Totally conservative integration method for N-body problem and its equilibrium solutions

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Abstract

We design a precise integration method for the general N-body problem (GNBP) which maintains all conserved quantities. Our method prove to retain the same orbits of equilibrium solutions as the original GNBP. This method is based on a kind of 2nd-order symplectic integrators, logarithmic Hamiltonian leapfrog ($(\text{LogH})_2$), and involves an energy-preserving parameter. Until our method, no integration method was known to hold both preservation of all conserved quantities and the existence of triangular Lagrangian solutions.

Introduction

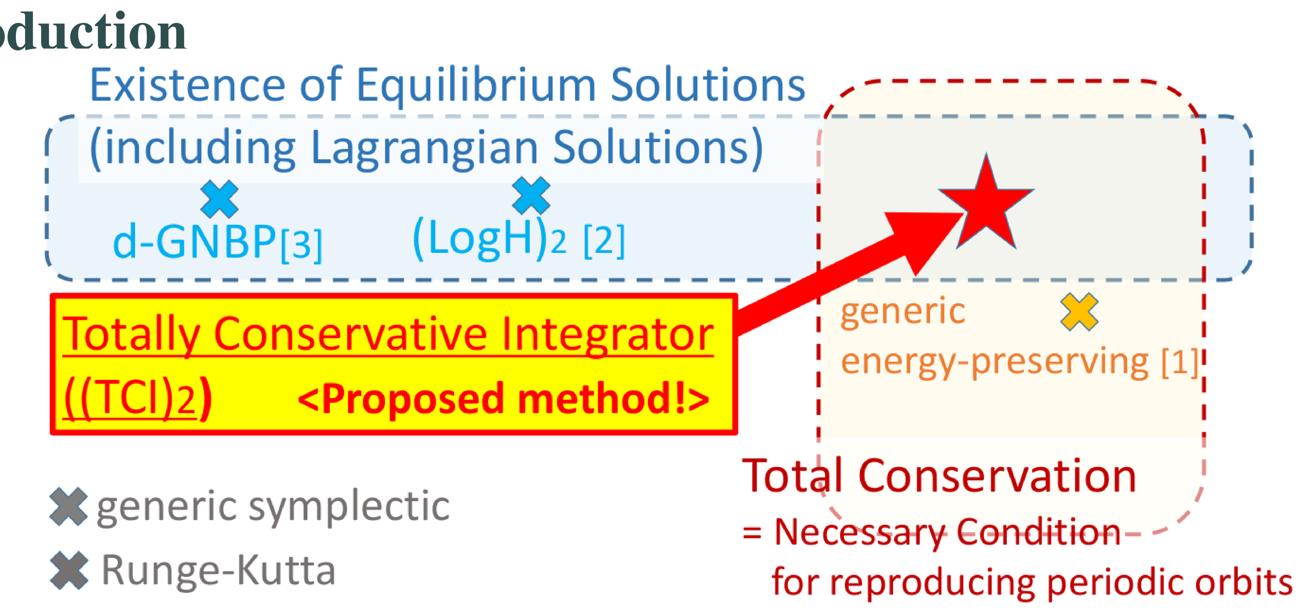

Figure 1: Various integrators for GNBP

Totally conservative integration method for general 3-body problem (G3BP)

For simplicity, we describe only a 2nd-order **totally conservative integration method ($(\text{TCI})_2$)** for G3BP in the relative frame:

