Billiards in confocal quadrics as a pluri-Lagrangian system

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Part 1, based on:

Yu. S. *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*. J. Geometric Mechanics, 2013, **5**, 365–379,

A. Sridhar, Yu. S. *Commutativity in Lagrangian and Hamiltonian mechanics*, J. Geometry and Physics, 2018 (to appear)

Continuous time: principal action

- non-degenerate Lagrange function $L : TM \to \mathbb{R}$,
- corresponding Hamilton function $H : T^*M \to \mathbb{R}$,
- *action functional* of a continuous path $q : [t_1, t_2] \to M$:

$$\mathcal{S}[q] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt,$$

- *principal action function* $S : M \times M \times \mathbb{R} \to \mathbb{R}$ – critical value of action functional:

$$S(q_1, q_2, t) = \min \left\{ \mathcal{S}[q] \;\middle|\; q : [0, t] \to M,\; q(0) = q_1,\; q(t) = q_2 \right\}.$$

Principal action is a generating function of Hamiltonian flow:

$$\frac{\partial S(q_1, q_2, t)}{\partial q_1} = -p(0), \quad \frac{\partial S(q_1, q_2, t)}{\partial q_2} = p(t),$$

$$\frac{\partial S(q_1, q_2, t)}{\partial t} = -H.$$

(Hamilton-Jacobi equations).

Commuting actions vs. commuting Hamiltonians

Let L_1, L_2 be two non-degenerate Lagrangians, S_1, S_2 their principal action functions.

For $q_0, q_{12} \in M$ and $t_1, t_2 > 0$, let

$$S_{12}(q_0, q_{12}, t_1, t_2) = \min_{q_1 \in M} \Big(S_1(q_0, q_1, t_1) + S_2(q_1, q_{12}, t_2) \Big),$$
$$S_{21}(q_0, q_{12}, t_2, t_1) = \min_{q_2 \in M} \Big(S_2(q_0, q_2, t_2) + S_1(q_2, q_{12}, t_1) \Big).$$

Definition. *Principal actions of Lagrangians L_1, L_2 commute, if*

$$S_{12}(q_0, q_{12}, t_1, t_2) = S_{21}(q_0, q_{12}, t_2, t_1).$$

Theorem. *Principal actions of Lagrangians L_1, L_2 commute if and only if corresponding Hamiltonians Poisson commute, $\{H_1, H_2\} = 0$.*

Commuting discrete Lagrangians

Discrete time Lagrangians $\Lambda_i : M \times M \to \mathbb{R}$ $(i = 1, 2)$ – generating functions of symplectic maps

$$F_i : T^*M \ni (q_0, p_0) \mapsto (q_i, p_i) \in T^*M,$$

$$p_0 = -\frac{\partial \Lambda_i(q_0, q_i)}{\partial q_0}, \quad p_i = \frac{\partial \Lambda_i(q_0, q_i)}{\partial q_i}.$$

Definition. *Discrete Lagrangians Λ_1, Λ_2 commute, if the following two functions coincide identically:*

$$S_{12}(q_0, q_{12}) = \min_{q_1 \in M} \Big(\Lambda_1(q_0, q_1) + \Lambda_2(q_1, q_{12}) \Big)$$

and

$$S_{21}(q_0, q_{12}) = \min_{q_2 \in M} \Big(\Lambda_2(q_0, q_2) + \Lambda_1(q_2, q_{12}) \Big).$$

Commuting Lagrangians vs. commuting maps

Theorem. *If Lagrangians* Λ_1, Λ_2 *commute, then maps* F_1, F_2 *commute:*

$$F_1 \circ F_2 = F_2 \circ F_1.$$

Conversely, if maps F_1, F_2 *commute, then*

$$S_{12}(q_0, q_{12}) - S_{21}(q_0, q_{12}) = \text{const}.$$

Corner equations

Minimizers q_1, q_2 in S_{12}, S_{21}, are solutions of *corner equations*:

$$\frac{\partial\Lambda_1(q_0,q_1)}{\partial q_1}+\frac{\partial\Lambda_2(q_1,q_{12})}{\partial q_1}=0, \qquad (E_1)$$

$$\frac{\partial\Lambda_2(q_0,q_2)}{\partial q_2}+\frac{\partial\Lambda_1(q_2,q_{12})}{\partial q_2}=0. \qquad (E_2)$$

