q-Stokes phenomenon on basic hypergeometric series

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1. Connection Problem

A **q-difference linear equation** with polynomial coefficients $(0 < |q| < 1)$:

$$\sum_{j=0}^{n} a_j(x) u(q^j x) = 0$$

We take **local fundamental solutions** around $x = 0, \infty$

$\mathbf{y_0}(x) = (y_0^{(1)}(x), ..., y_0^{(n)}(x))$: local solutions around $x = 0$

$\mathbf{y}_\infty(x) = (y_\infty^{(1)}(x), ..., y_\infty^{(n)}(x))$: local solutions around $x = \infty$

The **connection matrix** $P(x)$ is a $n \times n$ matrix with

$$\mathbf{y}_\infty(x) = \mathbf{y_0}(x) P(x).$$

The matrix elements of $P(x)$ are **pseudo constants** $P_{ij}(xq) = P_{ij}(x)$.
In this talk, we study only **single valued functions**.

Therefore $P_{ij}(xe^{2\pi i}) = P_{ij}(x)$: **elliptic functions** on $\mathbb{C}^\times / q^{\mathbb{Z}}$

Some **formal power series solutions are divergent**.
We consider **hypergeometric** cases in this talk.

1.2. Basic notations

0) **q-shifted factorial** :

$$(a_1,\dots,a_r;q)_n=\prod_{i=1}^{n}(a_i;q)_n,\quad (a;q)_n=(1-a)(1-qa)\cdots(1-q^{n-1}a).$$

1) **generalized q-hypergeometric series**:

$$ {}_r\phi_s\left(a_1,\dots,a_r;b_1,\dots,b_s;q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n(q;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n.$$

2) **Theta function**:

$$\theta_q(x):=\sum_{n=-\infty}^{\infty}q^{n(n-1)/2}x^n=(q,-x,-q/x;q)_\infty.$$

$$e_c(x):=\frac{\theta(x)}{\theta(cx)},\ \text{for } c\in\mathbb{C}^\times$$

$e_c(x)$ is **single-valued** and $e_c(xq)=ce_c(x)$.
A **multi-valued function** $h(x)=x^\gamma$ satisfies $h(xq)=q^\gamma h(x)$.

1. Basic hypergeometric equations

The basic hypergeometric series $y(x) = {}_r\phi_s(a_1, ..., a_r; b_1, ..., b_s; q, x)$ satisfies a **q-analogue of Goursat's equation** (**Jackson** [J]):

$$[xP_r(\sigma) - Q_s(\sigma)]y(x) = 0.$$

$\sigma_q = \sigma$: a **q-difference operator** $\sigma y(x) = y(xq)$.
$P_r(\sigma)$: a polynomial of σ of degree $\leqq r$
$Q_s(\sigma)$: a polynomial of σ of degree $\leqq s + 1$.

We set $0 \leqq s \leqq r$, $m + s + 1 \leqq r$; $b_0 = q$; $m + s + 1 + \mu = r$:

$$P_r(\sigma) = (-\sigma)^\mu \prod_{j=1}^{r}(1 - a_j\sigma),\ Q_s(\sigma) = -\left(-\frac{\sigma}{q}\right)^m \prod_{k=0}^{s}(1 - \frac{b_k}{q}\sigma).$$

We assume that

$$a_1 a_2 \cdots a_r \neq 0,$$

This means that **$x = \infty$ is a regular singular point**.
In this case the order of the q-analogue of Goursat's equation is r.

2. Differential case: Mellin transform

shifted factorial: $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$

Generalized hypergeometric series

$$y(x) = {}_pF_q\left(\begin{matrix}a_1, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} x^n$$

satisfies **Goursat's equation** ($\vartheta = x\dfrac{d}{dx}$):

$$\left[\vartheta(\vartheta + b_1 - 1)\cdots(\vartheta + b_q - 1) - x(\vartheta + a_1)\cdots(\vartheta + a_p)\right] y = 0.$$

The Mellin Transform

$$g(s) := \mathcal{M}(f; s) = \int_0^{\infty} x^{s-1} y(x)\, dx.$$

satisfies the following **first order difference equation**:

$$g(s+1) = -\frac{s(b_1 - s - 1)\cdots(b_q - s - 1)}{(a_1 - s - 1)\cdots(a_p - s - 1)} g(s)$$

$g(s)$ can be represented by a product quotient of the **Gamma functions**.

Remark.

$$g_1(s+1) = a_1(s)g_1(s),\ \ g_2(s+1) = a_2(s)g_2(s)$$
$$\Rightarrow g_1(s+1)g_2(s+1) = a_1(s)a_2(s)g_1(s)g_2(s).$$

For ${}_2F_1$, we obtain **Barnes' integral** by **the inverse Mellin Transform**

$${}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}x^s\,dt.$$

Remark.

Hjalmar Mellin:
Zur Theorie der Gammafunction. *Acta Math.* **8**, 37–80. **19 Fèvrier 1886**.

Salvatore Pincherle (Bologna):
Sopra una trasformazione delle equazioni differenziali lineari in equazioni lineari alle differenze, e viceversa, *R. Ist. Lomb. Sci. Lett. Rend.* **19**, 559–562. **21 febbrajo 1886**.

