Surface discretization & Lie sphere geometry

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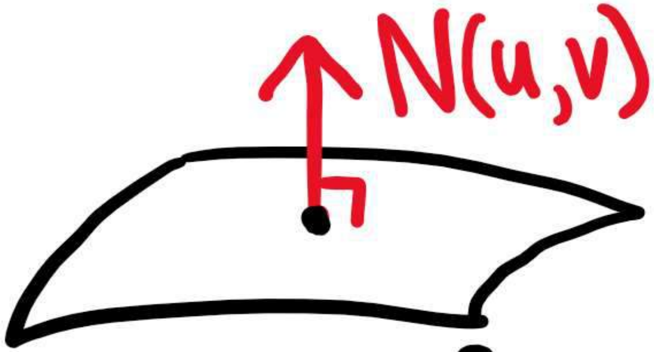
Goal:

Discretize surfaces while preserving mathematical structures

Smooth case:



$x(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$

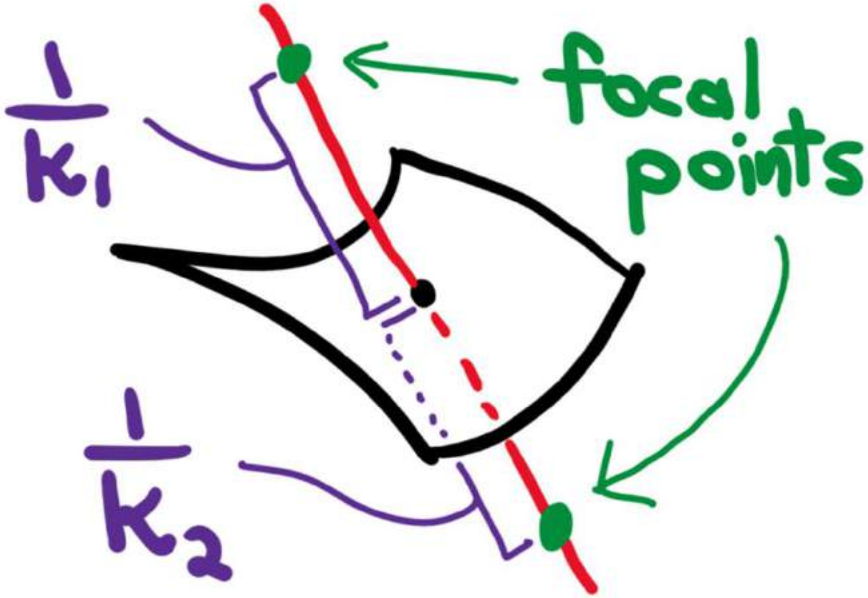
Legendre immersion

[(x,N) is an immersion]

$\Rightarrow$ can have singularities

$$I = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad II = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \begin{matrix} k_j \\ K \\ H \end{matrix} = \begin{matrix} \text{eigenvalues} \\ \det \\ \frac{1}{2} \cdot \text{trace} \end{matrix} \left(I^{-1} II \right)$$

Smooth case:



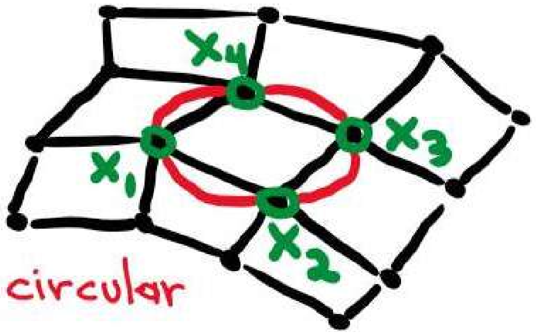
Steiner formula:

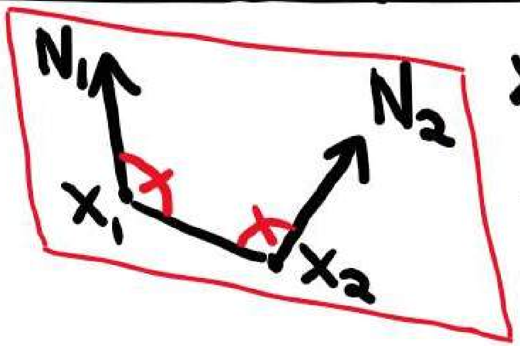
$$A^t = \int_{\Sigma^2} (1 - 2tH + t^2 K)\, dA_x$$

dA_x is the area form for x

area A^t of $x^t = x + t\mathsf{N}$

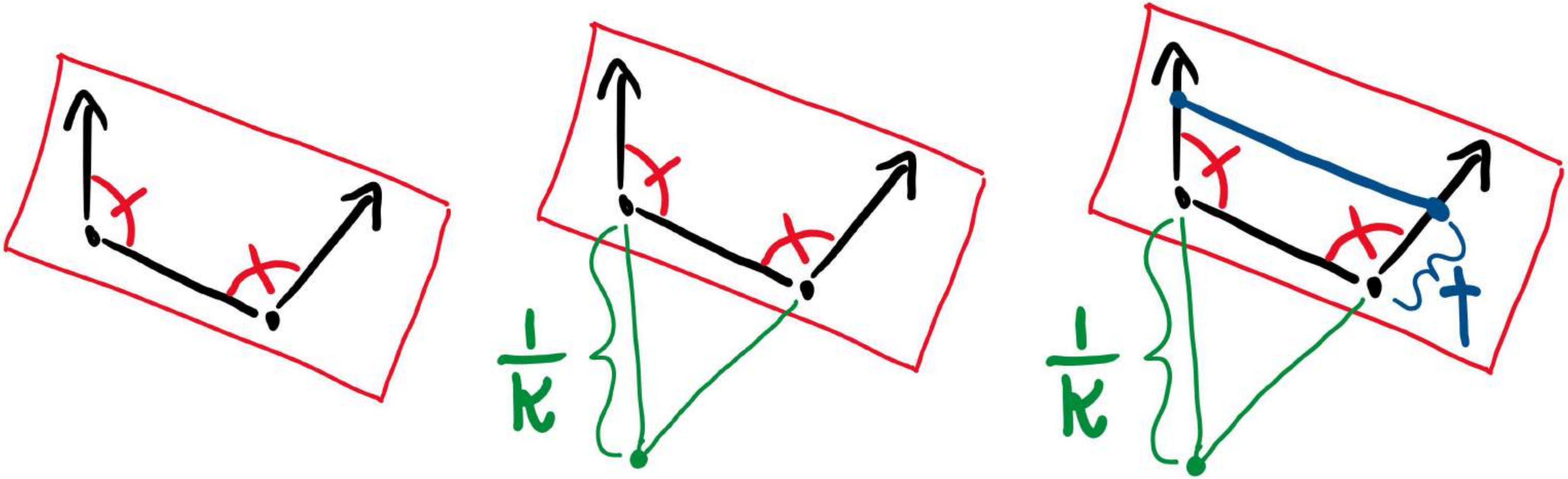
Discretize surfaces, keeping math'l structures





$x(m,n): \mathbb{Z}^2 \to \mathbb{R}^3$

$\Rightarrow$ notions of k_j, K, H



$\Rightarrow k, K, H$ defined for discrete surfaces

$x : \mathbb{R}^2 \to \mathbb{R}^3$, $(u,v) \mapsto x(u,v)$ isothermic,

i.e. conformal curvature line coordinates,

i.e. $\mathrm{I} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{11} \end{pmatrix}$, $\mathrm{II} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}$

$\Leftrightarrow$ for general curvature line coordinates,

$$\mathrm{I} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \ \mathrm{II} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}, \ \frac{g_{11}}{g_{22}} = \frac{a(u)}{b(v)}$$

$\Leftrightarrow$ for general curvature line coordinates,

$$\mathrm{I} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \mathrm{II} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}, \ \lim_{\epsilon \to 0} cr_\epsilon = \frac{-a(u)}{b(v)} .$$

Smooth case: **Minimal surfaces in $\mathbb{R}^3$**

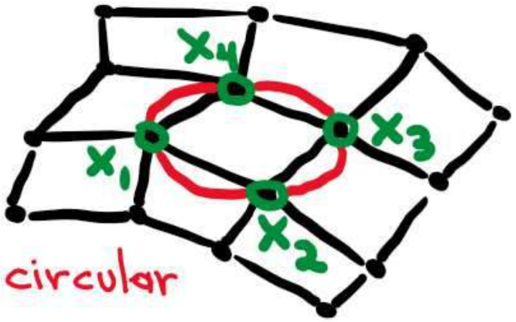
Weierstrass representation

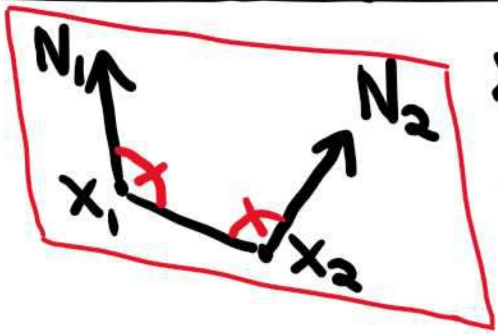
$$x_u = \mathrm{Re}\left\{(2g, 1-g^2, i+ig^2)\frac{1}{g_u}\right\}$$