$$\begin{cases} \mathbf{v}_{12}^{(l+1)}+\mathbf{v}_{23}^{(l+1)}-\mathbf{v}_{13}^{(l+1)}=\mathbf{0}, \\ \mathbf{q}_{ij}^{(l+\frac12)}=\mathbf{q}_{ij}^{(l)}+\dfrac{\Delta s}{2}\dfrac{\mathbf{v}_{ij}^{(l)}}{T_R(\mathbf{v}^{(l)})-E^{(0)}},\ 1\le i<j\le 3, \quad t^{(l+\frac12)}=t^{(l)}+\dfrac{\Delta s}{2}\dfrac{1}{T_R(\mathbf{v}^{(l)})-E^{(0)}}, \\ \displaystyle\sum_{i=1}^{j-1} m_i\left(\frac{U_R(\mathbf{q}^{(l+\frac12)})}{\alpha^{(l+\frac12)}\Delta s}(\mathbf{v}_{ij}^{(l+1)}-\mathbf{v}_{ij}^{(l)})-\frac{M\mathbf{q}_{ij}^{(l+\frac12)}}{|\mathbf{q}_{ij}^{(l+\frac12)}|^3}\right)=\sum_{i=j+1}^{3} m_i\left(\frac{U_R(\mathbf{q}^{(l+\frac12)})}{\alpha^{(l+\frac12)}\Delta s}(\mathbf{v}_{ji}^{(l+1)}-\mathbf{v}_{ji}^{(l)})-\frac{M\mathbf{q}_{ji}^{(l+\frac12)}}{|\mathbf{q}_{ji}^{(l+\frac12)}|^3}\right),\ j=2,3, \\ \mathbf{q}_{ij}^{(l+1)}=\mathbf{q}_{ij}^{(l+\frac12)}+\dfrac{\Delta s}{2}\dfrac{\mathbf{v}_{ij}^{(l+1)}}{T_R(\mathbf{v}^{(l+1)})-E^{(0)}},\ 1\le i<j\le 3,\ t^{(l+1)}=t^{(l+\frac12)}+\dfrac{\Delta s}{2}\dfrac{1}{T_R(\mathbf{v}^{(l+1)})-E^{(0)}}, \end{cases}$$

where Δs: a fictitious time step, the total kinetic energy: $T_R(\mathbf{v})\equiv\dfrac{\sum_{i=1}^{2}\sum_{j=i+1}^{3}m_im_j|\mathbf{v}_{ij}|^2}{2(m_1+m_2+m_3)}$, the total potential: $U_R(\mathbf{q})\equiv-\displaystyle\sum_{i=1}^{2}\sum_{j=i+1}^{3}\frac{m_im_j}{|\mathbf{q}_{ij}|}$, the total energy: $E^{(0)}\equiv T_R(\mathbf{v}^{(0)})+V_R(\mathbf{q}^{(0)})$, the total potential: $M=m_1+m_2+m_3$, and energy-preserving parameter: $\boldsymbol{\alpha}^{(l+1/2)}$. In the case of $\alpha^{(l+\frac12)}=1$, $(\text{TCI})_2$ coincides with $(\text{LogH})_2$ [2].

$$(\text{LogH})_2 \xrightarrow{\Delta s\to 0} \textbf{G3BP in the relative frame}: \begin{cases} \mathbf{v}_{12}+\mathbf{v}_{23}-\mathbf{v}_{13}=\mathbf{0};\ \dfrac{d\mathbf{q}_{i,j}}{dt}=\mathbf{v}_{i,j},\ 1\le i<j\le 3; \\ \displaystyle\sum_{i=1}^{j-1} m_i\left(\frac{d\mathbf{v}_{i,j}}{dt}+\frac{M\mathbf{q}_{i,j}}{|\mathbf{q}_{i,j}|^3}\right)=\sum_{i=j+1}^{3} m_i\left(\frac{d\mathbf{v}_{j,i}}{dt}+\frac{M\mathbf{q}_{j,i}}{|\mathbf{q}_{j,i}|^3}\right),\ j=2,3. \end{cases}$$

