On a "Quasi" Integrable Discrete Ibragimov-Shabat Equation

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The Ibragimov-Shabat Equation

The Ibragimov-Shabat equation was first introduced by the ones that have given its name, [1], and has the form:

$$u_t = u_{xxx} + 3u_{xx}u^2 + 9u_x^2 u + 3u_x u^4. \tag{1}$$

It was originally proposed that such equation was integrable due to infinite conserved quantities, and later it was discovered that it is, in fact, linearizable.

Discretization

One is able to map the linear PDE

$$v_t = v_{xxx} \tag{2}$$

to the Ibragimov-Shabat equation by the transformation [2]:

$$v(x,t) = u(x,t)\exp\left(\int_{-\infty}^{x} u(x',t)^2 dx'\right). \tag{3}$$

First we discretize (2) on the spatial coordinates:

$$\frac{\partial v_n(t)}{\partial t} = \frac{v_{n+3} - 3v_{n+2} + 3v_{n+1} - v_n}{\epsilon^3} \tag{4}$$

Now we make use the discretized version of (3):

$$v_n = u_n e^{S_n}, \qquad S_n = \epsilon\sum_{-\infty}^{n} u_j^2,$$

which, when applied on (4), becomes:

$$u_{n,t} + S_{n,t}u_n = \frac{1}{\epsilon^3}\left(e^{\epsilon u_{n+1}^2+\epsilon u_{n+1}^2+\epsilon u_{n+3}^2}u_{n+3} - 3e^{\epsilon u_{n+1}^2+\epsilon u_{n+2}^2}u_{n+2} + 3e^{\epsilon u_{n+1}^2}u_{n+1} - u_n\right)$$

Such equation in fact reproduced correctly (1) in the continuum limit of $n \to \infty$, $\epsilon \to 0$ with $n\epsilon = x$ finite. Explicitly:

$$u_n \to u$$
$$u_{n\pm 1} \to u \pm \epsilon u_x + \frac{\epsilon^2}{2}u_{xx} \pm \ldots$$
$$S_n = \epsilon\sum_{-\infty}^{n} u_j^2 \to \int_{-\infty}^{x} u(x',t)^2 dx'$$

Dividing v_n^2 by v_{n-1}^2:

$$\frac{v_n^2}{v_{n-1}^2} = \frac{u_n^2}{u_{n-1}^2}\exp\left(2\epsilon u_n^2\right) \tag{5}$$

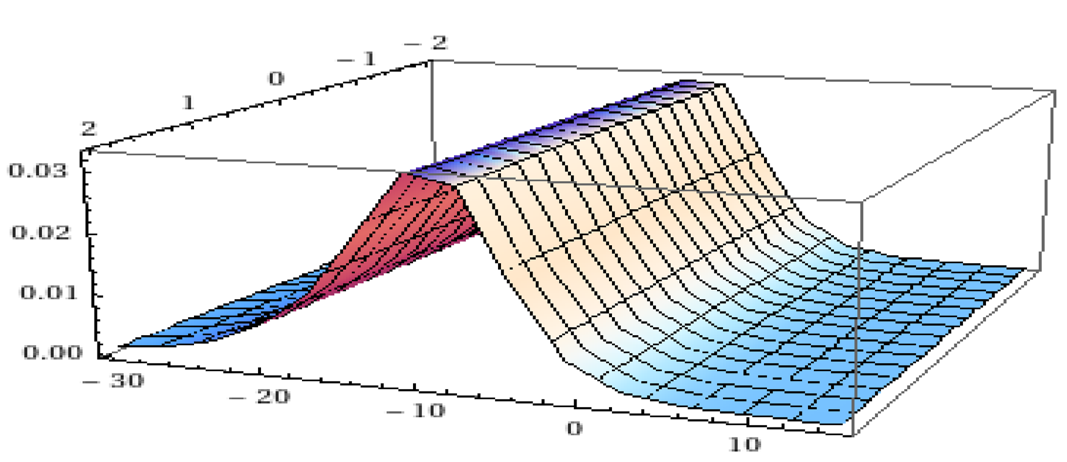
From (5) we are able to define the recursion relation:

$$X_{n-1} = \lambda X_n e^{2\epsilon X_n}, \qquad \lambda \equiv \frac{v_{n-1}^2}{v_n^2}, \qquad X_n \equiv u_n^2. \tag{6}$$