Discrete case:

$$x_j - x_i = \mathrm{Re}\left\{(g_j+g_i, 1-g_jg_i, i+ig_jg_i)\frac{a_{ij}}{g_j-g_i}\right\}$$

Discretize surfaces, keeping math'l structures





$x(m,n): \mathbb{Z}^2 \to \mathbb{R}^3$

$\Rightarrow$ notions of k_j, K, H

$a(\overline{x_1x_2}) = a(\overline{x_3x_4}),\ a(\overline{x_1x_4}) = a(\overline{x_2x_3})$

- isothermic surfaces: $cr(x_1, x_2, x_3, x_4) = \dfrac{a(\overline{x_1x_2})}{a(\overline{x_1x_4})} \in \mathbb{R}$
- Weierstrass-type rep's (DPW-type rep's)

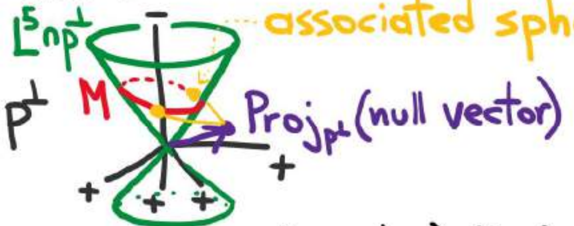
Consider a Legendre immersion x in a 3-dim'l spaceform M $(=\mathbb{R}^3, S^3, \mathbb{H}^3)$ with unit normal field N.

x and N can be represented in $\mathbb{R}^{4,2}$ (with signature $(++++--)$) as follows:

Spaceform is

$$M = \{ x \in L^5 \mid (x,p)=0, (x,q)=-1 \}$$

Spheres in M are represented by null vectors:



Then N (as a null vector) lies in

$$\mathcal{N} = \{ y \in L^5 \mid (y,q)=0, (y,p)=-1 \}.$$

The surface now lifts to a map

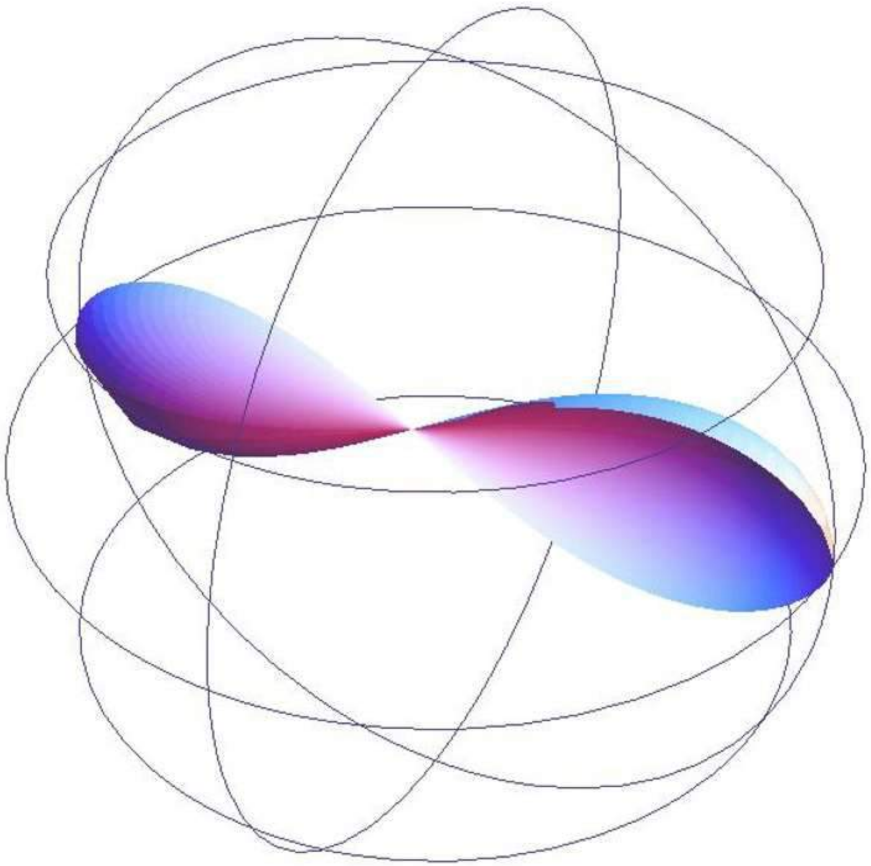
$$(u,v) \longmapsto \mathrm{span}\{x,N\}$$

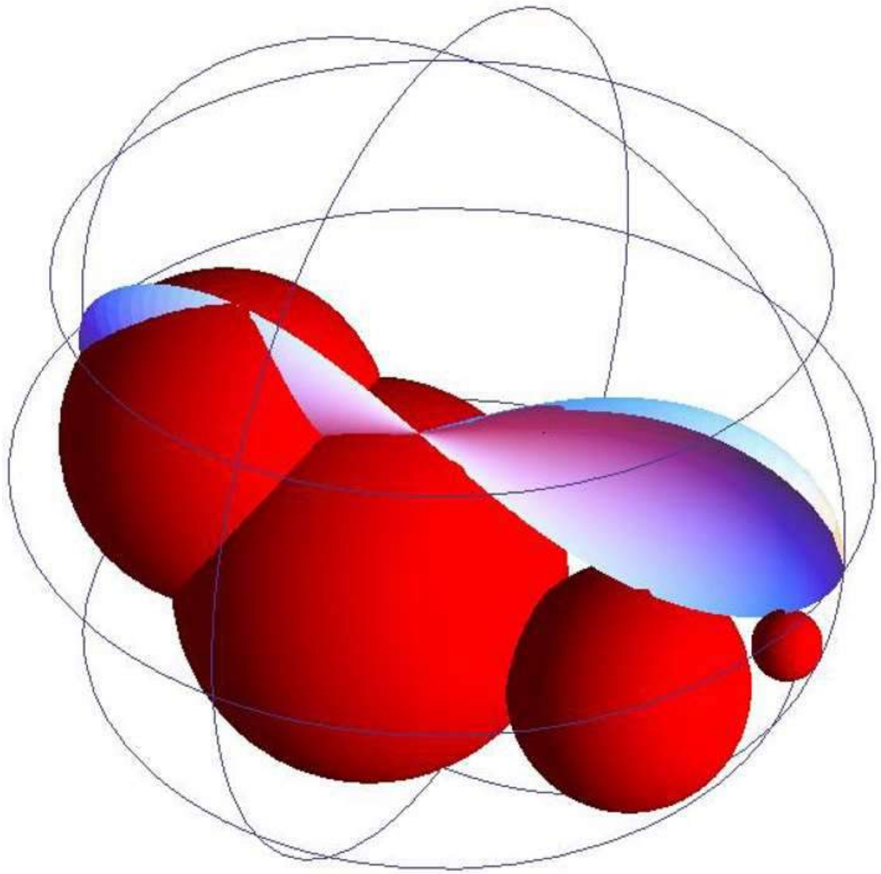
into the space of null planes in $\mathbb{R}^{4,2}$.

<u>Remark</u> Changing p and q, we can project into different spaceforms.

The sections of the lift $\mathrm{span}\{x,N\}$ are the sphere congruences of x — but they are surfaces in $\mathbb{R}^{4,2}$, so we can consider if they are isothermic (just like we did in $\mathbb{R}^3$).

Smooth case:





Conclusions:

- iso'c sphere congruences $\rightarrow$ flat connections Γ^λ on $\mathbb{R}^2 \times \mathbb{R}^{4,2}$, for $\lambda \in \mathbb{R}$
- can consider conserved quantities (c.q.) of all the Γ^λ
- can use any 3-dim'l spaceforms, including Lorentzian ones.

polynomial c.q.'s: $\Gamma^\lambda P(\lambda) = 0$

$$P(\lambda) = \sum_{j=0}^{n} P_j \cdot \lambda^j$$

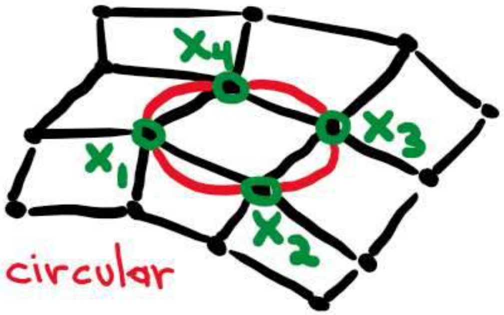
Thm (Burstall, Hertrich-Jeromin, R–, Santos)
Discrete CMC surfaces in 3D spaceforms can be characterized as those iso'c surfaces having a linear c.q. ($\perp p$).