Crucial lemma. *Let Lagrangians Λ_1, Λ_2 commute. If q_1, q_2 satisfy corner equations (E_1), (E_2), then the following two corner equations are satisfied, as well:*

$$\frac{\partial\Lambda_1(q_0,q_1)}{\partial q_0}-\frac{\partial\Lambda_2(q_0,q_2)}{\partial q_0}=0, \qquad (E_0)$$

$$\frac{\partial\Lambda_1(q_2,q_{12})}{\partial q_{12}}-\frac{\partial\Lambda_2(q_1,q_{12})}{\partial q_{12}}=0. \qquad (E_{12})$$

Corner equations

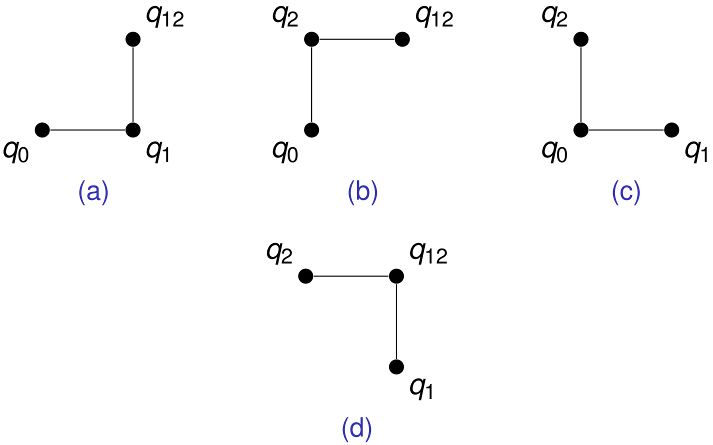


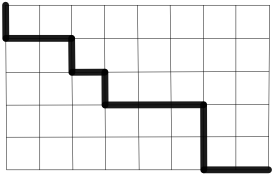
Figure: Four corner equations: (a) (E_1), (b) (E_2), (c) (E_0), (d) (E_{12}).

Pluri-Lagrangian problem, discrete time, $d = 1$

A *discrete 1-form* $\mathcal{L}$ is a skew-symmetric function on directed edges of $\mathbb{Z}^m$, depending on a field $q : \mathbb{Z}^m \to M$ (where M is a manifold):

$$\mathcal{L}(n, n+e_i) = \Lambda_i(q, q_i) \quad \Leftrightarrow \quad \mathcal{L}(n, n-e_i) = -\Lambda_i(q_{-i}, q).$$

A *discrete curve* Γ in $\mathbb{Z}^m$ is a concatenation of edges $\mathfrak{e}_k$.



Action functional along Γ:

$$S_\Gamma = \sum_{k \in \mathbb{Z}} \mathcal{L}(\mathfrak{e}_k).$$

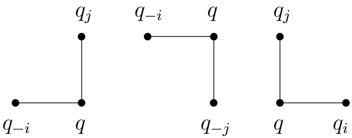
Problem. Find functions $q : \mathbb{Z}^m \to M$ delivering critical points for the functional S_Γ along *any* discrete curve Γ in $\mathbb{Z}^m$.

Pluri-Lagrangian EL equations = corner equations

Function $q : \mathbb{Z}^m \to M$ solves pluri-Lagrangian problem iff *corner equations* are satisfied everywhere:

$$
\begin{aligned}
\frac{\partial \Lambda_i(q_{-i}, q)}{\partial q} + \frac{\partial \Lambda_j(q, q_j)}{\partial q} &= 0, \\
\frac{\partial \Lambda_i(q_{-i}, q)}{\partial q} - \frac{\partial \Lambda_j(q_{-j}, q)}{\partial q} &= 0, \\
\frac{\partial \Lambda_i(q, q_i)}{\partial q} - \frac{\partial \Lambda_j(q, q_j)}{\partial q} &= 0
\end{aligned}
$$

($2m$ equations at each point $n \in \mathbb{Z}^m$, four equations per elementary square σ_{ij}).



Corner equations vs. usual Euler-Lagrange equations

If $q:\mathbb{Z}^m\to M$ solves a pluri-Lagrangian problem, then symplectic maps $F_i:(q,p)\mapsto(q_i,p_i)$ defined by

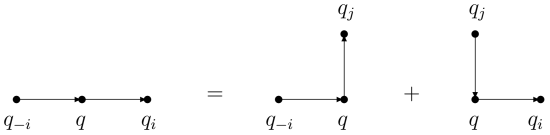
$$p=-\frac{\partial\Lambda_i(q,q_i)}{\partial q},\quad p_i=\frac{\partial\Lambda_i(q,q_i)}{\partial q_i},$$

commute, $F_i\circ F_j=F_j\circ F_i$.