$$\xrightarrow[\mathbf{p}_i=\frac{m_i}{M}\left(-\sum_{j=1}^{i-1}m_j\mathbf{v}_{ji}+\sum_{j=i+1}^{3}m_j\mathbf{v}_{ij}\right)]{\mathbf{q}_i=\frac{1}{M}\left(-\sum_{j=1}^{i-1}m_j\mathbf{q}_{ji}+\sum_{j=i+1}^{3}m_j\mathbf{q}_{ij}\right)} \textbf{G3BP in the inertial barycentric frame}: \begin{cases} \dfrac{d\mathbf{p}_i}{dt}=m_i\left(\displaystyle\sum_{k=1}^{i-1}\frac{m_k(\mathbf{q}_k-\mathbf{q}_i)}{|\mathbf{q}_k-\mathbf{q}_i|^3}-\sum_{k=i+1}^{3}\frac{m_k(\mathbf{q}_i-\mathbf{q}_k)}{|\mathbf{q}_i-\mathbf{q}_k|^3}\right), \\ \dfrac{d\mathbf{q}_i}{dt}=\dfrac{\mathbf{p}_i}{m_i},\ i=1,2,3. \end{cases}$$

Conservation of G3BP ($(\text{LogH})_2$ vs $(\text{TCI})_2$)

Conserved quantities of $(\textbf{TCI})_2$

- Conserved quantities of $(\textbf{LogH})_2 = (\textbf{TCI})_2$ **with** $\boldsymbol{\alpha}^{(l+1/2)}=1$
  - Conserved quantities **by Coordinate transformation** $(\mathbf{q}_{ij},\mathbf{v}_{ij})\to(\mathbf{q}_i,\mathbf{p}_i)$
    - * the total linear momentum : $\boldsymbol{\ell}=0$ * the position of the center of mass : $\boldsymbol{c}=0$
  - * the total angular momentum : $\boldsymbol{a}\equiv\dfrac{1}{M}\displaystyle\sum_{i=1}^{2}\sum_{j=i+1}^{3}m_im_j\mathbf{q}_{ij}\times\mathbf{v}_{ij}$
- * the total energy : $E\equiv T_R(\mathbf{v})+U_R(\mathbf{q})\longleftarrow$ This is accomplished by adjusting $\alpha^{(l+1/2)}$.