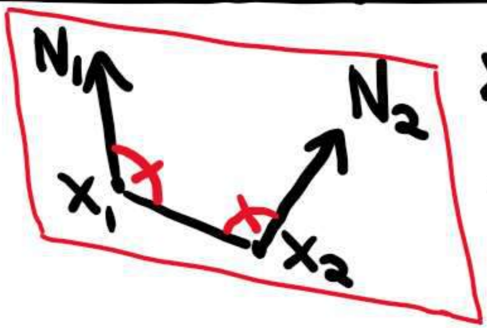
Thm (Burstall, Hertrich-Jeromin, R–)
Both smooth and discrete linear Weingarten surfaces can be characterized as those with iso'c sphere congruences having constant c.q.'s.

Thm [BHR] The c.c.q.'s in the previous theorem come in pairs and are fully constant. Furthermore:

- If one of them is lightlike, then the surface has a Weierstrass-type representation.
- If both of them are lightlike, then the surface is flat in $\mathbb{H}^3$.

Discretize surfaces, keeping math'l structures





$x(m,n): \mathbb{Z}^2 \to$ 3D spaceforms

$\Rightarrow$ notions of k_j, K, H

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- isothermic surfaces: $cr(x_1,x_2,x_3,x_4) = \dfrac{a(\overline{x_1x_2})}{a(\overline{x_1x_4})} \in \mathbb{R}$
- Weierstrass-type rep's (DPW-type rep's)
- Lie sphere geometry unification (iso'c sphere cong's, assoc'd flat conn's Γ^λ for $\lambda \in \mathbb{R}$, c.q.'s $=$ $\parallel$-sections of all Γ^λ)

$\rightarrow$ special surfaces (p.c.q.'s), linear Weingarten surfaces (c.c.q.'s) $\dashrightarrow$ min'l, CMC, flat, CGC surfaces

Smooth case: Christoffel transformations

x a surface in $\mathbb{R}^3$

- x^* is defined on the same domain as x
- x^* has the same conformal structure as x,
- and x and x^* have parallel tangent planes with opposite orientations at corresponding points.

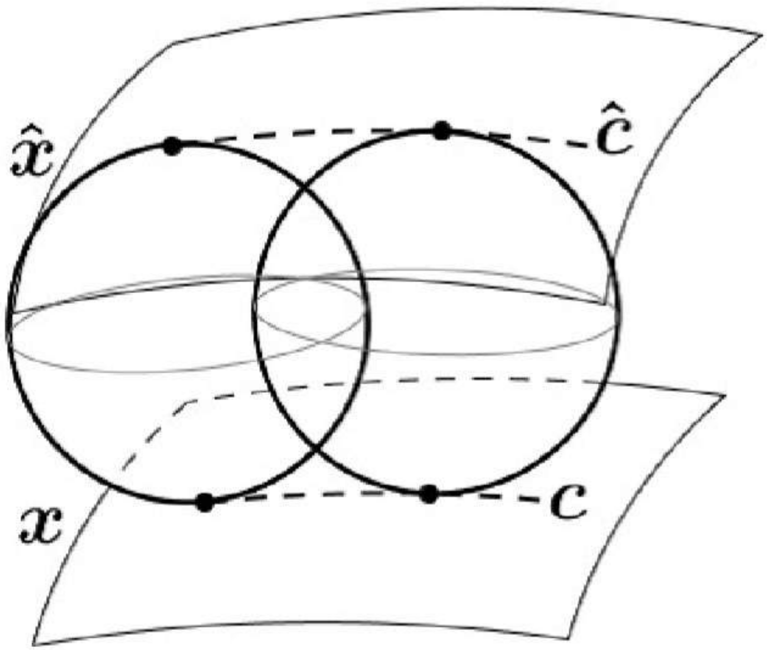
Lemma

Away from umbilics of x, the Christoffel transform x^* exists if and only if x is isothermic.

Smooth case: Darboux transformations

Geometrically, a Darboux transformation of an isothermic surface is one such that

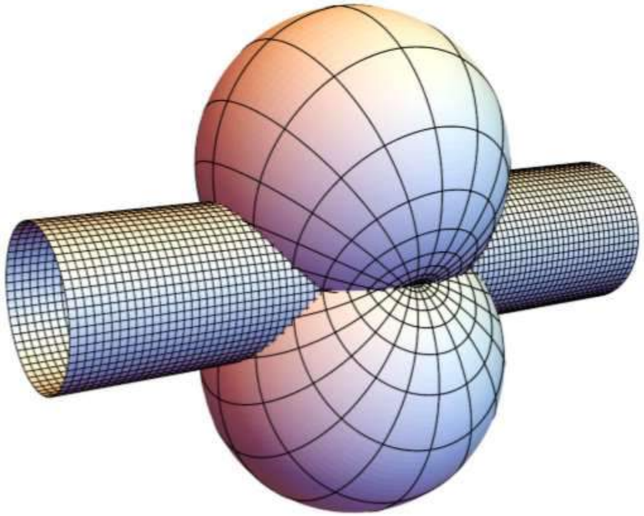
- there exists a sphere congruence enveloped by the original surface and the transform,
- the correspondence, given by the sphere congruence, from the original surface to the other enveloping surface (i.e. the transform), preserves curvature lines,
- this correspondence preserves conformality.

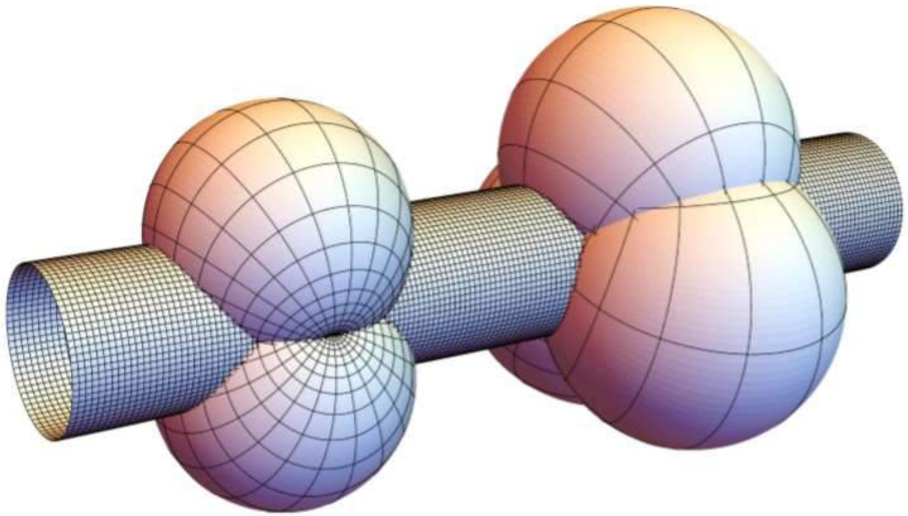


Smooth case:

Lemma *If the initial isothermic surface $x = x(u,v)$ has a polynomial conserved quantity P of order n, then any Darboux transform $\hat{x} = \hat{x}(u,v)$ has a polynomial conserved quantity $\hat{P}$ of order at most $n+1$.*

The Darboux transform in Lemma is a Bäcklund transform exactly when it is of type at most n.

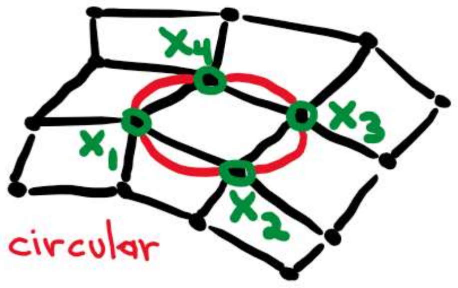


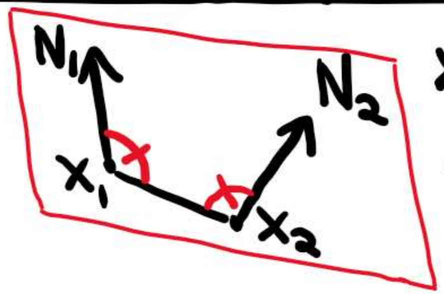


This transformation theory works just as well in the discrete case.

For example, the above lemma holds in the discrete case too [BHRS].

