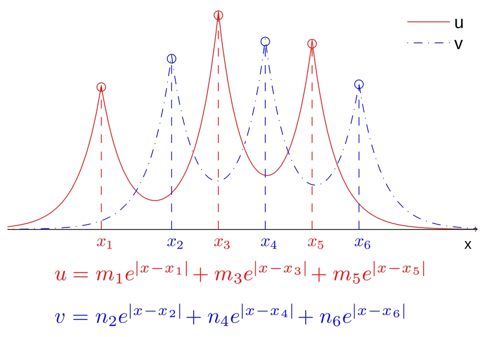

Deformations of Orthogonal Functions: Peakon and Toda

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A class of nonlinear integrable PDEs admit some special weak solutions called "peakons", which are characterised by ODE systems, namely peakon lattices. The celebrated Toda lattice was originally obtained as a simple model for describing a chain of particles with nearest neighbor exponential interaction. For some initial value problems, these lattices can be explicitly solved by use of inverse spectral method involving certain "orthogonality", approximation problems and continued fractions.

Interestingly, it is implied in [1, 6] that there exists certain intimate connection between the Camassa-Holm (CH) peakon lattice (i.e. the ODE system describing the CH peakons) and the finite Toda lattice, which are both indeed related to some deformations of ordinary orthogonal polynomials.

Consider a family of polynomials $\{P_n(x)\}$ orthogonal with respect to the measure $\nu(x)dx$ satisfying

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + u_nP_{n-1}(x).$$

- If $\nu(x;t)dx = e^{xt}\nu(x;0)dx$, then

  ⇒Toda lattice:

$$\dot{u}_k = u_k(b_k - b_{k-1}),\qquad \dot{b}_k = u_{k+1} - u_k.$$

- If $\nu(x;t)dx = e^{\frac{t}{x}}\nu(x;0)dx$, then

  ⇒CH peakon lattice:

$$\dot{x}_k = u(x_k), \qquad \dot{m}_k = -m_k\langle u_x\rangle(x_k),$$

  where

$$u(x;t) = \sum_{k=1}^{N} m_k(t)e^{-|x-x_k(t)|}$$

  are peakon solutions of CH eq.

$$m_t + um_x + 2mu_x = 0,\ m = u - u_{xx}.$$

	Known Result	Our Work		
Peakon lattice $e^{\frac{t}{x}}\mu(x;0)$	*CH*	2mCH	Novikov	DP
Toda lattice $e^{tx}\mu(x;0)$	*A-Toda*	K-vM	B-Toda	C-Toda
Lax x-part	*OPs*	sOPs	PSOPs	CBOPs
"τ-structure"	$\det((c_{i+j}))$	$\det((c_{i+j}))$	$Pf((c_{i,j}))$	$\det((c_{i,j}))$
Ref	[1, 6]	[2]	[3, 4]	[5]

Partial-skew-orthogonal polynomials $\{P_n(x)\}$:

- Partial-skew-orthogonality:

$$\iint P_{2n}(x)y^m\frac{x-y}{x+y}\nu(x)\nu(y)dxdy = \frac{\tau_{2n+2}}{\tau_{2n}}\delta_{2n+1,m},$$
$$\iint P_{2n+1}(x)y^m\frac{x-y}{x+y}\nu(x)\nu(y)dxdy = -\frac{\tau_{2n+2}}{\tau_{2n+1}}\beta_m,\ 0\le m\le 2n+1.$$

- Four-term recurrence:

$$z(P_n + u_nP_{n-1}) = P_{n+1} + (b_n+u_n)P_n - u_n(b_n+u_{n+1})P_{n-1} - u_n^2u_{n-1}P_{n-2}.$$

- If $\nu(x;t)dx = e^{\frac{t}{x}}\nu(x;0)dx$, then

  ⇒ Novikov peakon lattice:

$$\dot{x}_k = u(x_k)^2,\qquad \dot{m}_k = -m_ku(x_k)\langle u_x\rangle(x_k),$$

  where

$$u(x;t) = \sum_{k=1}^{N} m_k(t)e^{-|x-x_k(t)|}$$

  are peakon solutions of Novikov eq.

$$m_t + m_xu^2 + 3muu_x = 0,\ m = u - u_{xx}.$$

- If $\nu(x;t)dx = e^{xt}\nu(x;0)dx$, then

  ⇒ B-Toda lattice:

  Nonlinear form:

$$\dot{u}_k = u_k(b_k - b_{k-1}),\qquad \dot{b}_k = u_{k+1}(b_{k+1}+b_k) - u_k(b_k+b_{k-1}).$$

  Bilinear form (Hirota&Iwao&Tsujimoto 2001)

$$\ddot{\tau}_k\tau_k - (\dot{\tau}_k)^2 = \dot{\tau}_{k-1}\tau_{k+1} - \dot{\tau}_{k+1}\tau_{k-1}.$$

Cauchy biorthogonal polynomials $\{\Phi_n(x)\}$:

- Biorthogonality:

$$\iint \frac{\Phi_k(x)\Phi_l(y)}{x+y}\nu(x)\nu(y)dxdy = h_k\delta_{k,l}.$$

- Four-term recurrence:

$$z\left(\mathcal{A}_k\Phi_{k-1}(z) + \Phi_k(z)\right) = \Phi_{k+1}(z) + \mathcal{B}_k\Phi_k(z) + \mathcal{C}_k\Phi_{k-1}(z) + \mathcal{D}_k\Phi_{k-2}(z)$$

  with

$$\mathcal{A}_k = \sqrt{\frac{b_ku_k}{b_{k-1}}},\quad \mathcal{B}_k = \frac{1}{2}b_k + \sqrt{\frac{b_ku_k}{b_{k-1}}},\quad \mathcal{C}_k = -u_k - \frac{1}{2}\sqrt{b_ku_kb_{k-1}},\quad \mathcal{D}_k = -u_{k-1}\sqrt{\frac{b_ku_k}{b_{k-1}}}.$$

- If $\nu(x;t)dx = e^{\frac{t}{x}}\nu(x;0)dx$, then

  ⇒ DP peakon lattice:

$$\dot{x}_k = u(x_k),\quad \dot{m}_k = -2m_k\langle u_x\rangle(x_k),$$

  where

$$u(x;t) = \sum_{k=1}^{N} m_k(t)e^{-|x-x_k(t)|}$$

  are peakon solutions of DP eq.

$$m_t + um_x + 3mu_x = 0,\ m = u - u_{xx}.$$

- If $\nu(x;t)dx = e^{xt}\nu(x;0)dx$, then

  ⇒ C-Toda lattice:

  Nonlinear form:

$$\dot{u}_k = u_k(b_k - b_{k-1}),\qquad \dot{b}_k = 2(\sqrt{u_kb_kb_{k-1}} - \sqrt{u_{k+1}b_{k+1}b_k}).$$

  Bilinear form:

$$\dot{\tau}_k\tau_{k-1} - \tau_k\dot{\tau}_{k-1} = (\sigma_k)^2,\qquad \ddot{\tau}_k\tau_k - (\dot{\tau}_k)^2 = -2\sigma_{k+1}\sigma_k.$$

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