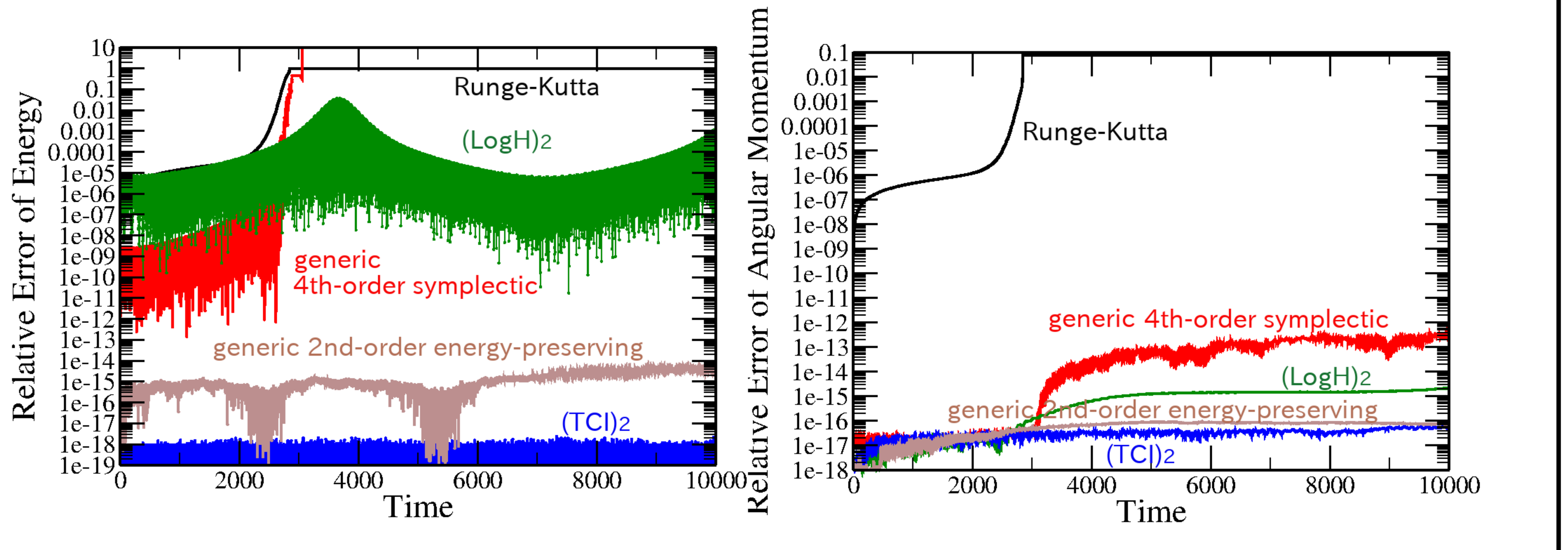

Figure 2: Relative errors of the total energy and total angular momentum for a periodic solution different from Lagrangian solutions.

The above relation for conservation is fulfilled for GNBP.

Triangular Lagrangian Solutions of $(\text{TCI})_2$

For any integer l, and arbitrary real numbers Θ, and Φ, we assume

$$\mathbf{q}_{12}^{(k)}=\mathfrak{q}^{(k)}R_3(\Theta)\,R_1(\Phi),\ \mathbf{q}_{13}^{(k)}=\mathfrak{q}^{(k)}R_3(\Theta\pm\pi/3)\,R_1(\Phi),\ \mathbf{q}_{23}^{(k)}=\mathbf{q}_{13}^{(k)}-\mathbf{q}_{12}^{(k)},\ k=l,l+1/2,l+1,$$
$$\mathbf{v}_{12}^{(k)}=\mathfrak{v}^{(k)}R_3(\Theta)\,R_1(\Phi),\ \mathbf{v}_{13}^{(k)}=\mathfrak{v}^{(k)}R_3(\Theta\pm\pi/3)\,R_1(\Phi),\ \mathbf{v}_{23}^{(k)}=\mathbf{v}_{13}^{(k)}-\mathbf{v}_{12}^{(k)},\ k=l,l+1,$$

(double sign corresponds),

where $R_3(\theta)$, and $R_1(\phi)$ are respectively the rotational matrix about the 3rd, and the 1st axes. Then, $(\text{TCI})_2$ reduces to the **discrete-time general 2-body problem (d-G2BP)**:

$$\begin{cases} \mathfrak{q}^{(l+\frac12)}=\mathfrak{q}^{(l)}+\dfrac{\frac{\Delta\sigma_{LT}}{2}\mathfrak{v}^{(l)}}{\frac{(\mathfrak{v}^{(l)})^2}{2}-\mathcal{E}_{LT}^{(0)}},\ t^{(l+1/2)}=t^{(l)}+\dfrac{\frac{\Delta\sigma_{LT}}{2}}{\frac{(\mathfrak{v}^{(l)})^2}{2}-\mathcal{E}_{LT}^{(0)}}, \\ \mathfrak{v}^{(l+1)}=\mathfrak{v}^{(l)}-\alpha^{(l+\frac12)}\dfrac{\Delta\sigma_{LT}\mathfrak{q}^{(l+\frac12)}}{|\mathfrak{q}^{(l+\frac12)}|^2}, \\ \mathfrak{q}^{(l+1)}=\mathfrak{q}^{(l+\frac12)}+\dfrac{\frac{\Delta\sigma_{LT}}{2}\mathfrak{v}^{(l+1)}}{\frac{(\mathfrak{v}^{(l+1)})^2}{2}-\mathcal{E}_{LT}^{(0)}},\ t^{(l+1)}=t^{(l+1/2)}+\dfrac{\frac{\Delta\sigma_{LT}}{2}}{\frac{\left(\mathfrak{v}^{(l+1)}\right)^2}{2}-\mathcal{E}_{LT}^{(0)}}, \end{cases}$$

where $\Delta\sigma_{LT}\equiv\dfrac{M\Delta s}{m_1m_2+m_1m_3+m_2m_3}$, and $\mathcal{E}_{LT}\equiv\dfrac{|\mathfrak{v}|^2}{2}-\dfrac{M}{|\mathfrak{q}|}$.

