Integrable discretisation of hodograph-type systems, Abelian integrals and Whitham equations

by

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[with B.G. Konopelchenko, Proc. R. Soc. London A (2014)]

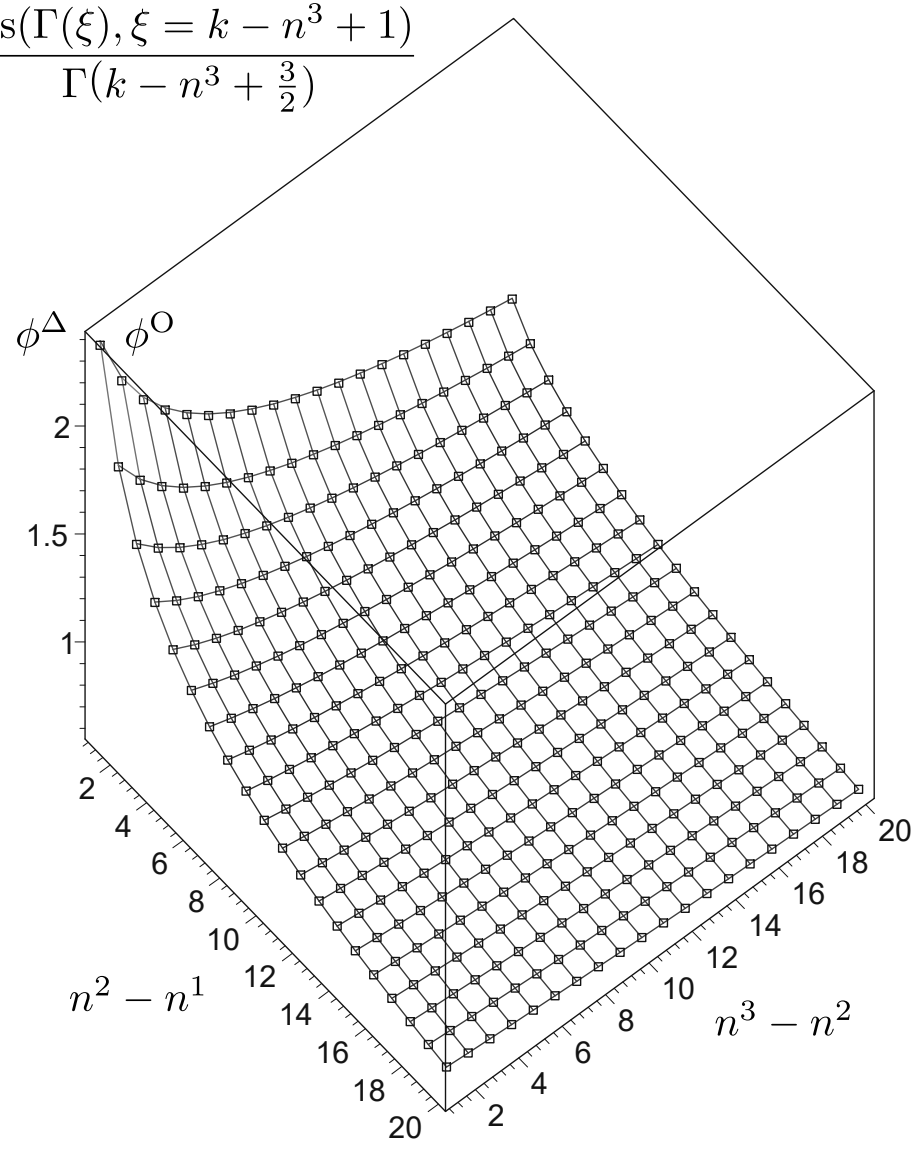
$$\phi^{\Delta} = \frac{1}{\sqrt{\delta}} \sum_{k=n^2}^{n^3-1} \frac{\Gamma(k-n^1+1)}{\Gamma(k-n^1+\frac{3}{2})} \frac{\Gamma(k-n^2+\frac{1}{2})}{\Gamma(k-n^2+1)} \frac{\mathrm{res}(\Gamma(\xi), \xi = k-n^3+1)}{\Gamma(k-n^3+\frac{3}{2})}$$

?

$$\phi^{\mathrm{O}} = \frac{2}{\pi\sqrt{x^3-x^1}} \mathrm{K}\left(\sqrt{\frac{x^3-x^2}{x^3-x^1}}\right)$$

$$x^1 = \delta n^1, \quad x^3 = \delta n^3$$

$$x^2 = \delta(n^2+\tfrac{1}{2})$$



1. Systems of hydrodynamic type

Diagonal systems of hydrodynamic type:

$$u^i_t = \lambda^i(\boldsymbol{u})u^i_x, \qquad i = 1, \dots, N+1$$

λ^i: characteristic speeds

History.