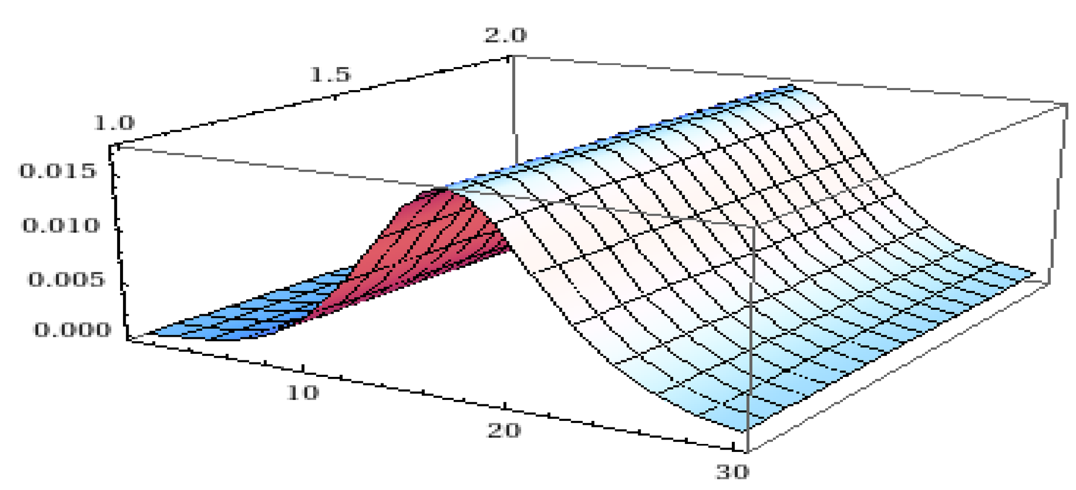
On is able to verify as a discrete solution:

$$v_n(t) = r^n \exp\left(\frac{\left(-1+e^k r\right)^3}{\epsilon^3}t + kn\right) \tag{7}$$

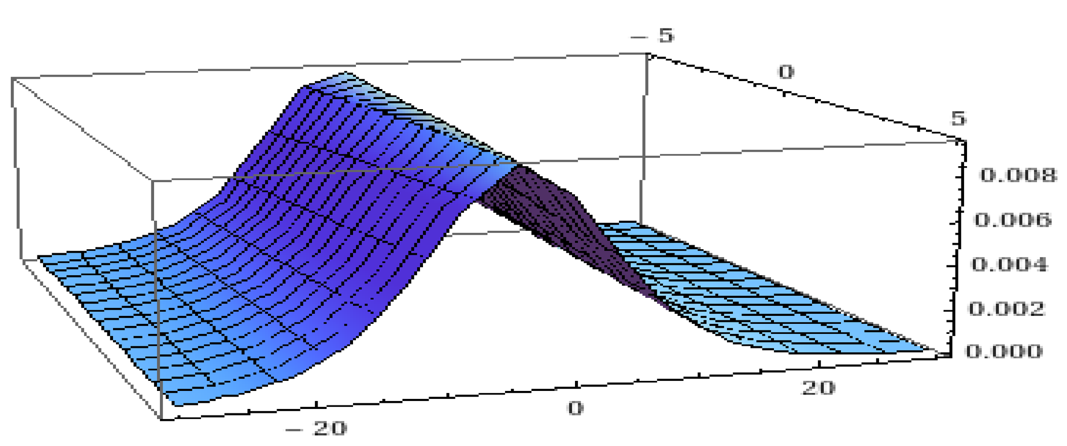
and then verify its phenomenological behavior.