$$\textbf{d-G2BP}\xrightarrow{\boldsymbol{\alpha}^{(l+1/2)}=1}(\textbf{LogH})_2\ \textbf{for G2BP}\xrightarrow{\Delta s\to 0}\textbf{G2BP}:\ \frac{d\mathfrak{q}}{dt}=\mathfrak{v},\ \frac{d\mathfrak{v}}{dt}=-\frac{M\mathfrak{q}}{|\mathfrak{q}|^3}$$

$(\text{LogH})_2$ for G2BP [2] maintains all conserved quantities of G2BP, namely, (i) energy : $\mathcal{E}_{LT}$, (ii) angular velocity : $\mathbf{J}_{LT}\equiv\mathfrak{q}\times\mathfrak{v}$, and (iii) Laplace vector : $e_{LT}\equiv\dfrac{\mathfrak{v}\times(\mathfrak{q}\times\mathfrak{v})}{M}-\dfrac{\mathfrak{q}}{|\mathfrak{q}|}$. Therefore, $(\text{LogH})_2$ for G2BP exactly traces the original G2BP. Further, **$\boldsymbol{\alpha}^{(l+1/2)}$ is adjusted to conserve $\boldsymbol{\mathcal{E}}_{LT}^{(0)}$. Accordingly, when $(\text{TCI})_2$ holds the triangular Lagrangian solutions, $(\text{TCI})_2$ has $\boldsymbol{\alpha}^{(l+1/2)}=1$, namely, $(\text{TCI})_2$ coincides with $(\text{LogH})_2$ for G2BP.** In a similar procedure, we prove that $(\text{TCI})_2$ has the collinear Lagrangian solutions of G3BP and equilibrium solutions of GNBP.

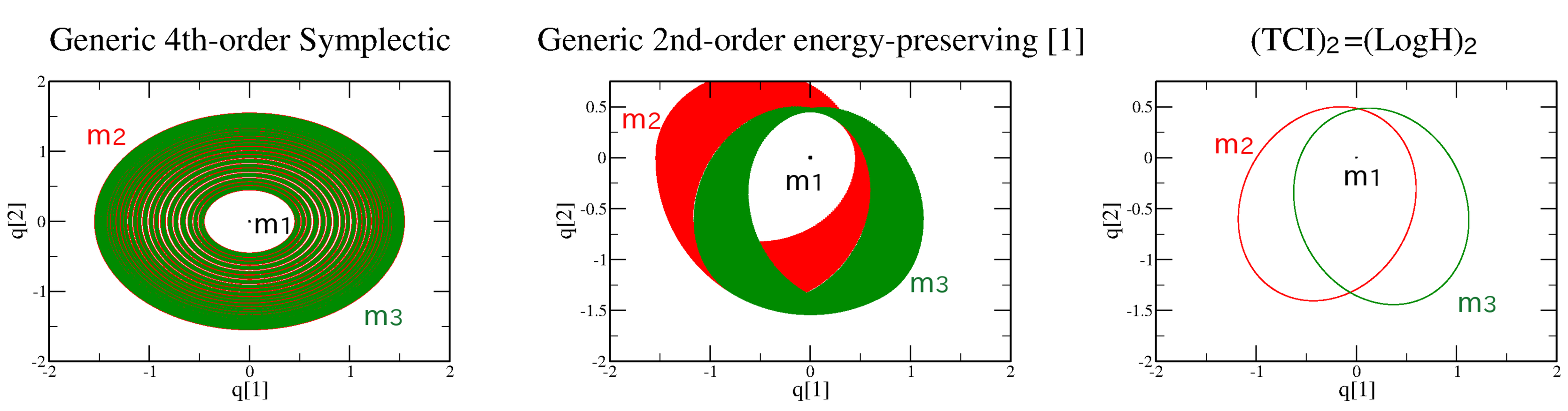

Figure 3: Orbits of a triangular Lagrangian solution in the inertial barycentric frame