• Dubrovin & Novikov '83 – '89: Theory of (more general) Hamiltonian quasi-linear systems of hydrodynamic type

• Various authors: Relation to classical differential geometry

• Tsarev '91: The geometry of semi-Hamiltonian systems of hydrodynamic type. Generalised hodograph method

2. One-phase Whitham (averaged KdV) equations (1974)

$$
\begin{aligned}
u^1_t &= \lambda^1(u^1,u^2,u^3)u^1_x \\
u^2_t &= \lambda^2(u^1,u^2,u^3)u^2_x \\
u^3_t &= \lambda^3(u^1,u^2,u^3)u^3_x,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda^1 &= \frac{u^1+u^2+u^3}{2} - (u^2-u^1)\frac{\mathsf{K}(s)}{\mathsf{E}(s)}, \qquad s^2 = \frac{u^3-u^2}{u^3-u^1} \\
\lambda^2 &= \frac{u^1+u^2+u^3}{2} + (u^3-u^2)\frac{(1-s^2)\mathsf{K}(s)}{\mathsf{E}(s)-(1-s^2)\mathsf{K}(s)} \\
\lambda^3 &= \frac{u^1+u^2+u^3}{2} - (u^3-u^2)\frac{\mathsf{K}(s)}{\mathsf{E}(s)-\mathsf{K}(s)},
\end{aligned}
$$

where K and E are complete elliptic integrals of the first and second kind.

3. The geometric setup

Classical conjugate net equations:

$$\psi_{u^i u^k} = A_{ik}\psi_{u^i} + A_{ki}\psi_{u^k}, \qquad i \neq k \in \{1, \ldots, N+1\}$$

Compatibility conditions (Darboux system):

$$A_{iku^l} = A_{kl}A_{ik} + A_{lk}A_{il} - A_{ik}A_{il} \qquad \text{(DS)}$$

Combescure transform $\psi_\star$:

$$\psi_{\star u^i} = \lambda^i \psi_{u^i}$$

Compatibility conditions (adjoint linear system):

$$\lambda^i_{u^k} = A_{ik}(\lambda^k - \lambda^i) \qquad \text{(ALS)}$$

4. Semi-Hamiltonian systems (Tsarev '90)

Theorem. The system of diagonal hydrodynamic-type equations

$$u^i_{t^1} = \lambda^i_1(\boldsymbol{u})u^i_x \qquad \text{(SH)}$$

admits compatible flows

$$u^i_{t^2} = \lambda^i_2(\boldsymbol{u})u^i_x, \quad u^i_{t^3} = \lambda^i_3(\boldsymbol{u})u^i_x, \quad \dots$$

iff the sets $\{\lambda^i_\alpha\}$, $\alpha = 1, 2, \dots$ constitute adjoint eigenfunctions, that is, obey (ALS) for an underlying Darboux system.

Theorem (Tsarev '90). The semi-Hamiltonian system (SH) may be linearized/implicitly solved via

$$\omega^i(\boldsymbol{u}) = x + \lambda^i_1(\boldsymbol{u})t^1, \qquad i = 1, \dots, N+1,$$

where $\{\omega^i\}$ is the general solution of (ALS). [Generalised hodograph method.]

5. Characteristic speeds

Fact. Special important characteristic speeds λ^i arise as

- limits of PDEs without dispersion or dissipation

- Whitham equations for slow modulations [Flaschka, Forest & McLaughlin '80]

These are expressed in terms of (hyper)elliptic (or even general Abelian) integrals.

Q: Where do the hyperelliptic integrals come from?