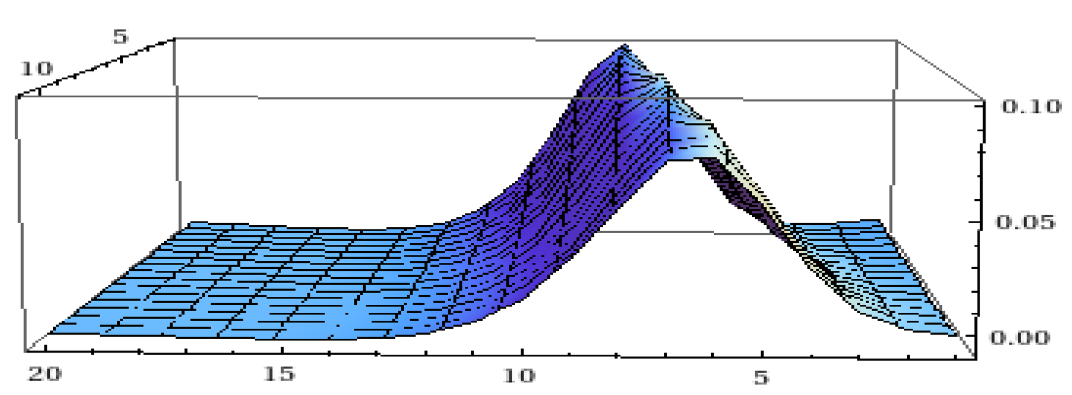
(a) Continuous solitron of first kind



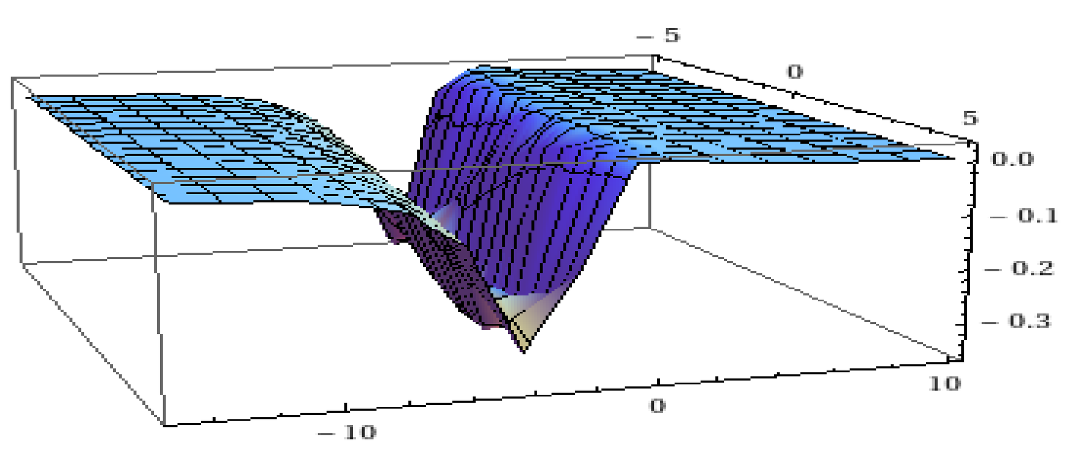
(b) Discrete solitron of first kind



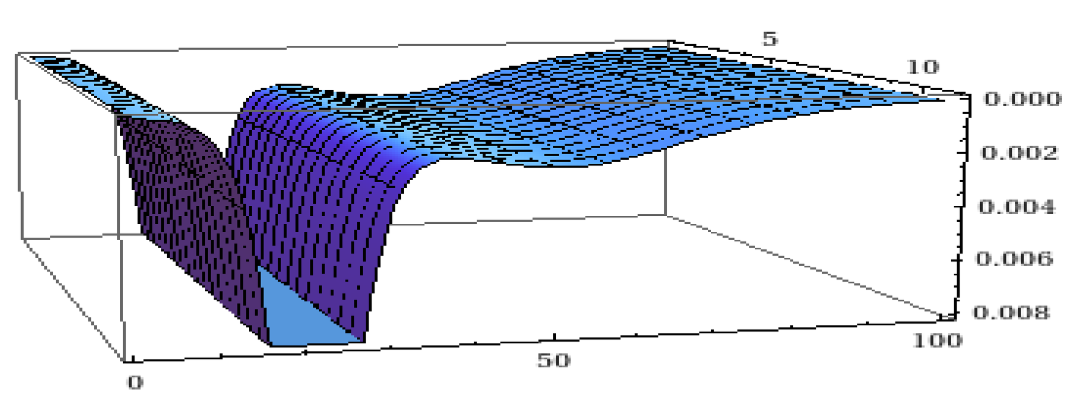
(a) Continuous solitron of second kind



(b) Discrete solitron of second kind



(a) Continuous solitron of third kind



(b) Discrete solitron of third kind

Acknowledgements

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References

[1] N. Ibragimov, A. Shabat; *Funkc. Anal. Prilozen.* **14 (4)**, 79 (1980)

[2] F. Calogero *J. of Math. Ph.* **28**, 538 (1987); **DOI**: 10.1063/1.527639