Discretize surfaces, keeping math'l structures





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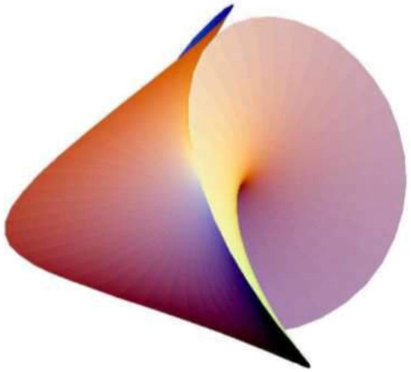
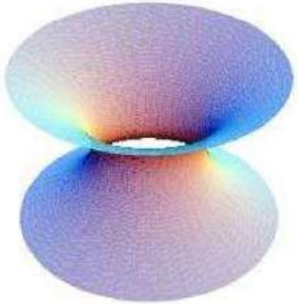
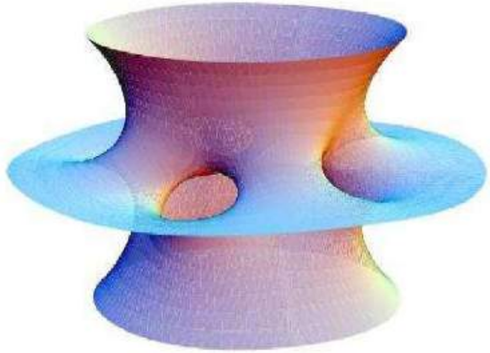
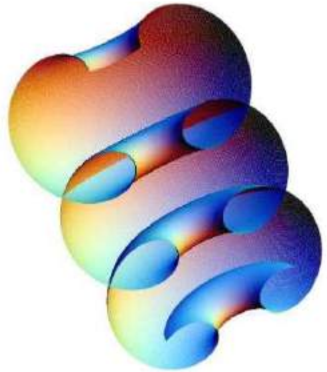
  $\to$ special surfaces (p.c.q.'s), linear Weingarten surfaces (c.c.q.'s) $\dashrightarrow$ min'l, CMC, flat, CGC surfaces
- transformation theory

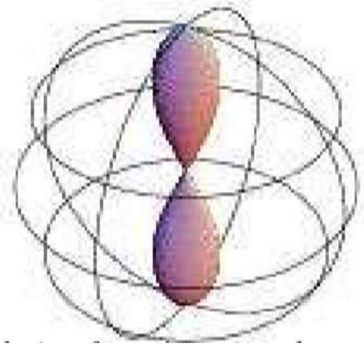
Smooth case:

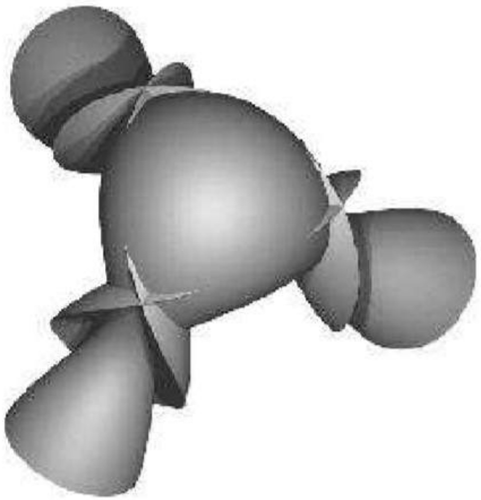


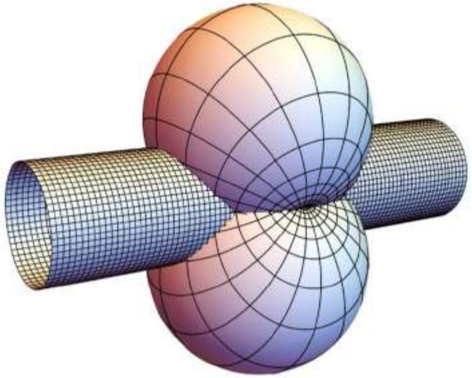


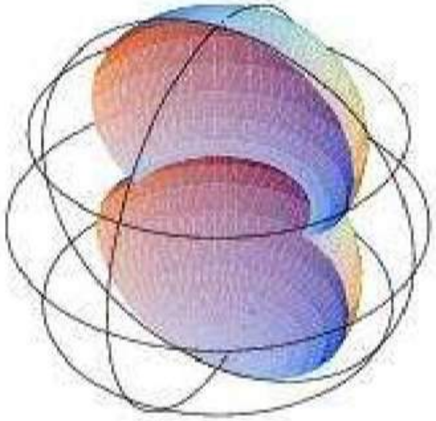


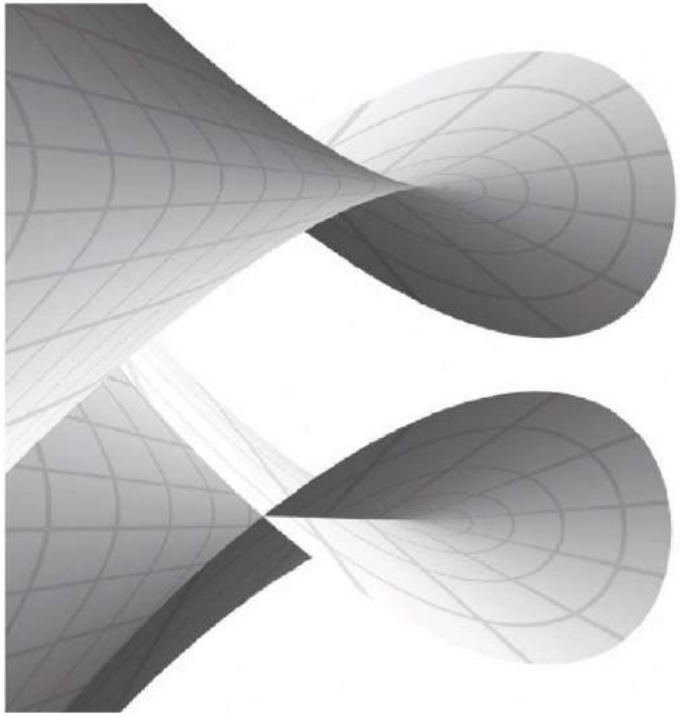












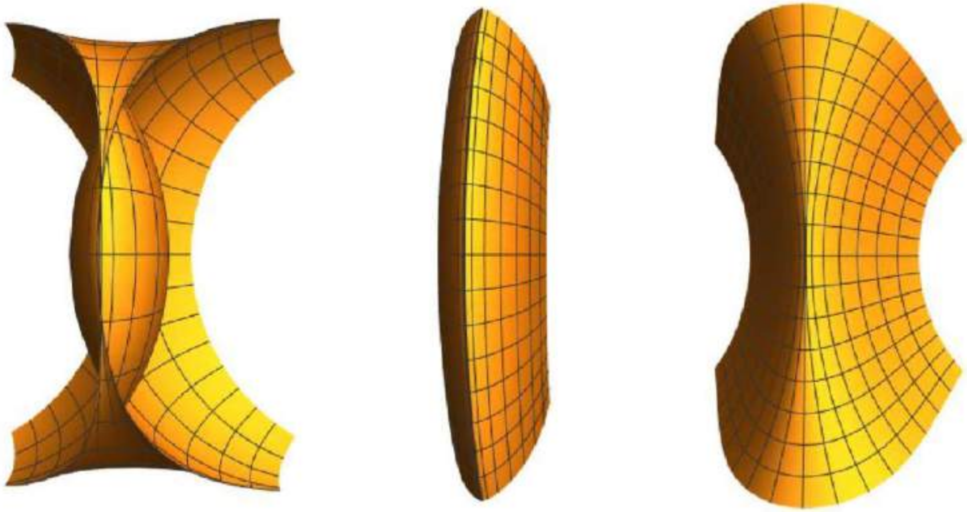


Figure 4.1: From left to right, cuspidal beaks ($\xi < 1/(2\sqrt{2})$), other type 2 degenerate singularity ($\xi = 1/(2\sqrt{2})$) and cuspidal lips ($\xi > 1/(2\sqrt{2})$), as in Example 4.3

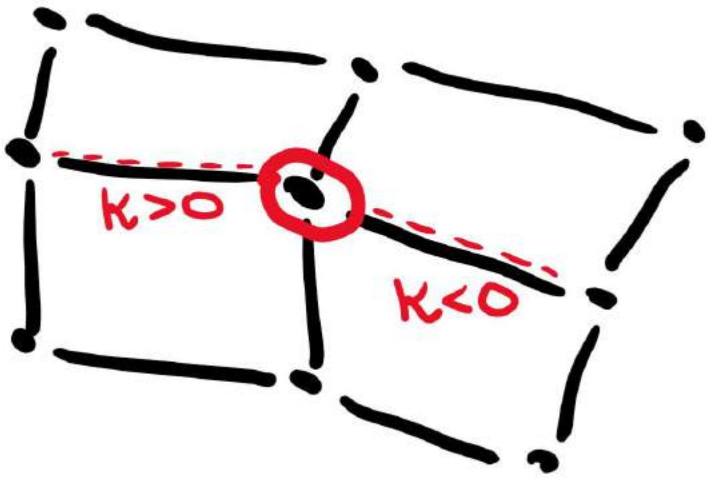
Smooth case:

Result by Teramoto , and Umehara, Murata

- Front = Legendre immersion
- Frontal = normal field well defined but not necessarily a Legendre immersion
- Nondegenerate singularity "=" local singular set is a single creased curve
- Teramoto's and Umehara-Murata's result: For any nondegenerate singularity on a front, one of the principal curvatures will diverge to infinity, with a sign change.

The essential idea behind work on singularities on discrete surfaces (with M. Yasumoto, T. Hoffmann, T. Sasaki, M. Yoshida):

Look for vertices where the principle curvature changes sign in one direction.