A: Euler-Poisson-Darboux systems (one explanation)
[Kudashev & Sharapov '91, Gurevich, Krylov & El '91, Tian '94]

6. The Euler-Poisson-Darboux system

Consider the particular class of compatible hyperbolic equations

$$\phi_{x^i x^k} = \frac{1}{x^i - x^k}(\epsilon^k \phi_{x^i} - \epsilon^i \phi_{x^k}) \qquad \text{(EPD)}$$

The hyperelliptic integrals

$$\oint_{b_\kappa} \frac{\zeta^k}{\sqrt{\prod_{i=1}^{2g+1}(\zeta - x^i)}} d\zeta, \qquad \begin{array}{r} \text{cycles } b_\kappa, \quad \kappa = 1, \dots, g \\ k = 0, \dots, g-1 \end{array}$$

are superpositions of separable solutions of (EPD) for $\epsilon^i = \frac{1}{2}$.

Example. Genus g=1. Then, the above elliptic integral is essentially given by

$$\phi^{\mathsf{O}} = \frac{2}{\pi\sqrt{x^3 - x^1}} \mathsf{K}\left(\sqrt{\frac{x^3 - x^2}{x^3 - x^1}}\right), \quad x^1 < x^2 < x^3$$

7. Generation of multi-phase flows

Fact [Kudashev '91, Gurevich, Krylov & El '91]. The characteristic speeds $\lambda^1, \lambda^2, \lambda^3$ for the one-phase Whitham equations may be formulated (with $u^i = x^i$) as

$$\lambda^i = \phi^0 - \frac{\phi^{\bigcirc}}{\phi^{\bigcirc}_{x^i}} \phi^0_{x^i}, \qquad i = 1, 2, 3,$$

where

$$\phi^0 = \frac{1}{2}(x^1 + x^2 + x^3)$$

constitutes the simplest non-constant solution of (EPD).

Key observation. The right-hand side of λ^i is precisely of the form of the classical Levy transform (Darboux-type transform) for conjugate nets with respect to x^i generated by $\phi^{\bigcirc}$ and applied to ϕ^0.

Theorem [Kodama, BGK & WKS '15]. The characteristic speeds of multi-phase KdV type may be generated by non-standard (!) applications of g Levy transformations. The solution of the associated semi-Hamiltonian system may be obtained in the same purely algebraic manner. [Cf. Ferapontov '00 for the "standard" application]

8. Discretisation

Remark. The fact that the above method extends to the semi-Hamiltonian system is due to the interpretation of both x and t in the context of the generalised hodograph method as eigenfunctions in the sense of soliton theory.

Plan. Use standard discretisation of

- conjugate net equations
- adjoint linear systems
- Darboux-type transformations (but non-standard application!)

to discretise the continuous theory.

9. The generalised hodograph transformation

Key idea. Consider commuting flows (nonlinear)

$$u^i_{t^\alpha} = \lambda^i_\alpha(\boldsymbol{u}) u^i_x, \qquad i = 1, \dots, N+1, \quad \alpha = 1, \dots, N$$

and observe that

$$du^i \wedge dx + \sum_{\alpha=1}^{N} \lambda^i_\alpha(\boldsymbol{u})\, du^i \wedge dt^\alpha = 0.$$

This is equivalent to the linear (!) system of hodograph type

$$x_{u^k} + \sum_{\alpha=1}^{N} \lambda^i_\alpha(\boldsymbol{u}) t^\alpha_{u^k} = 0, \quad i \neq k$$

which, in turn, may be solved in terms of eigenfunctions (as indicated before).

10. Discrete hodograph equations (BGK & WKS '14)

Discrete analogue:

$$\Delta_k x + \sum_{\alpha=1}^{N} \lambda^i_{\alpha[k]}(\boldsymbol{n})\Delta_k t^\alpha = 0, \qquad \Delta_k f = f_{[k]} - f,$$

where the quantities λ^i_α constitute adjoint eigenfunctions of an underlying standard discrete Darboux system, that is,

$$\Delta_k \lambda^i_\alpha = A_{ik}(\lambda^k_{\alpha[i]} - \lambda^i_{\alpha[k]}), \quad A_{ik[l]} = \cdots$$

Interpretation. Integrable discretisation of (in)dependent variables u^i, depending on the point of view.

Main theorem [BGK & WKS '14]. A canonical discretisation of the characteristic speeds of multi-phase KdV type may be generated by non-standard (!) applications of g discrete Levy transformations. The solution of the discrete generalised hodograph system may be obtained in the same purely algebraic manner.