Remark. Standard (single-time) discrete EL equations for F_i,

$$\frac{\partial\Lambda_i(q_{-i},q)}{\partial q}+\frac{\partial\Lambda_i(q,q_i)}{\partial q}=0,$$

are *consequences* of 2D corner equations:



Almost closedness of multi-time 1-form

The multi-time 1-form is almost closed on solutions of corner equations:

$$d\mathcal{L}(\sigma_{ij}) = \Lambda_i(q, q_i) + \Lambda_j(q_i, q_{ij}) - \Lambda_i(q_j, q_{ij}) - \Lambda_j(q, q_j) = \mathrm{const} =: \ell_{ij}.$$

All $\ell_{ij} = 0$ means that $\mathcal{L}$ is closed on solutions, i.e., extremal value of S_Γ does not depend on curve Γ connecting two given points in $\mathbb{Z}^m$.

Closedness of multi-time 1-form vs. spectrality

Let $F_\lambda : (q,p) \mapsto (\widetilde{q},\widetilde{p})$ be a *1-parameter family* of commuting symplectic maps, with Lagrange function $\Lambda(q,\widetilde{q};\lambda)$.

For a second such map, we write $F_\mu : (q,p) \mapsto (\widehat{x},\widehat{p})$.

Specifying previous result:

$$\Lambda(q,\widetilde{q};\lambda) + \Lambda(\widetilde{q},\widehat{\widetilde{q}};\mu) - \Lambda(q,\widehat{q};\mu) - \Lambda(\widehat{q},\widehat{\widetilde{q}};\lambda) = \ell(\lambda,\mu).$$

Theorem. Discrete 1-form $\mathcal{L}$ is closed on solutions of multi-time Euler-Lagrange equations iff

$$\partial\Lambda(q,\widetilde{q};\lambda)/\partial\lambda$$

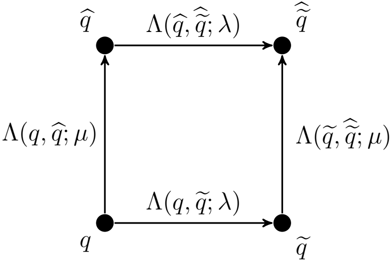
is a common integral of motion for all F_μ (*spectrality*; examples discovered by Kuznetsov-Sklyanin ’98).

Closedness of multi-time 1-form vs. spectrality (continued)

Proof. Due to skew-symmetry, $\ell(\lambda, \mu) = 0$ is equivalent to $\partial \ell(\lambda, \mu)/\partial \lambda = 0$, that is, to

$$\frac{\partial \Lambda(q, \widetilde{q}; \lambda)}{\partial \lambda} - \frac{\partial \Lambda(\widehat{q}, \widehat{\widetilde{q}}; \lambda)}{\partial \lambda} = 0.$$

This is equivalent to saying that $\partial \Lambda(q, \widetilde{q}; \lambda)/\partial \lambda$ is an integral of motion for F_μ. ■



Part 2, based on:

Yu. S. *Billiards in confocal quadrics as a pluri-Lagrangian system*. Theor. and Appl. Mech., 2016, **43**, 221–228.

Billiard in an ellipsoid

We consider the billiard in an ellipsoid

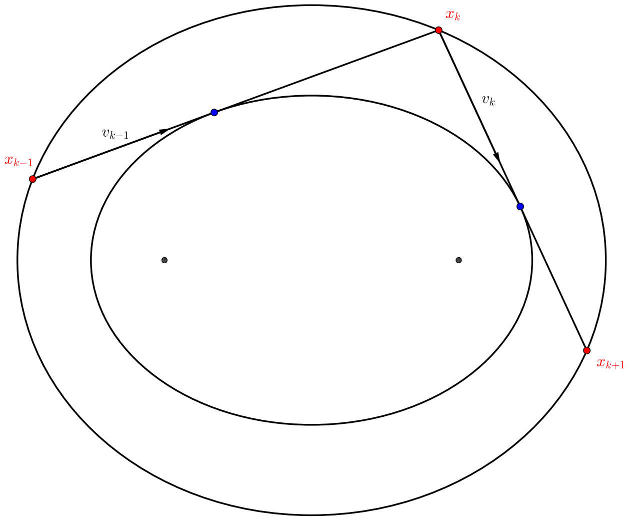
$$\mathcal{Q} = \left\{ x \in \mathbb{R}^n : \langle x, A^{-1}x \rangle = \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} = 1 \right\}.$$

Let $\{x_k\}_{k\in\mathbb{Z}}$, $x_k \in \mathcal{Q}$, be an orbit of this billiard. Denote by

$$v_k = \frac{x_{k+1} - x_k}{|x_{k+1} - x_k|} \in S^{n-1}$$

the unit vector along the line $(x_k x_{k+1})$.