Confluent and higher order case

Cornelis Simon Meijer
Expansion theorems for the G-function. XI. Expansions in series of generalized hypergeometric functions. *Indag. Math.* 18 (1956), 70–822.

Dorothy Wrinch: Case ${}_0F_n$
A generalized hypergeometric function with n parameters,
Phil. Mag. 41 (Ser. 6), (1921), 174-186.

LONDON, *March* 19/57.

When the cat's away the mice may play. You are the cat and I am the poor little mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it*. I took it up again a few days ago, and after a two or three days' fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it.....

Memoir and scientific correspondence of the late Sir George Gabriel Stokes, Vol.1, p.62

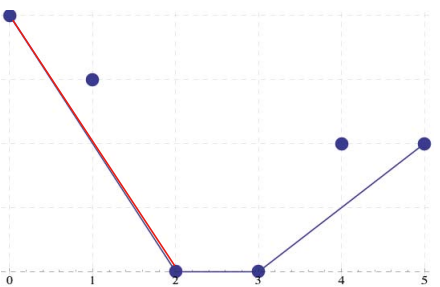
3. The Newton-Puiseux diagram and irregular singularity

The Newton-Puiseux diagram of a linear q-difference equation

$$a_n(x)u(xq^n) + a_{n-1}(x)u(xq^{n-1}) + \cdots + a_0(x)u(x) = 0$$

at the origin is a **lower convex hull** of

$$\{(j, \operatorname{ord}\, a_j(x)) \in \mathbb{R}^2 \,|\, 0 \leq j \leq n\}$$



segment: a line jointed with $(j, \operatorname{ord}\, a_j(x))$ and $(k, \operatorname{ord}\, a_k(x))$ $(j < k)$

slope of a segment:

$$\mu = \frac{\operatorname{ord}\, a_k(x) - \operatorname{ord}\, a_j(x)}{k - j}$$

length of a segment: $m = k - j$

3.1 Irregular singular points

General Theorem (Adams [A])

1) If partial characteristic exponents λ_j are non-resonant for any **segment with the slope μ and the length m**, the q-difference equation has m formal solutions

$$u_j(x) = \theta(x)^{\mu} \frac{\theta_q(x)}{\theta_q(\lambda_j x)} \sum_{k \geq 0} u_k x^k,$$

for each segment

2) When the slope of a segment is not integer $\mu = r/s$ $(s > 0)$, formal solutions are given by formal power series of $x^{1/s}$.

3) When a segment contains $(0, \mathrm{ord}\, a_0(x))$, **the power series are convergent**.

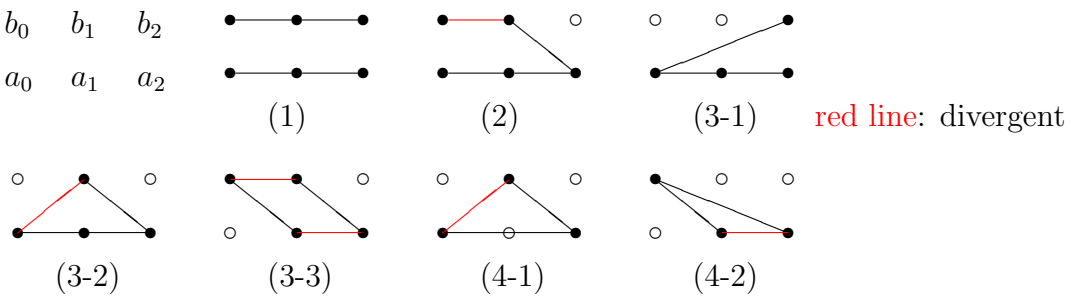
Problem:

1) Determine the **connection matrix** $P(x)$ for **q-Goursat's equation**.

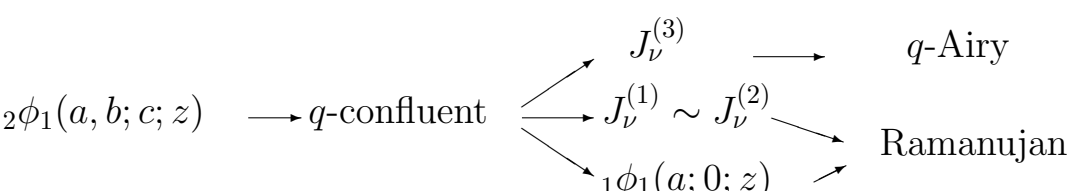
2) For a divergent power series, find a good **resummation method**

