_1	p_2	s_1	s_2
w_{θ_1}	q_1	q_2	$p_1 - \theta_1/q_1$	p_2	s_1	s_2
w_{θ_2}	q_1	q_2	p_1	$p_2 - \theta_2/q_2$	s_1	s_2
S_0	$\frac{s_1 p_1(q_1 p_1 - \theta_1)}{P_{12}(P_{12}+\kappa_\infty)}$	$\frac{s_2 p_2(q_2 p_2 - \theta_2)}{P_{12}(P_{12}+\kappa_\infty)}$	$-q_1 p_1/\bar{q}_1$	$-q_2 p_2/\bar{q}_2$	s_1	s_2
σ_1	$s_1^{-1} q_1$	$s_2^{-1} q_2$	$s_1 p_1$	$s_2 p_2$	s_1^{-1}	s_2^{-1}
σ_2	q_1/Q_{12}	q_2/Q_{12}	$Q_{12}(p_1 - P_{12})$	$Q_{12}(p_2 - P_{12})$	$\frac{s_1}{s_1 - 1}$	$\frac{s_2}{s_2 - 1}$
σ_3	q_1^{-1}	$-q_1^{-1} q_2$	$-q_1 P_{12}$	$-q_1 p_2$	s_1^{-1}	$s_1^{-1} s_2$
σ_4	q_2	q_1	p_2	p_1	s_2	s_1

where $Q_{12} = q_1 + q_2 - 1$, $Q_{12}^s = q_1/s_1 + q_2/s_2 - 1$ and $P_{12} = q_1 p_1 + q_2 p_2 + a_0$.

Theorem 3

(i) *Every generator except S_0 is lifted to a pseudo-isomorphism between rational projective varieties obtained by successive 10 blow-ups from $\mathbb{P}^2 \times \mathbb{P}^2$ such that the center of each blow-up, C_i $(i = 1, \ldots, 10)$ is two-dimensional for $i \neq 7, 9$ and one-dimensional for $i = 7, 9$.*

(ii) *Every generator including S_0 is lifted to a pseudo-isomorphism between rational projective varieties obtained by successive $10 + 11 = 21$ blow-ups from $\mathbb{P}^2 \times \mathbb{P}^2$ such that the center of each blow-up, where C_i $(i = 11, \ldots, 21)$ is two-dimensional for $i \neq 11, 12, 13$, zero-dimensional for $i = 14, \ldots, 19$ and one-dimensional for $i = 20, 21$.*

Remark 1

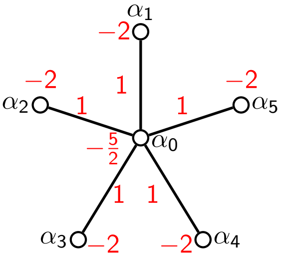
The space of Claim (i) is almost the same with Kimura's SIC for the DE [Kimura 93], where coordinates are slightly different. The blowup in Claim (ii) only appear when discrete symmetry is considered.

Root system for the 4D Garnier

Let $X_{\mathbf{a}}$ be the space of initial conditions obtained by the first 10 blow-ups. Define root vectors α_i ($i = 0, 1, \ldots, 5$) and co-root vectors as

$$\begin{aligned}
&\alpha_0 = \tfrac{1}{2}(H_1 + 2H_2 - 2E_1 - 2E_3 - 2E_5), \quad \alpha_1 = H_1 - E_9 - E_{10} \\
&\alpha_2 = H_1 - E_7 - E_8, \quad \alpha_3 = E_5 - E_6 \\
&\alpha_4 = E_1 - E_2, \quad \alpha_5 = E_3 - E_4
\end{aligned}$$

$$\begin{aligned}
&\check{\alpha}_0 = h_1 - e_1 - e_3 - e_5, \quad \check{\alpha}_1 = h_2 - e_9 - e_{10} \\
&\check{\alpha}_2 = h_2 - e_7 - e_8, \quad \check{\alpha}_3 = e_5 - e_6 \\
&\check{\alpha}_4 = e_1 - e_2, \quad \check{\alpha}_5 = e_3 - e_4.
\end{aligned}$$



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Then, w_{α_i} ($i \neq 0$) coincides with one of original w_i's and acts on the Néron-Severi bi-lattice as

$$w_{\alpha_i}(D) = D - 2\frac{\langle D, \check{\alpha}_i\rangle}{\langle \alpha_i, \check{\alpha}_i\rangle}\alpha_i, \quad w_{\alpha_i}(d) = d - 2\frac{\langle \alpha_i, d\rangle}{\langle \alpha_i, \check{\alpha}_i\rangle}\check{\alpha}_i$$

for $D \in H^2(X_{\mathbf{a}}, \mathbb{Z})$ and $d \in H_2(X_{\mathbf{a}}, \mathbb{Z})$. If we apply the above formula to w_{α_0} with $D = H_1$, we obtain

$$H_1 \mapsto H_1 + \frac{4}{5}(H_1 + 2H_2 - E_1 - E_3 - E_5),$$

which is not in $H^2(X_{\mathbf{a}}, \mathbb{Z})$ and hence impossible to realize.
On the other hand, Kac's translation:

$$T_i(D) = D + \langle D, \check{\delta}\rangle\alpha_i - (\langle D, \check{\alpha}_i\rangle + \langle D, \check{\delta}\rangle)\delta$$

can be realized as a birational map for $i = 1, 2, 3, 4, 5$. Moreover, T_0^2 can be realized as a birational map (but included in known symmetries).

Remark 2

Actions of Bäcklund transformations on the Néron-Severi bi-lattice of the SIC with 21 blow-ups are much more complicated.

Concluding remark

In this talk we constructed the space of initial conditions for some 4D Painlevé systems. Conversely and more importantly, it is expected to construct continuous / discrete 4D Painlevé systems by starting with various 4D rational varieties.