Issues:

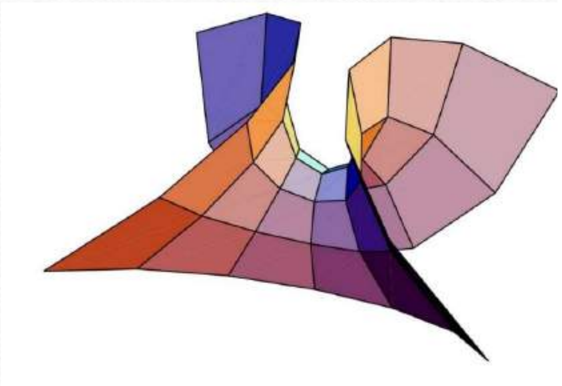
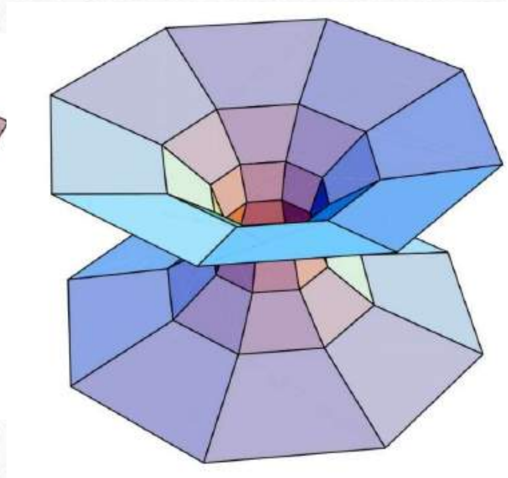
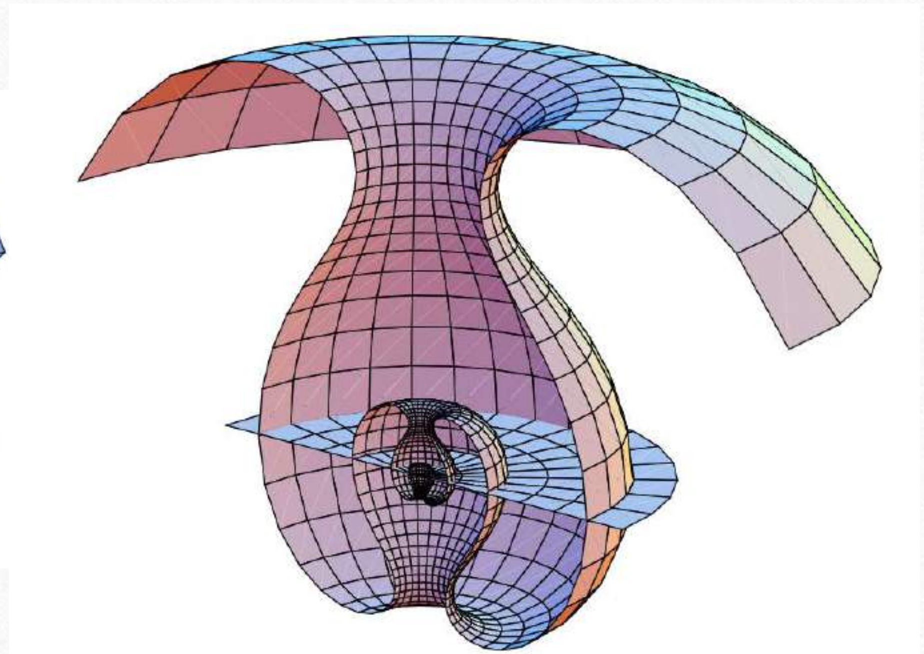
1) This does not always distinguish singularities from flat points.
2) It does not help us distinguish "types" of singularities.

When are discrete flat or parabolic or singular vertices necessarily singular?

- Smooth nonzero-CGC surfaces never have a principal curvature equal to zero.
- Parallel surfaces of smooth minimal surfaces in R^3 without umbilic points never have a principal curvature equal to zero.
- Parallel surfaces of smooth maximal surfaces in R^(2,1) without umbilic points never have a principal curvature equal to zero.

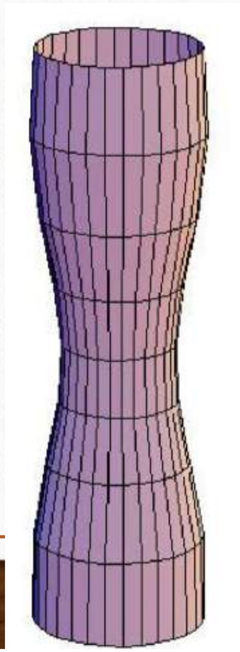
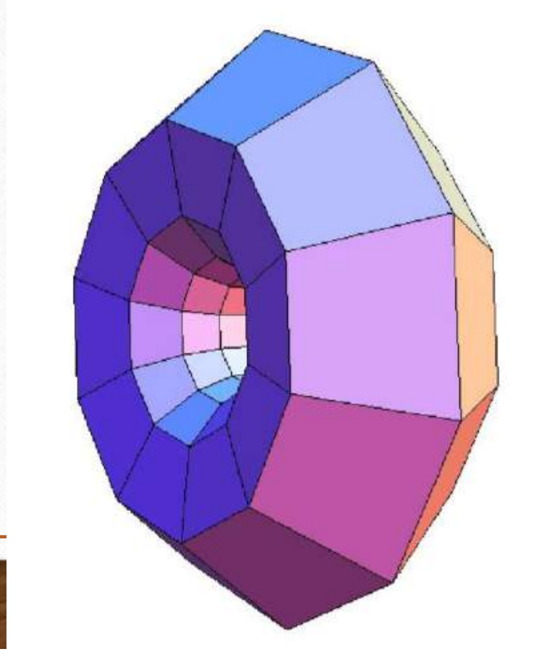
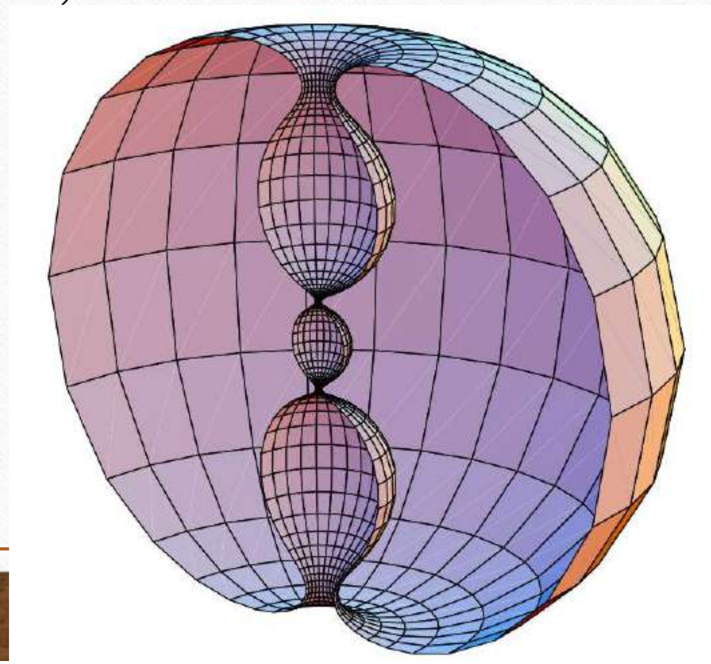
Examples of discrete Ω surfaces

- Minimal in R^3 and H^3

Examples of discrete Ω surfaces

- Minimal in R^3 and H^3
- Delaunay in R^3 and S^3 (including Clifford minimal)