Classification of q-hypergeometric equations of the second order

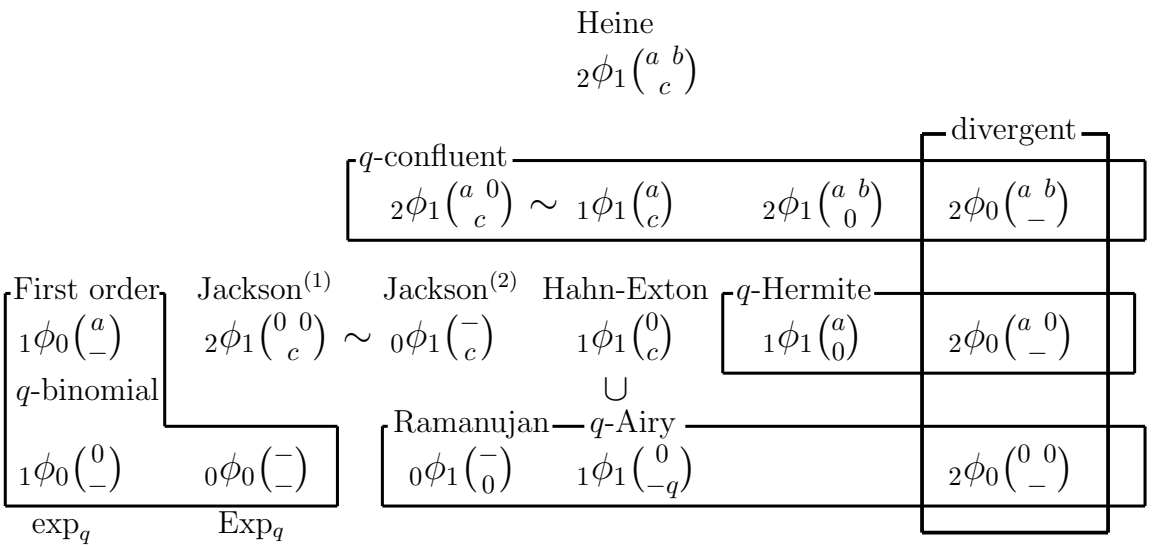
$$[(a_2+b_2x)\sigma_q^2+(a_1+b_1x)\sigma_q+(a_0+b_0x)]u(x)=0$$



Degeneration scheme



4.2. Degeneration of q-hypergeometric series



(1) Four q-confluent series appears in the same equation and

$$\textbf{Slater} : {}_1\phi_1\left(a; c; q, x\right) = (ax/c; q)_\infty {}_2\phi_1\left(c/a, 0; c; q, ax/c\right).$$

(2) ${}_0\phi_1\left(-, a^2q, q^2, a^2qx\right) = \left(x; q^2\right)_\infty {}_2\phi_1\left(0, 0; a^2q; q^2, x\right)$

(3) **q-Airy** and **Ramanujan** are essentially the same (**Morita**)

5.1. Connection Formula: regular singular case

Heine ${}_2\phi_1(a,b;c;x)$:

$$(c-abqx)u(xq^2)-[c+q-(a+b)qx]u(qx)+q(1-x)u(x)=0.$$

Local solution at $x=0$:

$$u_1={}_2\phi_1\left(a,b;c;x\right),\quad u_2=\frac{\theta(-x)}{\theta(-qx/c)}{}_2\phi_1\left(qa/c,qb/c;q^2/c;x\right).$$

Local solution at $x=\infty$:

$$v_1=\frac{\theta(-ax)}{\theta(-x)}{}_2\phi_1\left(a,aq/c;aq/b;cq/abx\right),\ v_2=(a\leftrightarrow b).$$

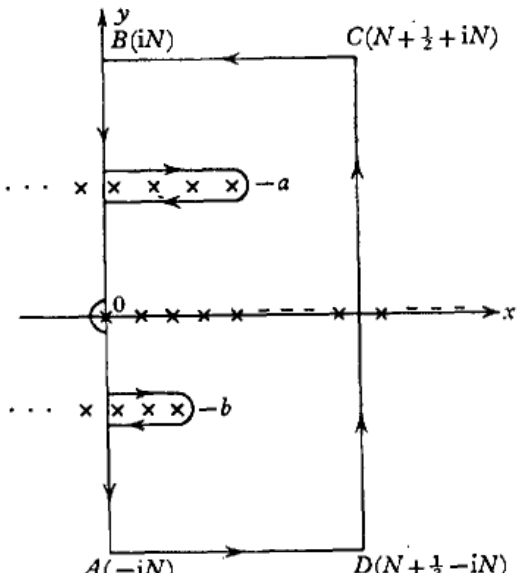
Theorem (Thomae, Watson) **Connection formula** of ${}_2\phi_1$

$$u_1=\frac{(b,c/a;q)_\infty}{(c,b/a;q)_\infty}v_1+\frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}v_2,$$
$$u_2=\frac{(qb/c,q/a;q)_\infty}{(q^2/c,b/a;q)_\infty}\frac{\theta(-qax/c)\theta(-x)}{\theta(-qx/c)\theta(-ax)}v_1+\frac{(qa/c,q/b;q)_\infty}{(q^2/c,a/b;q)_\infty}\frac{\theta(-qbx/c)\theta(-x)}{\theta(-qx/c)\theta(-bx)}v_2.$$

5.2. Integral representation of hypergeometric series

Barnes' integral

$$ {}_2F_1(a,b;c;-x) = \frac{\Gamma(c)}{2\pi i \Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} x^s \, dt. $$



We obtain the **connection formula** directly from Barnes' integral.