11. Discrete Euler-Poisson-Darboux system (BGK & WKS '14)

Q: How does one define the discrete ingredients of the above theorem?

Compatible discrete Euler-Poisson-Darboux system:

$$\Delta_{ik}\phi = \frac{\delta^k\epsilon^k\Delta_i\phi - \delta^i\epsilon^i\Delta_k\phi}{\delta^i(n^i+\nu^i) - \delta^k(n^k+\nu^k)}$$

ν^i: give rise to "location" of discretisation points, δ^i: lattice parameters

limit: $x^i = \delta^i(n^i+\nu^i), \quad \delta^i \to 0, \quad i = 1,\dots,2g-1$

Separable solutions:

$$\phi_{\text{sep}} = \prod_{i=1}^{2g+1} \rho^i(n^i;\zeta), \quad \rho^i = (\delta^i)^{-\epsilon^i}\frac{\Gamma(\xi^i - n^i - \nu^i + 1)}{\Gamma(\xi^i - n^i - \nu^i + \epsilon^i + 1)}, \qquad \xi^i = \frac{\zeta}{\delta^i}$$

Continuum limit: $\rho^i \to (\zeta - x^i)^{-\epsilon^i}$ as $\delta^i \to 0$

12. Formal structure of "discrete" hyperelliptic integrals

As in the continuous case, we now make the choice $\epsilon^i = \frac{1}{2}$, $\delta^i = \delta$.

Q: By analogy, we should consider contour integrals of the form

$$\oint_{b_\kappa} \zeta^k \prod_{i=1}^{2g+1} \rho^i(\zeta)\, d\zeta$$

but what does the above mean? Importantly, we also require that

$$\Delta_i \oint_{b_\kappa} = \oint_{b_\kappa} \Delta_i \; !$$

Denote the separable solution by

$$\varphi = \prod_{i=1}^{2g+1} p(\xi, n^i, \nu^i), \quad p(\xi, n, \nu) = \frac{\Gamma(\xi - n - \nu + 1)}{\sqrt{\delta}\,\Gamma(\xi - n - \nu + \frac{3}{2})}, \qquad \xi = \frac{\zeta}{\delta},$$

with $n^1 < n^2 < \cdots < n^{2g} < n^{2g+1}$.

13. Discretisation points

Key choice:

$$\nu^{2k} = \frac{1}{2}, \quad \nu^{2k+1} = 0$$

corresponding to the discretisation points $x^{2k} = \delta(n^{2k} + \frac{1}{2})$ and $x^{2k+1} = \delta n^{2k+1}$.

Result.

$$\varphi = \frac{1}{\delta^{g+\frac{1}{2}}} \prod_{k=0}^{g} \frac{\Gamma(\xi - n^{2k+1} + 1)}{\Gamma(\xi - n^{2k+1} + \frac{3}{2})} \prod_{k=1}^{g} \frac{\Gamma(\xi - n^{2k} + \frac{1}{2})}{\Gamma(\xi - n^{2k} + 1)}.$$

Example. Genus $g = 1$:

$$\varphi = \frac{1}{\delta^{\frac{3}{2}}} \frac{\Gamma(\xi - n^1 + 1)}{\Gamma(\xi - n^1 + \frac{3}{2})} \frac{\Gamma(\xi - n^2 + \frac{1}{2})}{\Gamma(\xi - n^2 + 1)} \frac{\Gamma(\xi - n^3 + 1)}{\Gamma(\xi - n^3 + \frac{3}{2})}.$$

Fact. The Gamma function is non-zero but has simple poles at non-positive integers.