Examples of discrete Ω surfaces

- Minimal in R^3 and H^3
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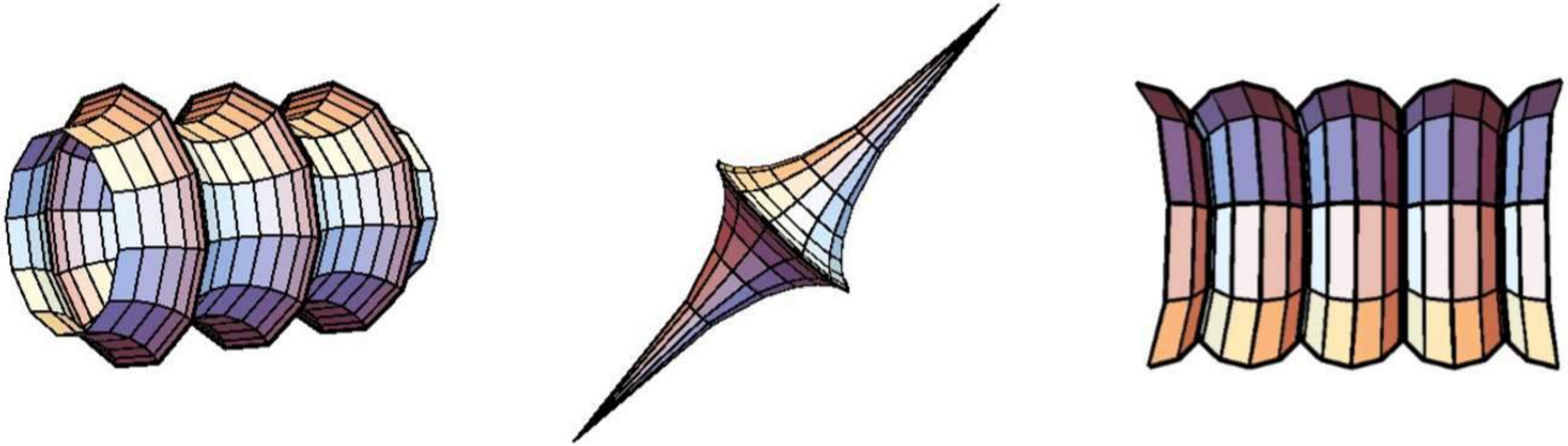
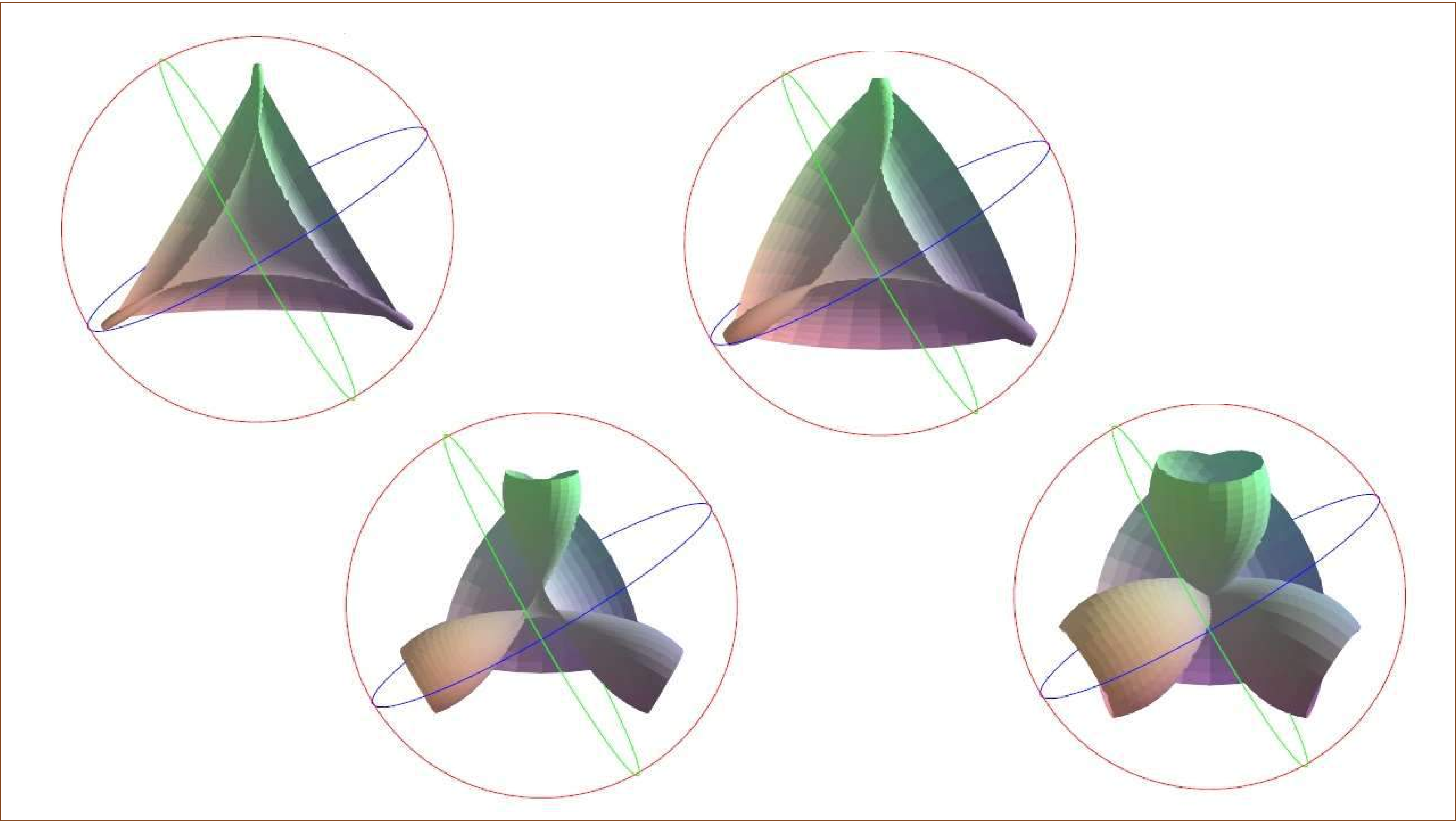
- Constant Gaussian curvature, in R^3



Examples of discrete Ω surfaces

- Minimal in R^3 and H^3
- Delaunay in R^3 and S^3 (including Clifford minimal)
- Constant Gaussian curvature, in R^3
- Linear Weingarten of Bryant type in H^3



Examples of discrete Ω surfaces

- Minimal in R^3 and H^3
- Delaunay in R^3 and S^3 (including Clifford minimal)
- Constant Gaussian curvature, in R^3
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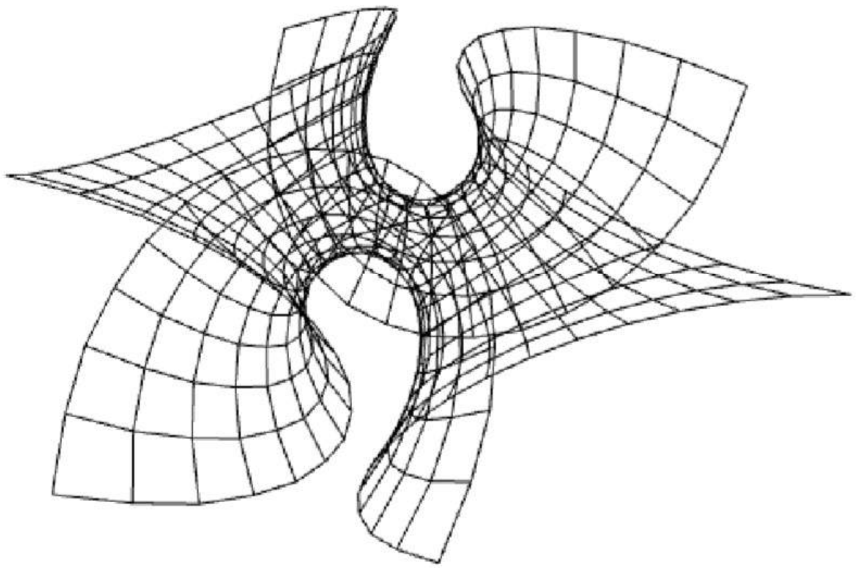
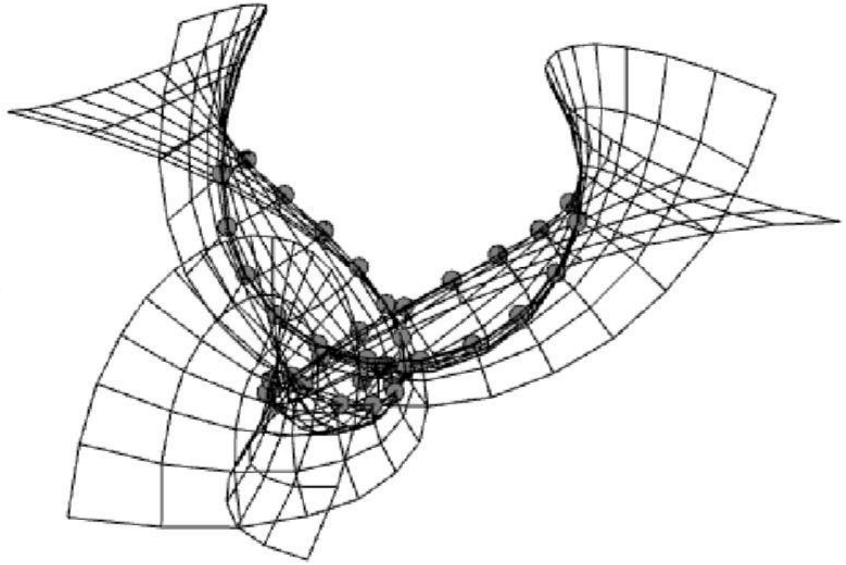
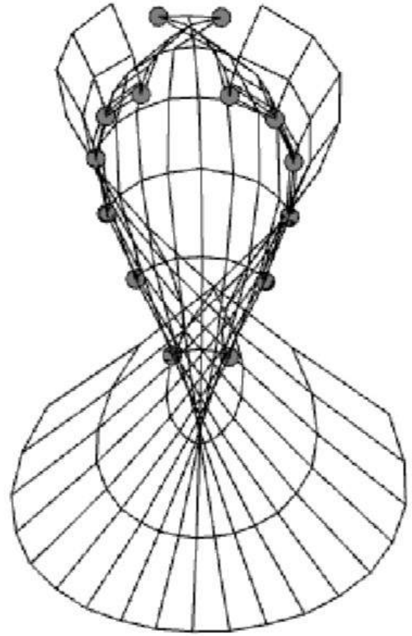
- Higher-order Enneper minimal surface in R^3, and parallel surface (with a 1/3 part)