5.3. q-analogue of the Barnes-Meijer integral

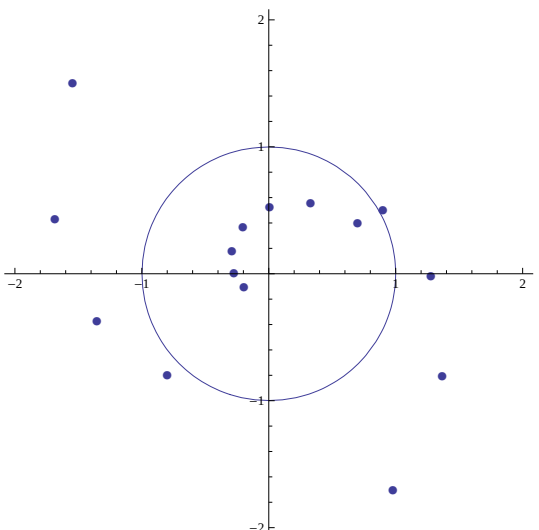
Take a **contour integral** (Slater, Askey-Roy, Gasper-Rahman)

$$\int_{|z|=\varepsilon} \frac{(a_1 z, a_2 z, ..., a_A z, b_1/z, ..., b_B/z; q)_\infty}{(c_1 z, c_2 z, ..., c_C z, d_1/z, ..., d_D/z; q)_\infty}\, dz.$$

The contour round $z = 0$ counterclockwisely and

$$z = 1/c_j q^k \ (j = 1, 2, ..., C; k \in \mathbb{Z}_{\geqq 0}), \quad z = d_l q^m \ (l = 1, 2, ..., D; m \in \mathbb{Z}_{\geqq 0})$$

$1/c_j q^k$ are outside、$d_l q^m$ are inside:



5.4. Proof of connection formula of ${}_2\phi_1$

We set $A = C = 2, B = D = 1$ and

$$a_1 = c, a_2 = q/z; b_1 = z; c_1 = a, c_2 = b; d_1 = 1$$

We get

$$P(t) = \frac{(ct, qt/z, z/t; q)_\infty}{(at, bt, 1/t; q)_\infty}$$

and the **contour integral** around $t = 0$:

$$\begin{aligned}\int_K P(t)\,dt &= \frac{(c, q/z, z; q)_\infty}{(q, a, b; q)_\infty} \times \sum_{n=0}^{\infty} \frac{(a, b, q/z; q)_n}{(q, c, b/a; q)_n} z^n \\ &= \frac{(c, q/z, z; q)_\infty}{(q, a, b; q)_\infty} {}_2\phi_1(a, b; c; q, z) = (1)\end{aligned}$$

The **contour integral** around $t = \infty$:

$$\int_K P(t)\,dt = \frac{(az, c/a, q/az; q)_\infty}{(q, a, b/a; q)_\infty} \times \sum_{n=0}^{\infty} \frac{(a, qa/c, az; q)_n}{(q, az, qa/b; q)_n} \left(\frac{cq}{abz}\right)^n + (a \leftrightarrow b) = (2)$$

Since (1)=(2), we obtain the **Thomae's connection formula**.

6. Connection problems and q-Stokes phenomenon

[I] **Apply q-Borel transform** [Zhang, Ramis, Sauloy]

1) The **q-Borel transformation** $\mathcal{B}_q^{\pm} : \mathbb{C}[[t]] \to \mathbb{C}[[\tau]]$ is defined by

$$\mathcal{B}_q^{\pm}\left[\sum_{n=0}^{\infty} a_n t^n\right] := \sum_{n=0}^{\infty} a_n q^{\pm n(n-1)/2} \tau^n.$$

q-Borel Transform of equations

Lemma As a linear operator on $\mathbb{C}[[t]]$, we have

$$\mathcal{B}_q^{\pm}(t^m \sigma_q^n f) = q^{\pm m(m-1)/2} \tau^m \sigma_q^{n\pm m} \mathcal{B}_q^{\pm}(f).$$

[II] When a solution is a **convergent power series**:

1) $\phi(\tau) = \mathcal{B}_q^{-}(f)(\tau)$ satisfies a **first order equation** and has a form

$$\varphi(\tau) = \frac{(a_1\tau, \cdots, a_k\tau; q)_\infty}{(b_1\tau, \cdots, b_m\tau; q)_\infty}, \quad (k < m)$$

2) The residue calculus of $\mathcal{L}_q^{-}(\phi)(t)$ gives a connectin formlua

$$\mathcal{L}_q^{-}(\varphi)(t) = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \varphi(\tau)\theta_q(t/\tau)\, \frac{d\tau}{\tau}.$$

[III] When a solution is a **divergent power series**:

1) The q-Borel transform $\mathcal{B}_q^+(f)$ satisfies an **equation of the same order**
2) $\mathcal{B}_q^+(f)$ may be convergent and reduces to the case [II]
3) Solve the connection problem for $\mathcal{B}_q^+(f)$ **on the Borel plane**.
4) Apply the **q-Laplace transform** $\mathcal{L}_{q;1}^{[\lambda]}$ $(\lambda \in \mathbb{C}^\times)$:

The q-Laplace transform $\mathcal{L}_{q;1}^{[\lambda]}$ is given by the Jackson integral

$$\mathcal{L}_{q;1}^{[\lambda]}(\varphi)(x) := \frac{1}{1-q}\int_0^{\lambda\infty} \frac{\varphi(\tau)}{\theta_q(\tau/x)}\frac{d_q\tau}{\tau} = \sum_{n\in\mathbb{Z}} \frac{\varphi(q^n\lambda)}{\theta_q(q^n\lambda/x)}.$$