Other Periodic Orbits of $(\text{TCI})_2$

$(\text{TCI})_2$ is merely 2nd-order accurate, but it can precisely reproduce some periodic orbits which $(\text{LogH})_2$, and generic geometric integrators cannot do. For example,

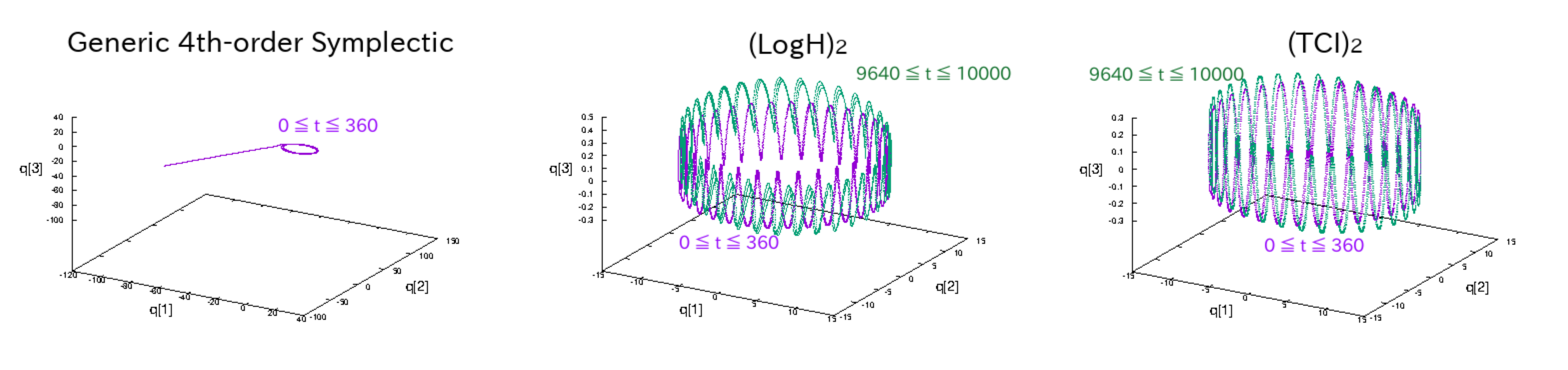

Figure 4: A Periodic Orbit of m_2 in the inertial barycentric frame

Conclusion

We modified $(\text{LogH})_2$ [2] for GNBP to design a new integration method, $(\text{TCI})_2$, which maintains all conserved quantities and holds equilibrium solutions including Lagrangian solutions. Although $(\text{TCI})_2$ is merely 2nd-order accurate, it can precisely trace some periodic orbits which generic geometric integrators, and $(\text{LogH})_2$ cannot do. Until $(\text{TCI})_2$, no integration method was known to maintain both preservation of all conserved quantities and the existence of equilibrium solutions.

References

[1] D. Greenspan, Discrete Newtonian gravitation and the three-body problem, *Fund. Phys.* **4** (1974), 299–310.

[2] S. Mikkola and K. Tanikawa, Explicit symplectic algorithms for timetransformed Hamiltonians, *Celest. Mech. Dyn. Astron.* **74** (1999a), 287–295.

[3] Y. Minesaki, An efficient conservative integrator with a chain regularization for the few-body problem, *Astron. J.* **150** (2015), 102.

Acknowledgments

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