Billiard in an ellipse



Billiard map

$$B: \begin{cases} x_{k+1} - x_k = \mu_k v_k, \\ v_k - v_{k-1} = \nu_k A^{-1} x_k. \end{cases}$$

Here μ_k, ν_k are found from $|v_k| = 1$ and $\langle x_k, A^{-1} x_k \rangle = 1$:

$$\mu_k = |x_{k+1} - x_k|,$$

$$\nu_k = \langle v_k - v_{k-1}, A(v_k - v_{k-1}) \rangle^{1/2}.$$

First (traditional) Lagrangian formulation

Eliminate v_k:

$$\frac{x_{k+1}-x_k}{\mu_k}-\frac{x_k-x_{k-1}}{\mu_{k-1}}=\nu_k A^{-1}x_k,$$

or

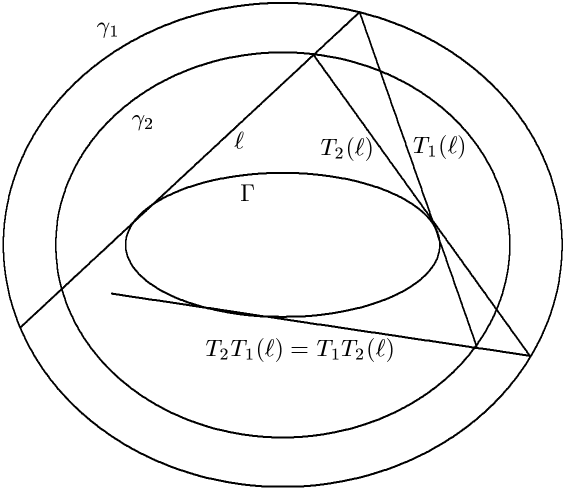
$$\frac{x_{k+1}-x_k}{|x_{k+1}-x_k|}-\frac{x_k-x_{k-1}}{|x_k-x_{k-1}|}=\nu_k A^{-1}x_k.$$

This is EL equation for the discrete Lagrange function

$$L:\mathcal{Q}\times\mathcal{Q}\to\mathbb{R},\quad L(x_k,x_{k+1})=|x_{k+1}-x_k|,$$

with Lagrange multiplier ν_k ensuring $x_k\in\mathcal{Q}$.

Commuting elliptic billiards



Billiard as a map on the space of lines

To formulate this correctly, B should be interpreted as a map on $\mathcal{L}$, the space of oriented lines in $\mathbb{R}^n$:

$$(x, v) \in \mathbb{R}^n \times S^{n-1} \quad \leftrightarrow \quad \ell = \{x + tv : t \in \mathbb{R}\} \in \mathcal{L}.$$

Replace here $x \in \ell$ by $x' = x - \langle x, v\rangle v \in \ell$, so that $\langle x', v\rangle = 0$,

$$x' \in T_v S^{n-1} \simeq T_v^* S^{n-1}.$$

Thus, one can identify

$$\mathcal{L} \simeq T^* S^{n-1},$$

and the billiard map can be considered as

$$B : T^* S^{n-1} \to T^* S^{n-1}.$$

Commuting elliptic billiards

Theorem. For any two quadrics $\mathcal{Q}_\lambda$ and $\mathcal{Q}_\mu$ from the confocal family

$$\mathcal{Q}_\lambda = \{x \in \mathbb{R}^n : Q_\lambda(x) = 1\},$$

where

$$Q_\lambda(x) := \langle x, (A + \lambda I)^{-1} x \rangle = \sum_{i=1}^{n} \frac{x_i^2}{a_i^2 + \lambda},$$

the corresponding maps $B_\lambda : T^* S^{n-1} \to T^* S^{n-1}$ and $B_\mu : T^* S^{n-1} \to T^* S^{n-1}$ commute.