$\star$ When $f(t) \in \mathbb{C}[[t]]$ is convergent, $\mathcal{L}_{q;1}^{[\lambda]}$ is an **inverse** of $\mathcal{B}_q^+$

$\star$ The RHS has poles at $x \in -\lambda q^{\mathbb{Z}}$. By the **$q$-Borel-Laplace resummation**,

$$\tilde{f}(x,\lambda) := \mathcal{L}_{q;1}^{[\lambda]} \circ \mathcal{B}_q^+(f)$$

is a holomorphic function $\{0 < |x| < \varepsilon\} \setminus -\lambda q^{\mathbb{Z}}$ and **asymptotic to** f.

The q-Stokes region is an open dense set in $\mathbb{C}^\times$, not an angle domain. For **q-divergence series**, the q-Stokes region **depends on a continuous parameter** λ.

6.2 Strategy

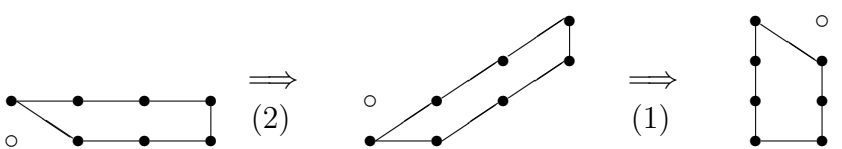
We change q-difference equation by the following two transformations:

Lemma 1. *(1) The* q**-Borel transformation** $\mathcal{B}_q^{\pm}$ *satisfies*

$$\mathcal{B}_q^{\pm}(t^m\sigma_q^n f) = q^{\pm m(m-1)/2}\tau^m\sigma_q^{n\pm m}\mathcal{B}_q^{\pm}(f).$$

(2) The **theta function** *shifts the power of* x*:*

$$x^m\sigma_q^n\left[\theta(cx)^{\pm 1}f(x)\right] = c^{\mp n}q^{\mp n(n-1)/2}x^{m\mp n}\theta(cx)^{\pm 1}\sigma_q^n f(x).$$



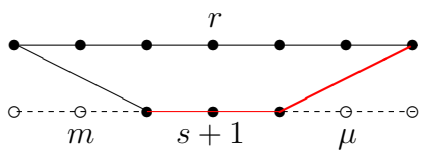
Remark.

In general, q-difference equations have **less transform** than differential equations. We can change q-difference equations to '**first order**' equations by the two transformations in the above lemma.

7. The case $x = \infty$ is a regular singular point

$0 \leqq s \leqq r$, $m + s + 1 \leqq r$; $b_0 = q$; $m + s + 1 + \mu = r$:

$$\left[x \prod_{j=1}^{r} (1 - a_j \sigma) - \left(-\frac{\sigma}{q} \right)^m \prod_{k=0}^{s} (1 - \frac{b_k}{q} \sigma) \right] y(x) = 0,$$



I) Around $x = \infty$, we have r **convergent hypergeometric** solutions

II) Around $x = 0$, we have

(1) m **convergent non-hypergeometric** solutions with ramification

(2) $s + 1$ **divergent hypergeometric** solutions

(3) μ **divergent non-hypergeometric** solutions with ramification

The cases $m = 0$, $s = -1$ or $\mu = 0$ may appear.

If $m = 0$, there exist $s + 1$ convergent solutions around $x = 0$.

If $m = 0$ and $s = -1$, there exist $\mu (= r)$ convergent solutions around $x = 0$.

Local solutions around the infinity $(j = 1, ..., r)$

$$y_{j,\infty}(x) = \frac{\theta(-a_j x)}{\theta(-x)}{}_{m+s}\phi_{r-1}\left(\begin{matrix} qa_j/b_0, qa_j/b_1, ..., qa_j/b_s, \mathbf{0} \\ qa_j/a_{j+1}, qa_j/a_{j+2}, ..., qa_j/a_{j-1}, \end{matrix}; q, \frac{q^\mu a_j^{r+1} b_0 b_1 \cdots b_s}{a_1 a_2 \cdots a_r x}\right)$$

Formal divergent solutions at the origin $(k = 0, ..., s)$

$$y_{k,0}(x) = \frac{\theta(-x)}{\theta(-qx/b_k)}{}_r\phi_{s-1}\left(\begin{matrix} qa_1/b_k, qa_2/b_k, ..., qa_r/b_k \\ qb_{k+1}/b_k, qb_{k+2}/b_k, ..., qb_{k-1}/b_k \end{matrix}; q, x\right)$$

Ramified convergent solutions at the origin $(k = 1, ..., m)$

$x = z^m$ and $q = p^m$. $\omega^m = 1$.

$$u_{k,0}(z) = \frac{1}{\theta_p(-p^{(m+1)/2}\omega^k z)} v_k(z), \ \ v_k(z) = \sum_{n=0}^{\infty} v_n^{(k)} z^n.$$

Ramified divergent solutions at the origin $(k = 1, ..., \mu)$

$\mu = r - s - 1$, $x = z^\mu$ and $q = p^\mu$. $\omega^\mu = 1$. $c^\mu = a_1 a_2 \cdots a_r / b_0 b_1 \cdots b_s$.

$$w_{k,0}(z) = \theta_p(-c\, p^{(1-\mu)/2}\omega^k z) h_k(z), \ \ h_k(z) = \sum_{n=0}^{\infty} h_n^{(k)} z^n.$$

7.2 Watson's formula: Covergent to Convergent

Assume that $m = 0$ [W].