Conclusion. Due to cancellation, $\varphi(\xi)$ has no zeros or poles on

$$\bigcup_{k=1}^{g+1} (n^{2k-1}-1, n^{2k}), \qquad n^{2g+2} = \infty.$$

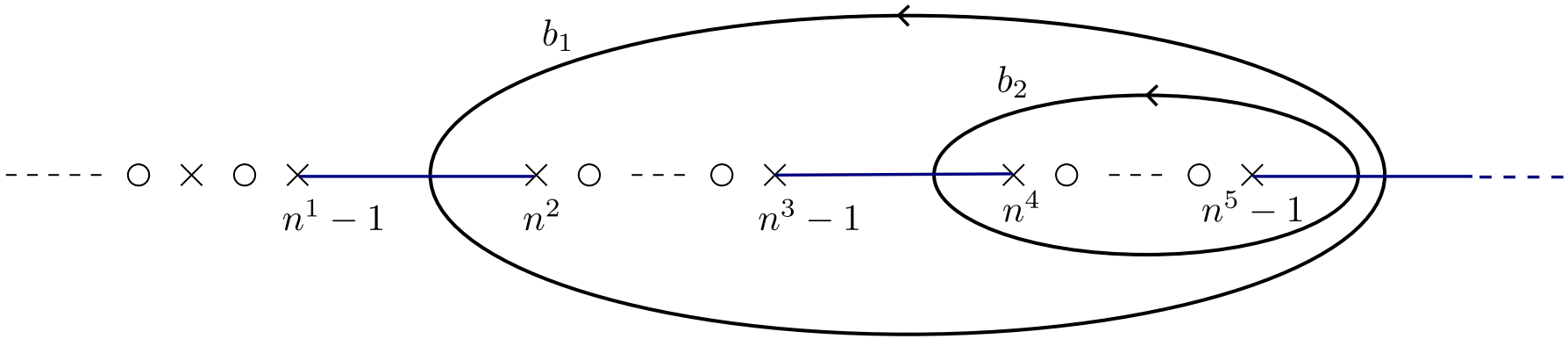
This guarantees the necessary property

$$\Delta_i \oint_{b_\kappa} = \oint_{b_\kappa} \Delta_i$$

in the following definitions.

Definition. The g cycles b_κ (on the ξ-plane) are closed paths of counterclockwise orientation passing through the pairs of intervals $(n^{2\kappa-1}-1, n^{2\kappa})$ and $(n^{2g+1}-1, \infty)$ for $\kappa = 1, \ldots, g$.

Example. Genus $g = 2$:



Blue: Branch cuts in Riemann surface theory

Definition. "Discrete" hyperelliptic integrals are (well-)defined by

$$\oint_{b_\kappa} \zeta^k \varphi(\xi)\, d\zeta, \qquad \xi = \frac{\zeta}{\delta}, \quad \kappa = 1, \ldots, g, \quad k = 0, \ldots, g-1$$

and constitute "discrete" analogues of the hyperelliptic integrals

$$\oint_{b_\kappa} \frac{\zeta^k}{\sqrt{\prod_{i=1}^{2g+1}(\zeta - x^i)}} d\zeta$$

16. Explicit form

Remark. Discrete hyperelliptic integrals may be evaluated explicitly in terms of the residues of the meromorphic integrand φ.

Example. Genus $g = 1$:

$$\phi^{\Delta} = \frac{1}{2\pi i} \oint_{b_1} \varphi(\xi)\, d\zeta = \delta \sum_{k=n^2}^{n^3-1} \operatorname{res}(\varphi(\xi), \xi = k)$$

so that

$$\phi^{\Delta} = \frac{1}{\sqrt{\delta}} \sum_{k=n^2}^{n^3-1} \frac{\Gamma(k-n^1+1)}{\Gamma(k-n^1+\frac{3}{2})} \frac{\Gamma(k-n^2+\frac{1}{2})}{\Gamma(k-n^2+1)} \frac{\operatorname{res}(\Gamma(\eta), \eta = k-n^3+1)}{\Gamma(k-n^3+\frac{3}{2})}.$$