Second ("dual") Lagrangian formulation

To find a Lagrange function $L: S^{n-1} \times S^{n-1} \to \mathbb{R}$ for the map $B: T^*S^{n-1} \to T^*S^{n-1}$, eliminate variables x_k:

$$\frac{A(v_{k+1} - v_k)}{\nu_{k+1}} - \frac{A(v_k - v_{k-1})}{\nu_k} = \mu_k v_k,$$

or

$$\frac{A(v_{k+1} - v_k)}{\langle v_{k+1} - v_k, A(v_{k+1} - v_k)\rangle^{1/2}} - \frac{A(v_k - v_{k-1})}{\langle v_k - v_{k-1}, A(v_k - v_{k-1})\rangle^{1/2}} = \mu_k v_k.$$

This is EL equation for the discrete Lagrange function

$$L: S^{n-1} \times S^{n-1} \to \mathbb{R}, \quad L(v_k, v_{k+1}) = \langle v_{k+1} - v_k, A(v_{k+1} - v_k)\rangle^{1/2},$$

with Lagrange multiplier μ_k ensuring $v_k \in S^{n-1}$.
Consequence: B preserves the canonical 2-form on T^*S^{n-1}.

A novel derivation of a result originally due to J. Moser.

Theorem. The maps $B_\mu : T^*S^{n-1} \to T^*S^{n-1}$ have a set of common integrals of motion

$$F_i(v,x) = v_i^2 + \sum_{j\neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i^2 - a_j^2}, \quad 1 \le i \le n.$$

Only $n-1$ of them are functionally independent, due to $\sum_{i=1}^n F_i = \langle v, v\rangle = 1$.

Proof. The value of the constant

$$dL(\lambda,\mu) := L(v,\widetilde{v};\lambda) + L(\widetilde{v},\widehat{\widetilde{v}};\mu) - L(\widehat{v},\widehat{\widetilde{v}};\lambda) - L(v,\widehat{v};\mu)$$

is easily determined on a concrete billiard trajectory along the big axis of $\mathcal{Q}_\lambda$, $\mathcal{Q}_\mu$, for which $v=(1,0,\ldots,0)$, $\widetilde{v}=\widehat{v}=-v$, and $\widehat{\widetilde{v}}=v$. Recall that

$$L(v,\widetilde{v};\lambda) = \langle \widetilde{v}-v, (A+\lambda I)(\widetilde{v}-v)\rangle^{1/2}.$$

There follows immediately that $dL(\lambda,\mu)=0$. Now find a generating function of common integral of motion for all B_μ:

$$\frac{\partial L(\underset{\sim}{v},v;\lambda)}{\partial\lambda} = \frac{\langle v-\underset{\sim}{v}, v-\underset{\sim}{v}\rangle}{\langle v-\underset{\sim}{v}, (A+\lambda I)(v-\underset{\sim}{v})\rangle^{1/2}}$$
$$= 2\langle x, (A+\lambda I)^{-1}v\rangle.$$

Thus,

$$Q_\lambda(x,v) := \langle x, (A+\lambda I)^{-1} v\rangle = \sum_{i=1}^{n} \frac{x_i v_i}{\lambda + a_i^2}$$

with $v \in S^{n-1}$, $x \in \mathcal{Q}_\lambda$, is an integral of motion for all maps $B_\mu : \mathcal{L} \to \mathcal{L}$.

To find $F_i(x,v)$ manifestly independent on the choice of $x \in \ell \cap \mathcal{Q}_\lambda$, observe: as soon as $Q_\lambda(x) = 1$, we have

$$\begin{aligned} Q_\lambda^2(x,v) &= Q_\lambda(v) - Q_\lambda(v) Q_\lambda(x) + Q_\lambda^2(x,v) \\ &= \sum_{i=1}^{n} \frac{F_i(v,x)}{\lambda + a_i^2}. \quad \blacksquare \end{aligned}$$

Conclusions

The notion of pluri-Lagrangian systems can serve as integrability of discrete and continuous variational systems.

- (Almost) closedness of the Lagrangian form ($d\mathcal{L} = \mathrm{const}$) on solutions of the pluri-Lagrangian system built-in.
- Closedness of the Lagrangian form ($d\mathcal{L} = 0$) on solutions is related to existence of integrals of motion (resp. conservation laws) in the discrete case, and to involutivity of Hamiltonians in the continuous case.
- Classification of pluri-Lagrangian systems looks promising.