Theorem 2. *We assume that* $0 \leqq s < r$. *Then*

$$ {}_r\phi_{r-1}\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s, 0, ..., 0 \end{matrix}; q, z\right) = \frac{(a_2, ..., a_{r+1}, b_s/a_1, ...b_s/a_1; q)_\infty}{(b_1, ..., b_s, a_2/a_1, ..., a_r/a_1; q)_\infty} $$
$$ \times \frac{\theta(-a_1 z)}{\theta(-z)} {}_{s+1}\phi_{r-1}\left(\begin{matrix} qa_1/b_0, qa_1/b_1, ..., qa_1/b_s \\ qa_1/a_2, ..., qa_1/a_r \end{matrix}; q, (a_1 q)^{r-s-1}\frac{b_0 b_1 \cdots b_s}{a_1 \cdots a_r z}\right) $$
$$ + \text{idem}(a_1; a_2, ..., a_r). $$

LHS is convergent when $|z| < 1$ *and RHS is convergent when* $s < r - 1$ *or* $s = r - 1$ *and* $|qb_1 \cdots b_s/a_1 \cdots a_r z| < 1$.

Proof is easy from a **contour integral**.
We consider a contour integral

$$ I = \frac{1}{2\pi i}\int_K \frac{(b_1 t, ..., b_s t, qt/z, z/t; q)_\infty}{(a_1 t, ..., a_r t, 1/t; q)_\infty}\frac{dt}{t}. $$

Here the contour K rounds once the origin counterclockwisely.

7.2 Ramified case

$m > 0$, $p^m = q$. We assume that $s + m \geqq r$.

$$
\begin{aligned}
I &= \int_{|\tau|=\varepsilon} \frac{\prod_{j=1}^{s}(b_j\tau;q)_\infty}{\prod_{k=1}^{r}(a_k\tau;q)_\infty}\theta_p(x/\tau)\,\frac{d\tau}{\tau} \\
&= \frac{(b_1/a_1,...,b_s/a_1;q)_\infty}{(q,a_2/a_1,...a_r/a_1;q)_\infty}\theta_p(a_1x) \\
&\qquad \times {}_{s+m}\phi_{r-1}\left(\begin{matrix} qa_1/b_1,...,qa_1/b_s,\mathbf{0}_m \\ qa_1/a_2,...,qa_1/a_r \end{matrix}; q, \frac{(-1)^r q^{r-s+(1-m)/2} b_1\cdots b_s}{a_1^{m-r+1}a_2\cdots a_r x}\right) \\
&\qquad\qquad + \mathrm{idem}(a_1;a_2,...,a_r).
\end{aligned}
$$

[**Useful relation**]

$$
\mathrm{Res}_{\tau=1/aq^n}\frac{1}{(a\tau;q)_\infty}\frac{d\tau}{\tau} = -\frac{(-1)^n q^{n(n+1)/2}}{(q;q)_\infty(q;q)_n},
$$

$$
\theta_p(x/\tau)|_{\tau\to 1/aq^n} = (ax)^{-mn}p^{-nm(nm-1)/2}\theta_p(ax),
$$

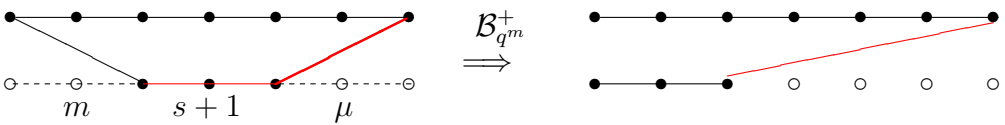
$$
(b\tau;q)_\infty|_{\tau\to 1/aq^n} = (-b/a)^n q^{-n(n+1)/2}(b/a;q)_\infty(aq/b;q)_n.
$$

7.3 q-Borel-Laplace transformation:

I) **Unramified case**

Any **divergent hypergeometric series** is **unramified**.

The q^m-Borel Laplace transform is **convergent hypergeometric series**.



II) **Ramified case**

Lemma 3. *We take a positive integer m. We consider*

$$\varphi(\xi) = \frac{\theta_q(a\xi)}{\theta_q(b\xi)} \sum_{n\geq 0} c_n \xi^{-n}.$$

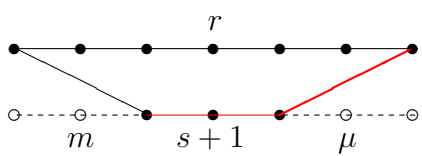
Then we obtain

$$\mathcal{L}^{[\lambda]}_{q^m;1}\varphi(x) = \frac{\theta_q(a\lambda)\theta_{q^m}(q^m a^m x/b^m\lambda)}{\theta_q(b\lambda)\theta_{q^m}(q^m x/\lambda)} \sum_{n\geq 0} c_n (q^m)^{-\frac{n(n-1)}{2}} \left(\frac{b^m}{a^m q^m x}\right)^n.$$