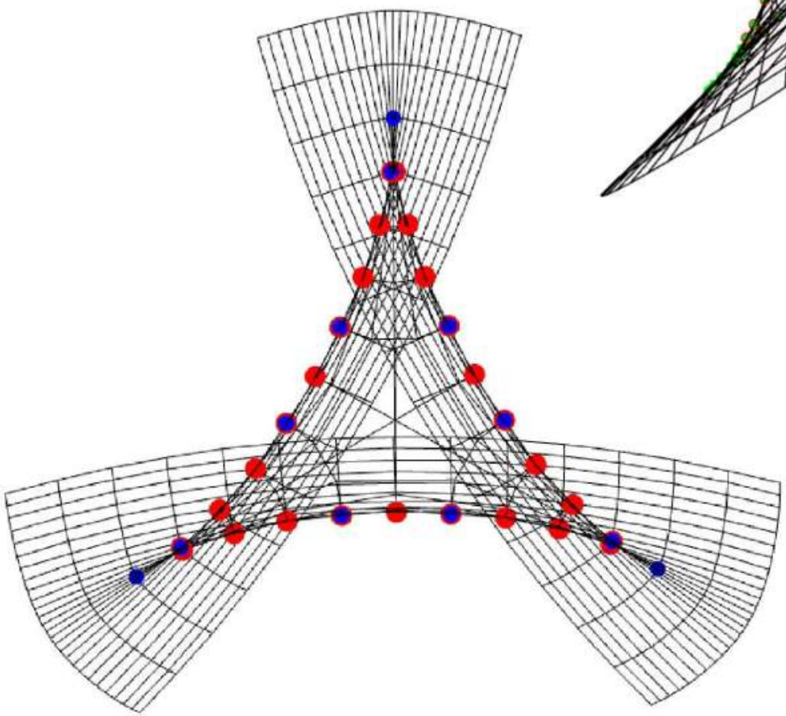


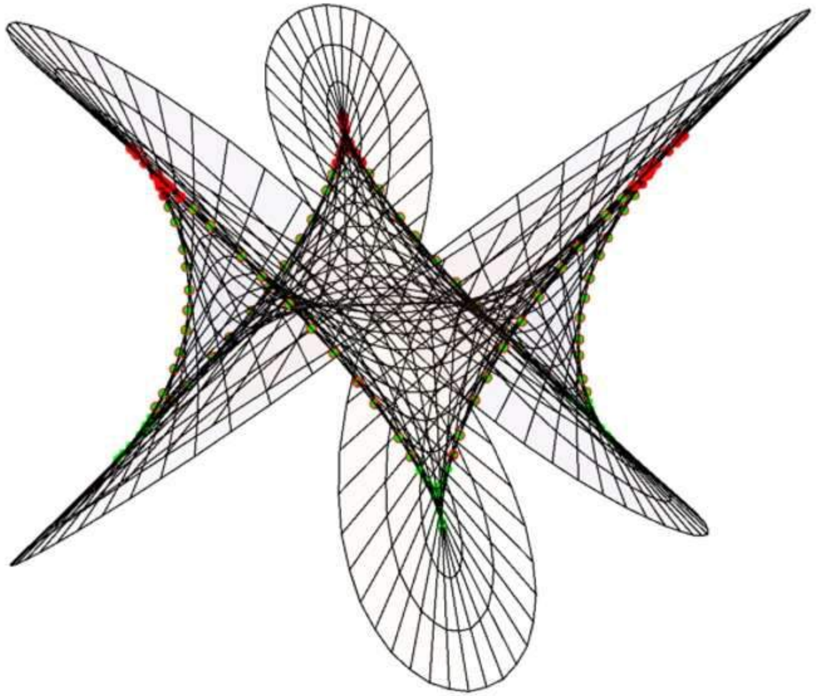


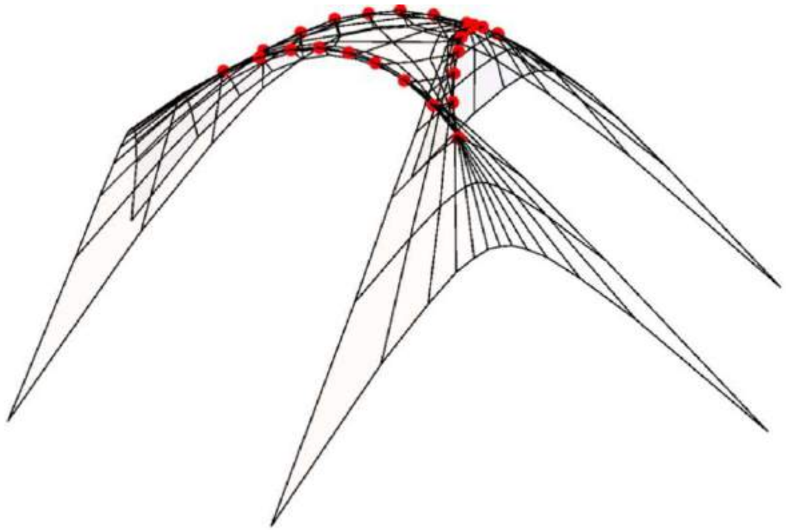


Examples of discrete Ω surfaces

- Minimal in R^3 and H^3
- Delaunay in R^3 and S^3 (including Clifford minimal)
- Constant Gaussian curvature, in R^3
- Linear Weingarten of Bryant type in H^3
- Higher-order Enneper minimal surface in R^3, and parallel surface (with a 1/3 part)
- Flat in H^3, higher-order Enneper-type maximal in R^(2,1), and parallel surface





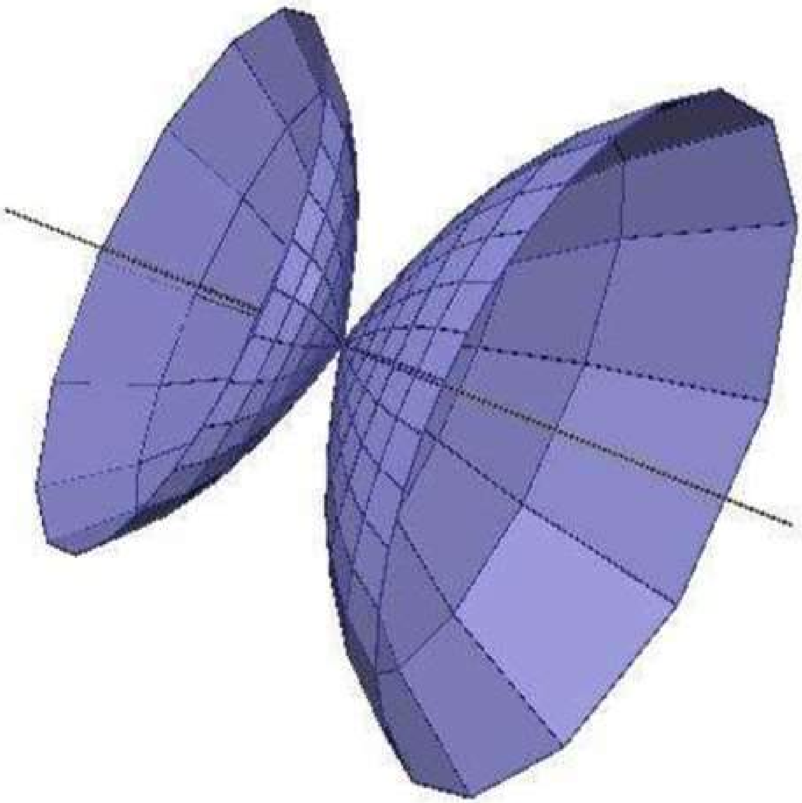


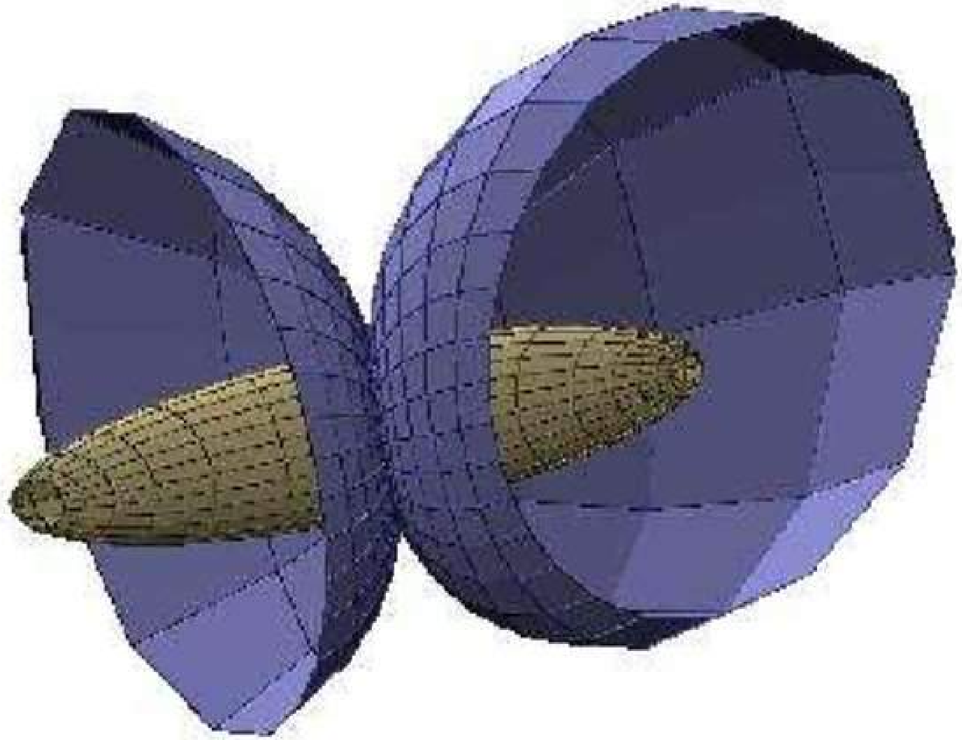
Examples of results about singularities on discrete Ω surfaces