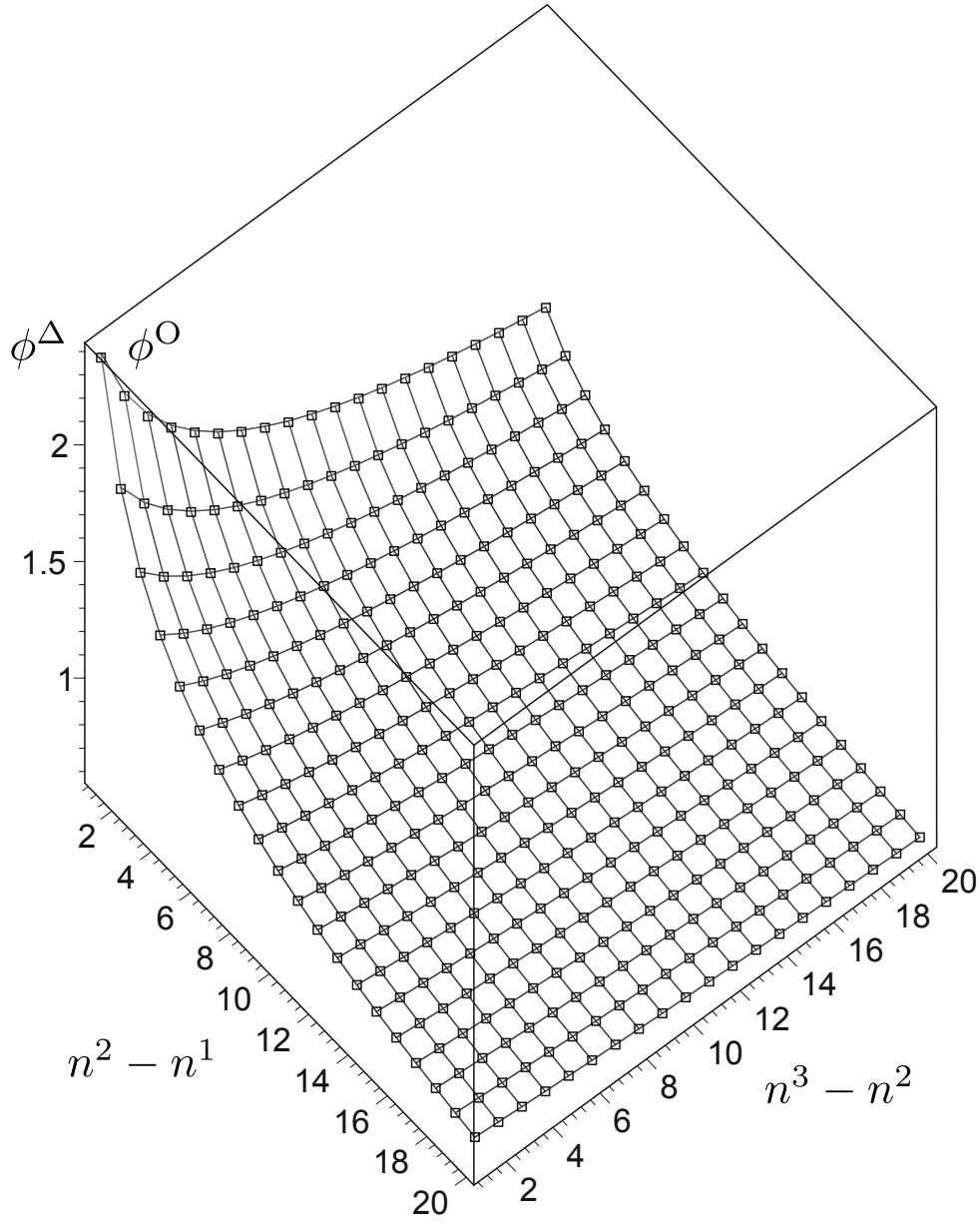
Comparison with the elliptic integral

$$\phi^{\bigcirc} = \frac{2}{\pi\sqrt{x^3 - x^1}} \mathsf{K}\left(\sqrt{\frac{x^3 - x^2}{x^3 - x^1}}\right)$$

evaluated at the points

$$x^1 = \delta n^1, \quad x^2 = \delta(n^2 + \tfrac{1}{2})$$
$$x^3 = \delta n^3$$

Remark. The relation between the discrete and continuous elliptic integrals is unique in the sense that, as a function of $\boldsymbol{n}$, the ratio $\phi^{\triangle}/\phi^{\bigcirc}$ is independent of the lattice parameter δ.



18. Extension to even genus and Abelian integrals

Concluding remarks. The preceding may be adapted to the case of

- hyperelliptic integrals associated with an even number $2g+2$ of branch points. These hyperelliptic integrals arise, for instance, in connection with the multi-phase averaged nonlinear Schrödinger (NLS) equations.

- Abelian integrals such as those associated with the trigonal curve

$$y^3 = \prod_{k=1}^{6} (x - x^k)$$

  which appear in the context of the Benney equations.

THANK YOU!