8. Main Theorem

The basic hypergeometric series $y(x) = {}_r\phi_s(a_1, ..., a_r; b_1, ..., b_s; q, x)$ satisfies a **q-analogue of Goursat's equation** :

$$\left[x \prod_{j=1}^{r} (1 - a_j \sigma) - \left(-\frac{\sigma}{q} \right)^m \prod_{k=0}^{s} (1 - \frac{b_k}{q} \sigma) \right] y(x) = 0,$$



Theorem 1. If $r = 2$, we have **seven different hypergeometric-type** equations. All of connection formula are determined. The q-Stokes regions are open dense domains $\mathbb{C}^\times \setminus \lambda q^{\mathbb{Z}}$.

Thomae [Th], Watson [W], Slater [S1], Zhang [Z02, Z03], Morita [M1], O.

Theorem 2. If $r = 3$ and $a_j \neq 0$, we have **ten different hypergeometric-type** equations. All of connection formula are determined. The q-Stokes regions are open dense domains $\mathbb{C}^\times \setminus \lambda q^{\mathbb{Z}}$. (Except $Q = 1$)

By Thomae [Th2], Watson [W], Morita[M2], Zhang, O. Adachi [Ad]

Thm:
We solved **connection problems** of q-hypergeometric equations of second order.

1) Heine's basic hypergeometric series ${}_2\varphi_1(a,b;c;q,x)$ (Thomae)
2) q-confluent hypergeometric series ${}_1\varphi_1(a;c;q,x)$ (Watson, Zhang)
3) q-Hermite-Weber function ${}_1\varphi_1(a;0;q,x)$ (**Zhang and O.**)
4) Jacson's first q-Bessel function ${}_2\varphi_1(0,0;a;q,x)$ (Zhang)
5) Jacson's second q-Bessel function ${}_0\varphi_1(-;a;q,x)$ (Zhang)
6) Hahn-Exton's q-Bessel function ${}_1\varphi_1(0;a;q,x)$ (Morita, **O.**)
7) The Ramanujan function ${}_0\varphi_1(-;0;q,x)$ (Ismail-Zhang, Morita)

Remark

[A] (2) (6) (7) has one irregular singular point. Therefore we should show Two different type of connection formula (**Convergent, Divergent**)

[B] (3) has **two irregular singular points**.

[C] (4) and (5) are essentially equivalent.

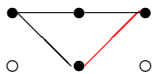
[D] For (1), (3), (4) and (5), we need to show only one connection formula.

[E] (7) is a special case $a = -q$ of (6).

9.1 Example: q-Stokes phenomenon of Hahn-Exton's q-Bessel

Modified **Hahn-Exton's q-Bessel function**:

$$qtv(tq) + [1 - (a+q)t]v(t) + atv(t/q) = 0.$$



Local solutions around $t = \infty$

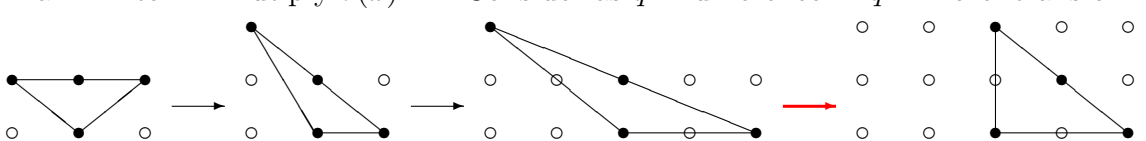
$$v_1^{(\infty)}(t) = {}_1\varphi_1(0; a; q, 1/t), \quad v_2^{(\infty)}(t) = \frac{\theta(qt/a)}{\theta(t)}\, {}_1\varphi_1(0; q^2/a; q, q/at).$$

Local (meromorphic) solutions around $t = 0$

$$\text{(C) } v_2^{(0)}(t) = \frac{1}{\theta(-aqt)} \sum_{m=0}^{\infty} c_m t^m, \quad \text{(D) } v_1^{(0)}(t) = \theta(-qt) \sum_{m=0}^{\infty} b_m t^m.$$

We need to take **$q^{1/2}$-Borel transformation** (Dreyfus, Eloy [DE] [El])

Hahn-Exton → mutiply $\theta(x)$ → Consider as $q^{1/2}$-difference T → $q^{1/2}$-Borel transform



The last equation is reduced to **the second Jackson q-Bessel** [Z03].
By the **$q^{1/2}$-Laplace transform** $\mathcal{L}^{[\lambda]}_{q^{1/2};1}$, we get the **Hahn-Exton** [O3].