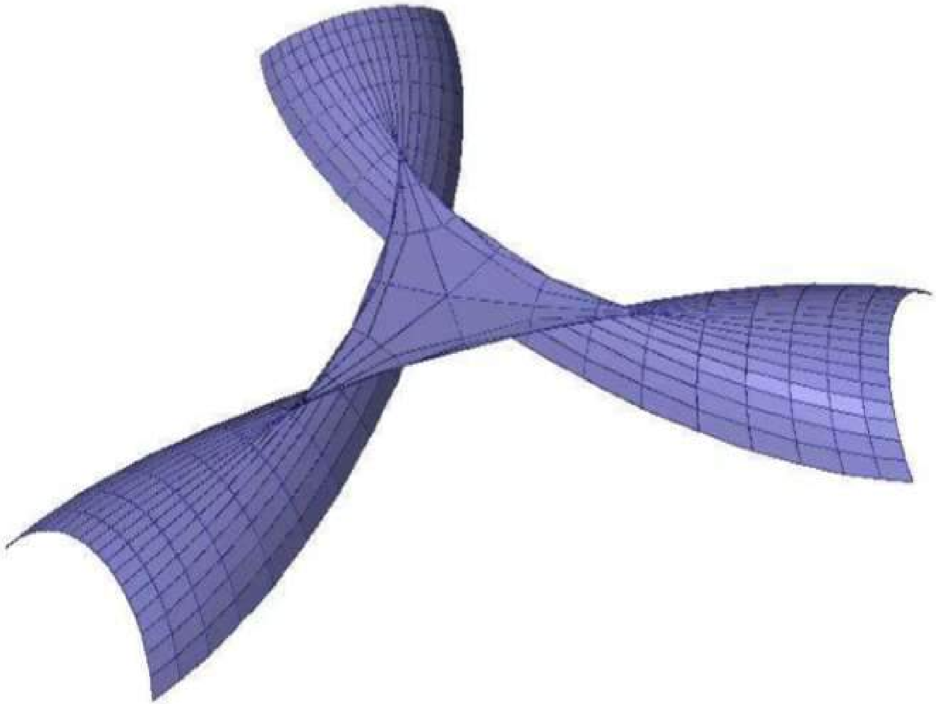
- Weierstrass representations for discrete surfaces in all cases where we expect them (Bobenko, Pinkall, Hertrich-Jeromin --->Hoffmann, Sasaki, Yoshida, Yasumoto & I).

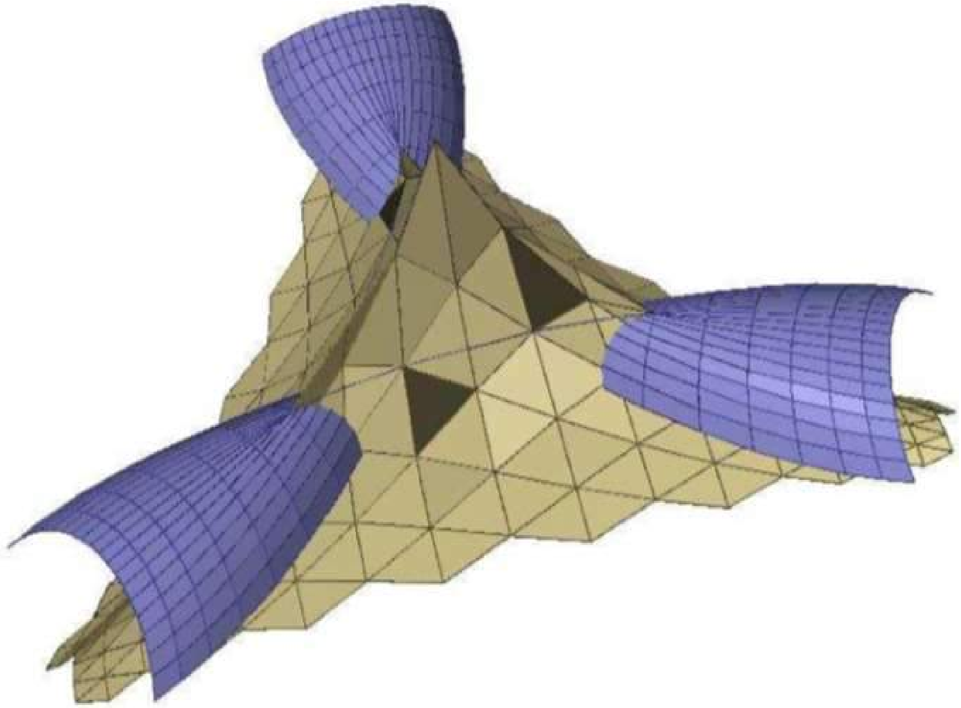
Examples of results about singularities on discrete Ω surfaces

- Weierstrass representations for discrete surfaces in all cases where we expect them.
- Smooth flat surfaces in H^3 have caustics that are also flat, with a Weierstrass representation for the caustics. We (Hoffmann, Sasaki, Yoshida, R--) proved a corresponding setting in the discrete case. Here the caustics have a discrete Weierstrass representation and have planar quadrilaterals, but not circular ones. In the discrete case as well, it's the collection of singular vertices on the parallel surfaces of the flat surface that form the caustic.









Examples of results about singularities on discrete Ω surfaces

- Weierstrass representations for discrete surfaces in all cases where we expect them.
- Weierstrass representation for caustics of discrete flat surfaces in H^3 matching with singular vertices on the parallel surfaces of the initial flat surface.
- Singularities on smooth maximal surfaces in R^(2,1) occur at lightlike points of the surface, where g (stereographic projection of the Gauss map) has absolute value 1. Yasumoto and I: singular vertices on discrete maximal surfaces have adjacent faces that are not spacelike, where the corresponding quadrilateral for the discrete g has a circumcircle that intersects S^1 (the last part Yasumoto established in a solo paper).

Examples of results about singularities on discrete Ω surfaces

- Weierstrass representations for discrete surfaces in all cases where we expect them.
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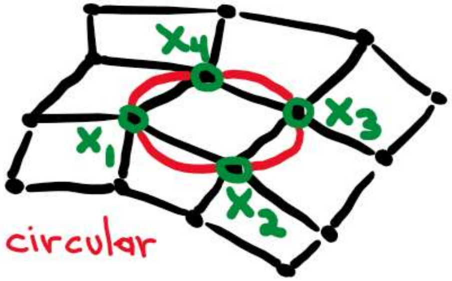
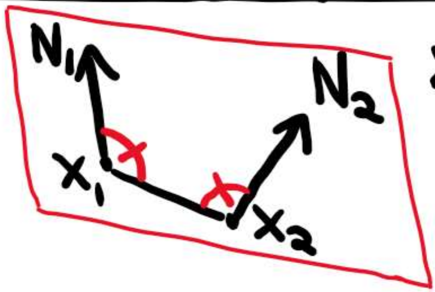
- Singular vertices on discrete maximal surfaces have adjacent faces that are not spacelike, where the corresponding circumcircle for the discrete g intersects S^1.
- Similar result (Yasumoto & I) for "spacelike" CMC 1 surfaces in de Sitter space S^(2,1).
- Still cannot distinguish a cuspidal edge from a swallowtail in the discrete case!

Discretize surfaces, keeping math'l structures





$x(m,n): \mathbb{Z}^2 \to$ 3D spaceforms

$\Rightarrow$ notions of k_j, K, H

$a(\overline{x_1x_2}) = a(\overline{x_3x_4}),\ a(\overline{x_1x_4}) = a(\overline{x_2x_3})$

- isothermic surfaces: $cr(x_1,x_2,x_3,x_4) = \dfrac{a(\overline{x_1x_2})}{a(\overline{x_1x_4})} \in \mathbb{R}$
- Weierstrass-type rep's (DPW-type rep's)
- Lie sphere geometry unification (iso'c sphere cong's, assoc'd flat conn's Γ^λ for $\lambda \in \mathbb{R}$, c.q.'s $=$ $\parallel$-sections of all Γ^λ)
  $\rightarrow$ special surfaces (p.c.q.'s), linear Weingarten surfaces (c.c.q.'s) $\dashrightarrow$ min'l, CMC, flat, CGC surfaces
- transformation theory
- singularities