9.2. Connection formula of the Hahn-Exton equation

We set $p^2 = q$ and

$$\tilde{v}_1^{(0)}(t,\lambda) = \theta(-qt)\mathcal{L}_p^{[\lambda]} \circ \mathcal{B}_p^+ \left[\sum_{m=0}^{\infty} b_m t^m \right].$$

Theorem [Connection formula of the Hahn-Exton equation]

$$\begin{aligned}\tilde{v}_1^{(0)}(t,\lambda) &= \frac{\theta_q(a\lambda)\theta_q(ap\lambda)}{(q/a;q)_\infty \theta_q(-ap\lambda^2)} \frac{\theta_q(-pt/\lambda^2)}{\theta_q(pt/\lambda)\theta_q(qt/\lambda)} v_1^{(\infty)}(t) \\ &\quad + \frac{\theta_q(q\lambda)\theta_q(qp\lambda)}{(a/q;q)_\infty \theta_q(-ap\lambda^2)} \frac{\theta_q(-pqt/a\lambda^2)}{\theta_q(pt/\lambda)\theta_q(qt/\lambda)} \frac{\theta_q(t)}{\theta_q(qt/a)} v_2^{(\infty)}(t).\end{aligned}$$

$$v_2^{(0)}(t) = \frac{1}{(q;q)_\infty (q/a;q)_\infty} v_1^{(\infty)}(t) + \frac{q}{a \cdot (q;q)_\infty (a/q;q)_\infty} v_2^{(\infty)}(t).$$

Remark. The second equation is obtained by Morita [M1].

We need a non-integer power $q^{1/n}$**-Borel-Laplace transform** for higher q-Bessel equations: $s = -1$.

9.3 Example: ${}_3\phi_2\left(a_1, a_2, a_3; b_1, 0; q, x\right)$

$$\left[x\prod_{j=1}^{3}(1-a_j\sigma)-(1-\sigma)(1-\frac{b_1}{q}\sigma)\right]y(x)=0.$$

Local solutions around the origin:

$$y_{1,0}(x) = {}_3\phi_2\left(\begin{matrix}a_1, a_2, a_3\\ b_1, 0\end{matrix}; q, x\right),$$

$$y_{2,0}(x) = \frac{\theta(-x)}{\theta(-qx/b_1)}{}_3\phi_2\left(\begin{matrix}a_1q/b_1, a_2q/b_1, a_3q/b_1\\ q^2/b_1, 0\end{matrix}; q, x\right),$$

$$y_{3,0}(x) = \theta(-a_1a_2a_3x/b_1q)w(x), \quad w(x)=\sum_{n=0}^{\infty}w_nx^n.$$

Local solutions around the infinity:

$$y_{1,\infty}(x) = \frac{\theta(-a_1x)}{\theta(-x)}{}_2\phi_2\left(\begin{matrix}a_1, a_1q/b_1\\ a_1q/a_2, a_1q/a_3\end{matrix}; q, \frac{q^2b_1}{a_2a_3x}\right)$$

$y_{2,\infty}(x), y_{3,\infty}(x)$ are cyclic

We set $v(\xi) = \mathcal{B}_q^+(w)$.

$$(1 + s_3\xi/a_1b_1q)(1 + s_3\xi/a_2b_1q)(1 + s_3\xi/a_3b_1q)v(\xi q) \\ =(1 + s_3\xi/b_1^2q)(1 + s_3\xi/b_1q^2)v(\xi).$$

Therefore

$$v(\xi) = \frac{(-s_3\xi/qa_1b_1, -s_3\xi/qa_2b_1, -s_3\xi/qa_2b_1)_\infty}{(-s_3\xi/q^2b_1, -s_3\xi/qb_1^2)_\infty}.$$

We set

$$\tilde{y}_3(x,\lambda) = \theta(-s_3x/b_1q)\mathcal{L}_{q;1}^{[\lambda]} \circ \mathcal{B}_q^+[w](x) = \theta(-s_3x/b_1q)\mathcal{L}_{q;1}^{[\lambda]}[v](x).$$

Theorem

$$\tilde{y}_3(x,\lambda) = \frac{(q, b_1/a_1, q/a_1; q)_\infty}{(a_2/a_1, a_3/a_1; q)_\infty} \frac{\theta(qa_2b_1/s_3\lambda, qa_3b_1/s_3\lambda)}{\theta(qb_1^2/s_3\lambda, q^2b_1/s_3\lambda)} \\ \times \frac{\theta(-s_3x/qb_1, qa_1b_1x/s_3\lambda)}{\theta(-a_1x, x/\lambda)} y_{1,\infty}(x) + \mathrm{idem}(a_1; a_2, a_3).$$

Connection formulae of $y_1(x), y_2(x)$ are obtained by Watson's formula.

We find connection formulae for **basic hypergeometric equations**.
Our methods can be applied to other equations.

[Problem]

1) Connection formula in the case **both $x = 0$ and $x = \infty$ are irregular**.

2) I forget $Q = 1$ case...

3) Find a connection formula of other **rigid q-difference equations**

4) Find a connection formula of **confluent q-Appell equations**

5) Apply to **q-Heun** or **q-Painlevé equations** (Vinet)

6) How about **confluent elliptic hypergeometric** equation? (Noumi)

7) Application to many fields in mathematical sciences (Sasamoto, ...)

8) Relation to **q-difference Galois theory** (Duval-Mitschi, Roques)

9) What is **q-WKB method**, **q-resurgent analysis** or **q-Écalle**? (Joshi)

Lucy Joan Slater

These photos were provided from the archives of the **Dalton Genealogical Society**.
Check out "**daltonhistory.org**"!

"**A Woman in a Man's World**" ed. John Dalton, 2013.

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